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Łojasiewicz exponents and Farey sequences

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Abstract Let I be an ideal of the ring of formal power series $\mathbf{K}[[x, y]]$ with coefficients in an algebraically closed field $\mathbf{K}$ of arbitrary characteristic. Let Φ denote the set of all parametrizations $\varphi = (\varphi_1, \varphi_2) \in \mathbf{K}[[t]]^2$, where $\varphi \neq (0, 0)$ and $\varphi(0, 0) = (0, 0)$. The purpose of this paper is to investigate the invariant

$$\mathcal{L}_0(I) = \sup_{\varphi \in \Phi} \left(\inf_{f \in I} \frac{\operatorname{ord} f \circ \varphi}{\operatorname{ord} \varphi} \right)$$

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called the *Łojasiewicz exponent* of I. Our main result states that for the ideals I of finite codimension the Łojasiewicz exponent $\mathcal{L}_0(I)$ is a Farey number i.e. an integer or a rational number of the form $N + \frac{b}{a}$, where a, b, N are integers such that $0 < b < a < N$.

Keywords Łojasiewicz exponent · Logarithmic distance · Newton diagram · Farey sequences

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1 Introduction

Let $\mathbf{K}$ be an algebraically closed field of arbitrary characteristic. Let t be a variable. A *parametrization* is a pair $\varphi(t) = (\varphi_1(t), \varphi_2(t)) \in \mathbf{K}[[t]]^2 \backslash \{(0,0)\}$ such that $\varphi_1(0) = \varphi_2(0) = 0$. We put $\operatorname{ord} \varphi = \inf\{\operatorname{ord} \varphi_1, \operatorname{ord} \varphi_2\}$, where $\operatorname{ord} \varphi_k$ stands for the *order* of the power series $\varphi_k = \varphi_k(t)$. For any ideal $I \subset \mathbf{K}[[x, y]]$ we consider the *Łojasiewicz exponent* $\mathcal{L}_0(I)$ (see [1,4,5,8,10]) defined by the formula

$$\mathcal{L}_0(I) = \sup_{\varphi \in \Phi} \left(\inf_{f \in I} \frac{\operatorname{ord} f \circ \varphi}{\operatorname{ord} \varphi} \right),$$

where Φ stands for the set of all parametrizations $\varphi = (\varphi_1, \varphi_2)$.

Note that $\mathcal{L}_0(I) < +\infty$ if and only if I is of finite codimension.

In the framework of the complex analytic geometry the notion of Łojasiewicz exponent was introduced and studied by Lejeune-Jalabert and Teissier [8]. They considered much more general notion including the Łojasiewicz exponent of holomorphic ideals in several variables. D'Angelo [4] defined this invariant independently and gave its applications to complex function theory on domains in $\mathbf{C}^n$. Recently Cassou-Noguès and Veys [2] introduced an algorithm to study ideals in $\mathbf{K}[[x, y]]$ which enables us to compute $\mathcal{L}_0(I)$ using a finite sequence of Newton diagrams.

Let $g \in \mathbf{K}[[x, y]]$ be an irreducible power series. We put

$$\mathcal{L}_0(I, g) = \inf_{f \in I} \left\{ \frac{\operatorname{ord} f \circ \varphi}{\operatorname{ord} \varphi} \right\},$$

where φ is a parametrization such that $g \circ \varphi = 0$. The notion does not depend on the choice of φ. If $\mathcal{L}_0(I) = \mathcal{L}_0(I, g)$ then we say that the Łojasiewicz exponent *is attained on the branch* $g = 0$.

Theorem 1 ([1, Theorem 6]) *Let $I \subset \mathbf{K}[[x, y]]$ be a proper ideal and let $f_1, \ldots, f_m$ be generators of I. Then there is an irreducible factor g of the power series $f_1, \ldots, f_m$ such that $\mathcal{L}_0(I)$ is attained on the branch $g = 0$.*

This result was proved by Chądzyński and Krasiński [3, Theorem 3] and independently by McNeal and Némethi [9, Theorem 1.1] for holomorphic ideals. The case of ideals in $\mathbf{K}[[x, y]]$, where $\mathbf{K}$ is of arbitrary characteristic is due to Brzostowski and Rodak

[1, Theorem 6]. In Sect. 2 of this note we give a very short proof of it. Let us write down the following corollary to Theorem 1.

Corollary 1 *If* $I \subset \mathbf{K}[[x, y]]$ *is of finite codimension then* $\mathcal{L}_0(I)$ *is a rational number.*

Our main result is

Theorem 2 *Let* I *be an ideal of* $\mathbf{K}[[x, y]]$ *of finite codimension. Then* $\mathcal{L}_0(I)$ *is a Farey number, i.e.,* $\mathcal{L}_0(I)$ *is an integer or a rational number of the form* $N + \frac{b}{a}$, *where* N, a, b *are integers such that* $0 < b < a < N$.

Theorem 2 gives a positive answer to Question 1 of [1]. It implies that the fractional parts of the Łojasiewicz exponents $\mathcal{L}_0(I)$ form the Farey sequences of order $\lfloor \mathcal{L}_0(I) \rfloor$ (see [7]), where $\lfloor z \rfloor$ denotes the integer part of the real number z.

The proof of Theorem 2 is given in Sect. 3.

The holomorphic version (I is a holomorphic ideal generated by two elements) was proved in [10, Theorem 3.4]. Its proof does not extend to the case of arbitrary characteristic.

2 Proof of Theorem 1

For any $f, g \in \mathbf{K}[[x, y]]$ we consider the intersection number

$$i_0(f, g) = \dim_{\mathbf{K}} \mathbf{K}[[x, y]]/(f, g),$$

where (f, g) is the ideal generated by f and g in $\mathbf{K}[[x, y]]$. Let

$$d(f, g) = \frac{i_0(f, g)}{\operatorname{ord} f \ \operatorname{ord} g}$$

for irreducible $f, g \in \mathbf{K}[[x, y]]$. Then $d(f, g)$ is a *logarithmic distance* on the set of all irreducible power series, that is

(D1) $d(f, f) = +\infty$,
(D2) $d(f, g) = d(g, f)$, and
(D3) $d(f, g) \geq \inf\{d(f, h), d(g, h)\}$ for f, g, h irreducible power series.

Only Property (D3) is non-trivial (see [6, Corollary 2.9]).

If $g \in \mathbf{K}[[x, y]]$ is irreducible then there exists a parametrization $\psi^o \in \mathbf{K}[[t]]^2$ such that $g \circ \psi^0 = 0$ and $\operatorname{ord} \psi^0 = \operatorname{ord} g$. Moreover, for any power series $f \in \mathbf{K}[[x, y]]$ we have $i_0(f, g) = \operatorname{ord}(f \circ \psi^o)$. If ψ is a parametrization such that $g \circ \psi = 0$ then there exists $\tau \in \mathbf{K}[[t]]$ of positive order such that $\psi = \psi^o \circ \tau$. The equality $\frac{\operatorname{ord}(f\circ\psi)}{\operatorname{ord}\psi} = \frac{\operatorname{ord}(f\circ\psi^o)}{\operatorname{ord}\psi^o} = \frac{i_0(f,g)}{\operatorname{ord} g}$ shows that the definition of $\mathcal{L}_0(I, g)$ is correct and can be rewritten as follows $\mathcal{L}_0(I, g) = \inf_{f\in I} \dfrac{i_0(f, g)}{\operatorname{ord} g}$.

If φ is a parametrization, then there exists an irreducible power series $g \in \mathbf{K}[[x, y]]$ such that $g \circ \varphi = 0$. This shows that

$$\mathcal{L}_0(I) = \sup \{\mathcal{L}_0(I, g) \ : \ g \text{ is irreducible}\}.$$

If $I = (f_1, \ldots, f_m)$, then

$$\mathcal{L}_0(I, g) = \inf_{1 \le k \le m} \frac{i_0(f_k, g)}{\operatorname{ord} g}. \tag{1}$$

Lemma 1 *Let $I = (f_1, \ldots, f_m)$ and let $\prod_i f_i = \prod_j h_j$ with $h_j \in \mathbf{K}[[x, y]]$ irreducible. Let $g \in \mathbf{K}[[x, y]]$ be an irreducible power series. Take an index k such that $d(g, h_k) = \sup_j \{d(g, h_j)\}$. Then $\mathcal{L}_0(I, g) \le \mathcal{L}_0(I, h_k)$.*

Proof Let us denote h_k by h. Then $d(g, h) \ge d(g, h_j)$ for any index j. After (D3) we get $d(h, h_j) \ge \inf\{d(g, h), d(g, h_j)\} = d(g, h_j)$. Therefore for any j we have $\frac{i_0(g,h_j)}{\operatorname{ord} g} \le \frac{i_0(h,h_j)}{\operatorname{ord} h}$ and consequently for any $i \in \{1, \ldots, m\}$ we get $\frac{i_0(g,f_i)}{\operatorname{ord} g} \le \frac{i_0(h,f_i)}{\operatorname{ord} h}$, which implies $\mathcal{L}_0(I, g) \le \mathcal{L}_0(I, h)$.

Now, we can prove Theorem 1.

Proof of Theorem 1 We keep the notations of Lemma 1. Fix an irreducible power series g. Then $\mathcal{L}_0(I, g) \le \mathcal{L}_0(I, h_k)$. Hence $\mathcal{L}_0(I) \le \sup_j\{\mathcal{L}_0(I, h_j)\}$. The inverse inequality is obvious. Therefore $\mathcal{L}_0(I) = \sup_j\{\mathcal{L}_0(I, h_j)\}$, which proves Theorem 1 □

3 Proof of Theorem 2

Let $f = \sum c_{\alpha\beta} x^\alpha y^\beta \in \mathbf{K}[[x, y]]$. The *Newton diagram* $\Delta(f)$ of f is by definition the convex hull of the set $\{(\alpha, \beta) \in \mathbf{N}^2 : c_{\alpha\beta} \ne 0\} + \mathbf{R}^2_{\ge 0}$. We use Teissier's notation ([8, p. 846]) denoting by $\left\{\frac{b}{a}\right\}$ the Newton diagram of $y^a + x^b$, for $a, b > 0$. The following properties of Newton diagrams are well-known

(N1) for generic $c_1, \ldots, c_m$, $\Delta(\sum_{i=1}^m c_i f_i)$ is the convex hull of the set $\bigcup_{i=1}^m \Delta(f_i)$,

(N2) if $f = 0$ is a branch different from the axes then $\Delta(f) = \left\{\frac{i_0(f, y)}{i_0(f, x)}\right\}$,

(N3) if $\Delta(f_1) = \left\{\frac{b_1}{a_1}\right\}$ and $\Delta(f_2) = \left\{\frac{b_2}{a_2}\right\}$ then $i_0(f_1, f_2) \ge \min\{a_1 b_2, a_2 b_1\}$, with equality if $a_1 b_2 \ne a_2 b_1$.

Property (N1) is a consequence of the definition of $\Delta(f)$. For Property (N2) see [11, Proposition 4.2]. Property (N3) follows from [11, Propositions 3.13, 3.8 (v)].

Let I be an ideal of $\mathbf{K}[[x, y]]$ with a finite Łojasiewicz exponent. Put $l = \mathcal{L}_0(I)$. Consider the set of ideals $J \subset \mathbf{K}[[x, y]]$ such that $\mathcal{L}_0(J) = l$ and let M be a maximal element of this set (with respect to the inclusion). Set $\operatorname{ord} M = \inf\{\operatorname{ord} f : f \in M\}$. Observe that replacing any system of generators of M by their general linear combinations we obtain generators of the same order, equal to $\operatorname{ord} M$.

Lemma 2 *If $f_1, \ldots, f_m$ is a system of generators of M of the same order then there exists $k \in \{1, ..., m\}$, such that f_k is irreducible and $\mathcal{L}_0(M, f_k) = \mathcal{L}_0(M)$.*

Proof Let $f_1, \ldots, f_m$ be a system of generators of M of the same order. By Theorem 1 the Łojasiewicz exponent of M is attained on an irreducible factor h of the product $f_1 \cdots f_m$.

Let $\overline{M}$ be the ideal generated by $f_1, \ldots, f_m$ and h. Since $M \subset \overline{M}$ we get $\mathcal{L}_0(M) \geq \mathcal{L}_0(\overline{M})$. On the other hand $\mathcal{L}_0(\overline{M}) \geq \mathcal{L}_0(\overline{M}, h) = \mathcal{L}_0(M, h) = \mathcal{L}_0(M)$, which implies $\mathcal{L}_0(\overline{M}) = \mathcal{L}_0(M)$. By the maximality of M we get $h \in M$. Let $k \in \{1, \ldots, m\}$ be an index such that h divides f_k. Then $\operatorname{ord} h \leq \operatorname{ord} f_k = \operatorname{ord} M \leq \operatorname{ord} h$, hence $\operatorname{ord} h = \operatorname{ord} f_k$ and $f_k = h \cdot u$, where $u(0,0) \neq 0$ which implies that f_k is irreducible.

Let us pass to the proof of Theorem 2.

Proof of Theorem 2 We keep the notation and assumptions introduced above. It suffices to prove that $\mathcal{L}_0(M)$ is an integer or $\mathcal{L}_0(M) = N + \frac{b}{a}$, where $0 < b < a < N$.

By Lemma 2 there exists an irreducible power series $h \in M$ of order $\operatorname{ord} h = \operatorname{ord} M$ such that $\mathcal{L}_0(M) = \mathcal{L}_0(M, h)$.

If $\operatorname{ord} h = 1$ then $\mathcal{L}_0(M, h)$ is an integer.

If $\operatorname{ord} h > 1$ then changing the system of coordinates if necessary, we may assume that $\operatorname{ord} h(0, y) < \operatorname{ord} h(x, 0)$. Let $a = \operatorname{ord} h(0, y)$ and $c = \operatorname{ord} h(x, 0)$. Then $\Delta(h) = \left\{\frac{c}{a}\right\}$, where $a = \operatorname{ord} h = \operatorname{ord} M$.

Replacing any system of generators of M by a sequence of their linear generic combinations we get a sequence $f_1, \ldots, f_m$ of generators of the same order such that $\Delta(f_1) = \cdots = \Delta(f_m)$. Let Δ be their common Newton diagram.

Since $h \in M$, we have $h = a_1 f_1 + \cdots + a_m f_m$, where $a_i \in \mathbf{K}[[x, y]]$. Substituting $x = 0$ we get $\operatorname{ord} M = \operatorname{ord} h(0, y) \geq \min_i\{\operatorname{ord} f_i(0, y)\} \geq \operatorname{ord} M$. Hence the diagram Δ intersects the vertical axis at $(0, a)$. By Lemma 2 at least one of $f_1, \ldots, f_m$ is irreducible. This implies that Δ has only one compact face. Since $\operatorname{ord} f_i = a$, we have $\Delta = \left\{\frac{d}{a}\right\}$, where $d \geq a$.

By (1) there is $k \in \{1, \ldots, m\}$ such that $\mathcal{L}_0(M, h) = \frac{i_0(f_k, h)}{\operatorname{ord} h}$.

If $d = a$ then by (N3) we get $i_0(f_k, h) = \min\{ac, a^2\} = a^2$. In this case $\mathcal{L}_0(M) = a$.

If $d > a$ then by (N3) we get $i_0(f_k, h) \geq \min\{ac, ad\} = a \min\{c, d\} \geq a(a + 1)$. Write $i_0(f_k, h) = aN + b$, where $0 \leq b < a$. Dividing this equality by a and taking integer parts we get $N = \left\lfloor \frac{i_0(f_k, h)}{a} \right\rfloor \geq \frac{a(a+1)}{a} = a + 1$. Therefore $\mathcal{L}_0(M) = N + \frac{b}{a}$, where $0 \leq b < a < N$, which completes the proof. □

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