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The Milnor number of plane irreducible singularities in positive characteristic

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Abstract

Let $\mu(f)$ (respectively, $c(f)$) be the Milnor number (respectively, the degree) of the conductor of an irreducible power series $f \in \mathbf{K}[[x,y]]$, where $\mathbf{K}$ is an algebraically closed field of characteristic $p \geqslant 0$. It is well known that $\mu(f) \geqslant c(f)$. We give necessary and sufficient conditions for the equality $\mu(f) = c(f)$ in terms of the semigroup associated with f, provided that $p > \operatorname{ord} f$.

Introduction

Let $\mathbf{K}$ be an algebraically closed field of characteristic $p \geqslant 0$ and let $f \in \mathbf{K}[[x,y]]$ be a reduced (without multiple factors) power series. Denote by $\overline{\mathcal{O}}$ the normalization of the ring $\mathcal{O} = \mathbf{K}[[x,y]]/(f)$ and consider the conductor ideal $\mathcal{C}$ of $\overline{\mathcal{O}}$ in $\mathcal{O}$. The integer $c(f) = \dim_{\mathbf{K}} \overline{\mathcal{O}}/\mathcal{C}$ is called the degree of the conductor. Since $\mathcal{O}$ is Gorenstein, we have $c(f) = 2\delta(f)$, where $\delta(f) = \dim_{\mathbf{K}} \overline{\mathcal{O}}/\mathcal{O}$ is the double point number. Recall that $\mu(f) = \dim_{\mathbf{K}} \mathbf{K}[[x,y]]/(\partial f/\partial x, \partial f/\partial y)$ is the Milnor number of f.

If $\operatorname{char} \mathbf{K} = 0$, then the Milnor formula holds : $\mu(f) = 2\delta(f) - r(f) + 1$, where $r(f)$ is the number of distinct irreducible factors of f (see [**10**, **12**]). If the characteristic $\operatorname{char} \mathbf{K}$ is arbitrary, then $\mu(f) \geqslant 2\delta(f) - r(f) + 1$ (see [**3**, **8**]) and the equality $\mu(f) = 2\delta(f) - r(f) + 1$ ($\mu(f) = c(f)$ if f is irreducible) means that f has no *wild* vanishing cycles. It is the case if f is Newton non-degenerate (see [**2**]) or if p is greater than the intersection number of f with its generic polar (see [**11**]).

The aim of this note is to give necessary and sufficient conditions for the equality $\mu(f) = c(f)$ in terms of the semigroup associated with the irreducible series f, provided that $p > \operatorname{ord} f$ (the order of f). Our result gives a partial answer to the question raised by Greuel and Nguyen [**7**].

1. *Main result*

Let f be an irreducible power series in $\mathbf{K}[[x,y]]$, where $\mathbf{K}$ is an algebraically closed field of characteristic $p \geqslant 0$. The semigroup $\Gamma(f)$ associated with the branch $f = 0$ is defined as the set of intersection numbers $i_0(f,h) = \dim_{\mathbf{K}} \mathbf{K}[[x,y]]/(f,h)$, where h runs over all power series such that $h \not\equiv 0 \pmod f$.

Let $\overline{\beta_0}, \ldots, \overline{\beta_g}$ be the minimal sequence of generators of $\Gamma(f)$ defined by the conditions

(i) $\overline{\beta_0} = \min(\Gamma(f)\backslash\{0\}) = \operatorname{ord} f$;
(ii) $\overline{\beta_k} = \min(\Gamma(f)\backslash \mathbf{N}\overline{\beta_0} + \cdots + \mathbf{N}\overline{\beta_{k-1}})$ for $k \in \{1, \ldots, g\}$;
(iii) $\Gamma(f) = \mathbf{N}\overline{\beta_0} + \cdots + \mathbf{N}\overline{\beta_g}$.

Let $e_k = \gcd(\overline{\beta_0}, \ldots, \overline{\beta_k})$ for $k \in \{1, \ldots, g\}$. Then $e_0 > e_1 > \cdots e_{g-1} > e_g = 1$ and $e_{k-1}\overline{\beta_k} < e_k\overline{\beta_{k+1}}$ for $k \in \{1, \ldots, g-1\}$. Let $n_k = e_{k-1}/e_k$ for $k \in \{1, \ldots, g\}$. Then $n_k > 1$ for $k \in \{1, \ldots, g\}$ and $n_k\overline{\beta_k} < \overline{\beta_{k+1}}$ for $k \in \{1, \ldots, g-1\}$.

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The degree of the conductor $c(f)$ is equal to the smallest element of $\Gamma(f)$ such that $c(f)+N \in \Gamma(f)$ for all integers $N \geqslant 0$. It is given by the conductor formula:

$$c(f)=\sum_{k=1}^{g}(n_k-1)\overline{\beta_k}-\overline{\beta_0}+1. \tag{1.1}$$

For the proof of the above equality we refer the reader to [6, Proposition 2.3].

The Milnor number $\mu(f)$ is not, in general, determined by $\Gamma(f)$. The following example is borrowed from [2]: take $f=x^p+y^{p-1}$ and $g=(1+x)f$, where $p>2$. Then $\Gamma(f)=\Gamma(g)$, $\mu(f)=+\infty$ and $\mu(g)=p(p-2)$. By a plane curve singularity we mean a non-zero power series of order greater than 1. The aim of this note is the following theorem.

THEOREM 1.1 (Main result). *Let $f \in \mathbf{K}[[x,y]]$ be an irreducible singularity and let $\overline{\beta_0},\dots,\overline{\beta_g}$ be the minimal system of generators of $\Gamma(f)$. Suppose that $p = char\,\mathbf{K} > ord\ f$. Then the following two conditions are equivalent*:

(i) $\overline{\beta_k} \not\equiv 0\ (mod\ p)$ *for* $k \in \{1,\dots,g\}$;
(ii) $\mu(f)=c(f)$.

We prove Theorem 1.1 in Section 3 of this note.

EXAMPLE 1. Let $f(x,y)=(y^2+x^3)^2+x^5y$. Then f is irreducible and $\Gamma(f)=4\mathbf{N}+6\mathbf{N}+13\mathbf{N}$ (see [6, Theorem 6.6]). By the conductor formula $c(f)=16$. Let $p=\operatorname{char}\mathbf{K}>\operatorname{ord} f=4$. If $p\neq 13$, then $\mu(f)=c(f)$ by Theorem 1.1. If $p=13$, then a direct calculation shows that $\mu(f)=17$.

EXAMPLE 2. Let $f=x^m+y^n+\sum_{n\alpha+m\beta>nm}c_{\alpha\beta}x^\alpha y^\beta$, where $1<n<m$ and $\gcd(n,m)=1$. Then $\Gamma(f)=\mathbf{N}n+\mathbf{N}m$ and $c(f)=(n-1)(m-1)$. We get $\mu(f)\geqslant(n-1)(m-1)$ with equality if and only if $n\not\equiv 0 \pmod p$ and $m \not\equiv 0 \pmod p$. To compute $\mu(f)$ one can use [5, Theorem 3].

2. *Factorization of the polar curve*

Let $f\in\mathbf{K}[[x,y]]$ be an irreducible singularity and let $\Gamma(f)=\mathbf{N}\overline{\beta_0}+\cdots+\mathbf{N}\overline{\beta_g}$ be the semigroup associated with f. Since f is unitangent $i_0(f,x)=\operatorname{ord} f$ or $i_0(f,y)=\operatorname{ord} f$. In the whole of this section, we assume that $i_0(f,x)=\operatorname{ord} f$. Let $n=\operatorname{ord} f$.

LEMMA 2.1. *Let $\psi=\psi(x,y)\in\mathbf{K}[[x,y]]$ be an irreducible power series such that $i_0(\psi,x)=ord\ \psi$. If $i_0(f,\psi)/ord\ \psi > e_{k-2}\overline{\beta_{k-1}}/n$ for $k\geqslant 2$, then $ord\ \psi\equiv 0\ (mod\ n/e_{k-1})$.*

Proof. For the proof, see [6, Lemma 5.6]. □

In what follows, we need a sharpened version of Merle's factorization theorem (see [9, Theorem 3.1]).

THEOREM 2.2. *Suppose that $ord\ f\not\equiv 0 (mod\,p)$. Then $\partial f/\partial y=\psi_1\cdots\psi_g$ in $\mathbf{K}[[x,y]]$, where*

(i) *$ord\ \psi_k = n/e_k - n/e_{k-1}$ for $k\in\{1,\dots,g\}$;*
(ii) *if $\phi\in\mathbf{K}[[x,y]]$ is an irreducible factor of ψ_k, $k\in\{1,\dots,g\}$, then*
   (a) *$i_0(f,\phi)/ord\ \phi = e_{k-1}\overline{\beta_k}/n$, and*
   (b) *$ord\ \phi\equiv 0\ (mod\,n/e_{k-1})$.*

Proof. The proof of the existence of the factorization $\partial f/\partial y = \psi_1 \cdots \psi_g$ with properties (i) and (ii)(a) given by Merle for the generic polar in the case $\mathbf{K} = \mathbf{C}$ works in our situation (see also [4]). To check (ii)(b) observe that $i_0(\partial f/\partial y, x) = n - 1$ and consequently $i_0(\phi, x) = \text{ord}\ \phi$ for any irreducible factor ϕ of $\partial f/\partial y$. Then use Lemma 2.1. □

3. *Proof of the main result*

We keep the notation and assumptions of Section 2. In particular, $f \in \mathbf{K}[[x, y]]$ is irreducible and $i_0(f, x) = \text{ord}\ f$. We let $n = \text{ord}\ f$. The following lemma is well known and may be deduced from the formula $\overline{\mathcal{O}} f'_y = \mathcal{C}\mathcal{D}_x$, where $\mathcal{D}_x$ is the different of $\overline{\mathcal{O}}$ with respect to the ring $\mathbf{K}[[x]]$(see [**14**, p. 10; **1**, Aphorism 5]).

Lemma 3.1. *Suppose that ord $f \not\equiv 0 \pmod p$. Then*

$$i_0\left(f, \frac{\partial f}{\partial y}\right) = c(f) + \textit{ord}\ f - 1.$$

Proof. Since $n \not\equiv 0 \pmod p$ the irreducible curve $f = 0$ has a good parametrization of the form $(t^n, y(t))$. Let $\beta_0 = n, \beta_1, \ldots, \beta_g$ be the characteristic of $(t^n, y(t))$. Then $\overline{\beta_0} = \beta_0$, $\overline{\beta_1} = \beta_1$ and $\overline{\beta_{k+1}} = n_k \overline{\beta_k} + \beta_{k+1} - \beta_k$ for $k \in \{1, \ldots, g-1\}$ (see [**14**, Section 3]).

Denote by $\mathbf{U}(n)$ the group of nth roots of unity in $\mathbf{K}$. A simple computation shows that

$$i_0\left(f, \frac{\partial f}{\partial y}\right) = \sum_{\epsilon \in \mathbf{U}(n) \setminus \{1\}} \text{ord}\ (y(t) - y(\epsilon t)) = \sum_{k=1}^{g} (e_{k-1} - e_k)\beta_k = \sum_{k=1}^{g} (n_k - 1)\overline{\beta_k}.$$

Now, the lemma follows from the conductor formula (1.1). □

Corollary 3.2. *If ord $f \not\equiv 0 \pmod p$, then $\mu(f) = c(f)$ if and only if $i_0(f, \partial f/\partial y) = \mu(f) + \textit{ord}\ f - 1$.*

If char $\mathbf{K} = 0$, then $i_0(f, \partial f/\partial y) = \mu(f) + i_0(f, x) - 1$ (see [**13**, Chapter II, Proposition 1.2]) for any reduced series $f \in \mathbf{K}[[x, y]]$, whence $\mu(f) = c(f)$ for irreducible f in characteristic zero.

Lemma 3.3. *Suppose that $p >$ ord f. Then $i_0(f, \partial f/\partial y) \leqslant \mu(f) + \textit{ord}\ f - 1$ with equality if and only if $\overline{\beta_k} \not\equiv 0 \pmod p$ for $k \in \{1, \ldots, g\}$.*

Proof. Let us begin with the following claim.
*C*laim 1: Suppose that $p > \text{ord}\ f$. Then for every irreducible factor ϕ of $\partial f/\partial y$ we have $i_0(\partial f/\partial x, \phi) + \text{ord}\ \phi \geqslant i_0(f, \phi)$ with equality if and only if $i_0(f, \phi) \not\equiv 0 \pmod p$.

Proof of Claim 1. Let ϕ be an irreducible factor of $\partial f/\partial y$. Then $\text{ord}\ \phi \leqslant \text{ord}\ (\partial f/\partial y) = \text{ord}\ f - 1$. Let $(x(t), y(t))$ be a good parametrization of $\phi = 0$. Then $\text{ord}\ x(t) = i_0(x, \phi) = \text{ord}\ \phi < \text{ord}\ f \leqslant p$ and, consequently, $\text{ord}\ x(t) \not\equiv 0 \pmod p$, which implies $\text{ord}\ x'(t) = \text{ord}\ x(t) - 1$. We have

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}(x(t), y(t)) x'(t).$$

Taking orders gives $\text{ord}\ (df(x(t), y(t))/dt) \geqslant \text{ord}\ f(x(t), y(t)) - 1$, with equality if and only if $\text{ord}\ f(x(t), y(t)) \not\equiv 0 \pmod p$, and $\text{ord}\ \frac{\partial f}{\partial x}(x(t), y(t)) x'(t) = \text{ord}\ \frac{\partial f}{\partial x}(x(t), y(t)) + \text{ord}\ x(t) - 1$.

Therefore, $\text{ord}\ \frac{\partial f}{\partial x}(x(t), y(t)) + \text{ord}\ x(t) \geqslant \text{ord}\ f(x(t), y(t))$ with equality if and only if $\text{ord}\ f(x(t), y(t)) \not\equiv 0 \pmod p$. Passing to the intersection numbers, we get the claim. □

Claim 2: Suppose that $p > \operatorname{ord} f$ and let $\partial f/\partial y = \psi_1 \cdots \psi_g$ be the Merle factorization of the polar $\partial f/\partial y$. Let ϕ be an irreducible factor of ψ_k. Then $i_0(f,\phi) \not\equiv 0 \pmod p$ if and only if $i_0(f,\phi) \not\equiv 0 \pmod{\overline{\beta_k}}$.

Proof of Claim 2. By Theorem 2.2(ii)(b), we can write $\operatorname{ord}\phi = m_k(n/e_{k-1})$, where $m_k \geqslant 1$ is an integer. Since $\operatorname{ord}\phi \leqslant \operatorname{ord}(\partial f/\partial y) = \operatorname{ord} f - 1 < p$, we have $\operatorname{ord}\phi \not\equiv 0 \pmod p$, which implies $m_k \not\equiv 0 \pmod p$. By Theorem 2.2(ii)(a), $i_0(f,\phi) = (e_{k-1}\overline{\beta_k}/n)\operatorname{ord}\phi = m_k\overline{\beta_k}$. Therefore, $i_0(f,\phi) \not\equiv 0 \pmod p$ if and only if $\overline{\beta_k} \not\equiv 0 \pmod p$. □

Now we continue with the proof of the lemma. Let P be the set of all irreducible factors of $\partial f/\partial y$. Then, by Claim 1,

$$\begin{aligned} i_0\left(f, \frac{\partial f}{\partial y}\right) &= \sum_{\phi\in P} e(\phi) i_0(f,\phi) \leqslant \sum_{\phi \in P} e(\phi) i_0\left(\frac{\partial f}{\partial x}, \phi\right) = \mu(f) + \operatorname{ord} \frac{\partial f}{\partial y} \\ &= \mu(f) + \operatorname{ord} f - 1, \end{aligned}$$

where $e(\phi) = \max\{e : \phi^e \text{ divides } \partial f/\partial y\}$ and with equality if and only if $i_0(f,\phi) \not\equiv 0 \pmod p$ for all $\phi \in P$. According to Claim 2, $i_0(f,\phi) \not\equiv 0 \pmod p$ for all $\phi \in P$ if and only if $\overline{\beta_k} \not\equiv 0 \pmod p$ for $k \in \{1,\ldots,g\}$ and the lemma follows. □

Remark 1. If $p < \operatorname{ord} f$, then the proof of Lemma 3.3 fails, even if $\operatorname{ord} f \not\equiv 0 \pmod p$. Take $f = x^{p+2} + y^{p+1} + x^{p+1}y$.

Proof of Theorem 1. Let $f \in \mathbf{K}[[x,y]]$ be an irreducible singularity. Suppose that $p = \operatorname{char}\mathbf{K} > \operatorname{ord} f$. Then, by Lemma 3.1, $\mu(f) = c(f)$ is equivalent to Teissier's formula $i_0(f, \partial f/\partial y) = \mu(f) + \operatorname{ord} f - 1$, which by Lemma 3.3 holds if and only if $\overline{\beta_k} \not\equiv 0 \pmod p$ for $k \in \{1,\ldots,g\}$. □

Conjecture. Let $f \in \mathbf{K}[[x,y]]$ be an irreducible singularity with the semigroup $\Gamma(f) = \mathbf{N}\overline{\beta_0} + \cdots + \mathbf{N}\overline{\beta_g}$. Then $\mu(f) = c(f)$ if and only if $\overline{\beta_k} \not\equiv 0 \pmod{\operatorname{char}\mathbf{K}}$ for $k \in \{0,\ldots,g\}$.

The conjecture is true if $\Gamma(f) = \mathbf{N}\overline{\beta_0} + \mathbf{N}\overline{\beta_1}$ (cf. Example 2 of this note).

References

1. S. S. Abhyankar, 'Historical ramblings in algebraic geometry and related algebra', *Amer. Math. Monthly* 83 (1976) 409–448.
2. Y. Boubakri, G.-M. Greuel and T. Markwig, 'Invariants of hypersurface singularities in positive characteristic', *Rev. Mat. Complut.* 25 (2010) 61–85.
3. P. Deligne, 'La formule de Milnor', *Groupes de Monodromie en Géométrie Algébrique*, Lecture Notes in Mathematics 340 (Springer, Berlin, 1973) 197–211.
4. F. Delgado de la Mata, 'A factorization theorem for the polar of a curve with two branches', *Compos. Math.* 92 (1994) 327–375.
5. M. Furuya and M. Tomari, 'A characterization of semi-quasihomogeneous functions in terms of the Milnor number', *Proc. AMS* 132, 1885–1890.
6. E. García Barroso and A. Płoski, 'An approach to plane algebroid branches', *Rev. Mat. Complut.* 28 (2015) 227–252.
7. G.-M Greuel and H. D. Nguyen, 'Some remarks on the planar Kouchnirenko's theorem', *Rev. Mat. Complut.* 25 (2012) 557–579.
8. A. Melle-Hernández and C.T. C. Wall, 'Pencils of curves on smooth surfaces', *Proc. London Math. Soc., III Ser.* 83 (2001) 257–278.
9. M. Merle, 'Invariants polaires des courbes planes', *Invent. Math.* 41 (1977) 103–111.
10. J. W. Milnor, *Singular points of complex hypersurfaces* (Princeton University Press, Princeton, NJ, 1968).
11. H. D. Nguyen, 'Invariants of plane curve singularities, and Plücker formulas in positive characteristic', Preprint, 2014, arXiv:1412.5007v1 [math. AG].
12. J. J. Risler, 'Sur l'idéal jacobien d'une courbe plane', *Bull. Soc. Math. France* 99 (1971) 305–311.

13. B. Teissier, ‘Cycles évanescents, section planes et condition de Whitney’, *Astérisque* 7 et 8 (1973) 285–362.
14. O. Zariski, *Le problème des modules pour les branches planes*, Centre de Mathématiques de l’École Polytechnique (Paris, 1973). With an appendix by Bernard Teissier, 2nd edn (Éditions Hermann, Paris, 1986).

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