Research Article
Evelia R. García Barroso* and Janusz Gwoździewicz

# Decompositions of the higher order polars of plane branches 

DOI: 10.1515/forum-2016-0049
Received February 22, 2016; revised April 3, 2016


#### Abstract

In [1] Casas-Alvero found decompositions of higher order polars of an irreducible plane complex analytic curve generalizing the results of Merle. We improve his result obtaining a finer decomposition where we find out a kind of branches that we call threshold semi-roots. The existence of threshold semi-roots is a new phenomenon observed for the higher order polars. The topological type and the number of these branches is determined by the topological type of the original curve.


Keywords: Irreducible plane curve, higher order polar, threshold semi-root
MSC 2010: Primary 32S05; secondary 32S99

Communicated by: Junjiro Noguchi

## 1 Introduction

In this paper we deal with plane complex analytic and algebroid curves. A curve $f(x, y)=0$, where $f$ is in the ring $\mathbb{C}\{x, y\}$ of convergent power series or in the ring $\mathbb{C}[[x, y]]$ of formal power series is called singular if $f$ has no linear terms and is called irreducible or a branch if $f$ is irreducible in the ring $\mathbb{C}[[x, y]]$. Each curve decomposes into a finite number of branches.

By the kth polar of $f(x, y)=0$ we mean the curve $\frac{\partial^{k}}{\partial y^{k}} f(x, y)=0$. In [1] Casas-Alvero found decompositions of higher order polars of an irreducible singular plane curve. Generalizing the results of [8], he proved that the irreducible components of the higher order polar curves of a plane branch $f(x, y)=0$ are branches that have characteristic contacts with $f(x, y)=0$ (see Section 4.1).

Casas-Alvero's decomposition of the $k$ th higher order polar curve of $f(x, y)=0$ involves writing $\frac{\partial^{k}}{\partial y^{k}} f(x, y)$ as a finite product of power series, not necessarily irreducible, called bunches, where each bunch is in turn the product of all irreducible factors of $\frac{\partial^{k}}{\partial y^{k}} f(x, y)$ having the same contact value with $f(x, y)=0$.

Note that with only the information about the contact value we cannot determine the equisingularity type (in the sense of Zariski) of the irreducible components of $\frac{\partial^{k}}{\partial y^{k}} f(x, y)=0$ from the equisingularity type of $f(x, y)=0$. It is well known that the equisingularity type of the polar curve can vary in a family of equisingular branches. The family $\left\{f_{a}=y^{3}+x^{11}+a x^{8} y\right\}_{a \in \mathbb{C}}$ (see [9, Exemple 3]) is equisingular; the first polar curve of $f_{a}(x, y)=0$ has two different smooth branches for $a \neq 0$, but it has a double smooth branch for $a=0$.

In this paper we refine Casas-Alvero's decomposition. We show that every Casas-Alvero's bunch $\Gamma$ of $\frac{\partial^{k}}{\partial y^{k}} f(x, y)$ is the product of two power series $\Gamma_{1} \cdot \Gamma_{2}$, where all irreducible factors of $\Gamma_{2}$ called threshold semi-roots, have the same Puiseux characteristic depending only on the Puiseux characteristic of $f(x, y)=0$. The remaining irreducible factors of $\Gamma$ constitute $\Gamma_{1}$. The existence of threshold semi-roots is a new phe-

[^0]nomenon observed for the higher order polars, because we note that the first order polar does not have such branches. We also prove that the number of Newton-Puiseux roots of $\Gamma_{1}=0$ and $\Gamma_{2}=0$ depends only on the Puiseux characteristic of $f(x, y)=0$.

In [7] the authors determine the possible components of the exceptional divisor $E$ of the minimal resolution of the branch $f(x, y)=0$ where the strict transform of $\frac{\partial}{\partial y} f(x, y)=0$ intersects $E$. For higher order polars the result of [7] remains true and we make it precise for threshold semi-roots: the strict transforms of branches defined by threshold semi-roots are smooth and intersect transversely (curvetta) the rupture components of the exceptional divisor $E$ (components of $E$ intersecting at least three other components). We observe that threshold semi-roots are not semi-roots (in the sense of Abhyankar).

The decomposition theorem of the first polar of a plane reduced curve $f(x, y)=0$ allowed to describe in [3] the phenomenon of Lipschitz-Killing curvature concentration on the Milnor fiber $f(x, y)=\lambda \subseteq \mathbb{C}^{2}$ for $|(x, y)|<\epsilon$ when $\lambda, \epsilon \rightarrow 0,|\lambda| \ll \epsilon$. This is a multiscale phenomenon (as the multiscale phenomenon shown in Example 4.2) depending only on the equisingularity type of the curve. It would be expected that the decomposition of the higher order polars presented in this paper will help in the description of the metric and topological properties of the fibers of singular complex analytic morphisms.

In order to refine Casas-Alvero's factorization we deal with Newton-Puiseux roots of $\frac{\partial^{k}}{\partial y^{k}} f(x, y)$. For any characteristic exponent $q$ of $f$ we count the number of roots that have a contact $q$ with $f$. Moreover, our approach allows to find the coefficients $c_{q}$ of the monomial $x^{q}$ in these roots. The Newton-Puiseux roots with $c_{q}=0$ are the roots of $\Gamma_{1}=0$ and the others are the roots of $\Gamma_{2}=0$.

All the results of this paper remain true if we replace $\mathbb{C}$ by any algebraically closed field $\mathbb{K}$ of characteristic zero.

## 2 Formal Puiseux power series

Denote by $\mathbb{C}[[x]]^{*}$ the set of formal Puiseux power series. The order of any nonzero formal Puiseux power series is the minimal degree of its terms. By convention the order of the zero formal Puiseux power series is $+\infty$. For every $\phi, \psi \in \mathbb{C}[[x]]^{*}$ we define $O(\phi, \psi)$ to be the order of the difference $\phi-\psi$ and we call it the contact order of $\phi$ and $\psi$. It is well known that for any $\phi_{1}, \phi_{2}, \phi_{3} \in \mathbb{C}[[x]]^{*}$ the Strong Triangle Inequality (STI) $O\left(\phi_{1}, \phi_{3}\right) \geq \min \left\{O\left(\phi_{1}, \phi_{2}\right), O\left(\phi_{2}, \phi_{3}\right)\right\}$ holds.

Let $\alpha \in \mathbb{C}[[x]]^{*}$ and let $r$ be a positive rational number. The set $B=\left\{\psi \in \mathbb{C}[[x]]^{*}: O(\alpha, \psi) \geq r\right\}$ is called a pseudo-ball of height $r$. Note that any two pseudo-balls of height $r$ are either disjoint or are equal. To prove it observe that by STI if $O\left(\alpha_{1}, \phi\right), O\left(\alpha_{2}, \phi\right), O\left(\alpha_{1}, \psi\right) \geq r$, then $O\left(\alpha_{2}, \psi\right) \geq r$. Hence if the pseudo-balls $\left\{\psi \in \mathbb{C}[[x]]^{*}: O\left(\alpha_{1}, \psi\right) \geq r\right\}$ and $\left\{\psi \in \mathbb{C}[[x]]^{*}: O\left(\alpha_{2}, \psi\right) \geq r\right\}$ have a non-empty intersection, they are equal.

Take a pseudo-ball $B$ of height $r$. Every formal Puiseux power series $\gamma(x) \in B$ has the form

$$
\gamma(x)=\lambda_{B}(x)+c_{\gamma} x^{r}+\text { higher order terms, }
$$

where $\lambda_{B}(x)$ is obtained from an arbitrary $\alpha(x) \in B$ by omitting all its terms of order bigger than or equal to $r$. We call the number $c_{y}$ the leading coefficient of $\gamma$ with respect to $B$ and denote it $\mathrm{lc}_{B} \gamma$. Remark that $c_{y}$ can be zero.

Hereinafter, for brevity, formal Puiseux power series will be called Puiseux series.

## 3 Newton-Puiseux roots of higher order polars

Let $f(x, y) \in \mathbb{C}[[x, y]]$ be such that $1<\operatorname{ord} f(0, y)=n<+\infty$. Fix a positive integer $k<n$. Then the order of $\frac{\partial^{k}}{\partial y^{k}} f(0, y)$ equals $n-k$. The Newton-Puiseux factorizations of $f(x, y)$ and $\frac{\partial^{k}}{\partial y^{k}} f(x, y)$ have the form

$$
f(x, y)=u(x, y) \prod_{i=1}^{n}\left(y-\alpha_{i}(x)\right)
$$

and

$$
\begin{equation*}
\frac{\partial^{k}}{\partial y^{k}} f(x, y)=\tilde{u}(x, y) \prod_{j=1}^{n-k}\left(y-\gamma_{j}(x)\right) \tag{3.1}
\end{equation*}
$$

where $u(x, y), \tilde{u}(x, y)$ are units in $\mathbb{C}[[x, y]]$ and $\alpha_{i}(x), \gamma_{j}(x)$ are Puiseux series of positive order called NewtonPuiseux roots of $f(x, y)=0$ and $\frac{\partial^{k}}{\partial y^{k}} f(x, y)=0$, respectively. We denote by Zer $g$ the set of Newton-Puiseux roots of $g(x, y)=0$ for any $g(x, y) \in \mathbb{C}[[x, y]]$.

Let $B$ be a pseudo-ball. We put

$$
F_{B}(z):=\prod_{j: \alpha_{j} \in B}\left(z-\operatorname{l~}_{B} \alpha_{j}\right)
$$

Remark that the above polynomial is equal up to multiplication by a constant, to the polynomial introduced in [4, Lemma 3.3] (see also [6, Formula (2.2)]).
Lemma 3.1. Let B be a pseudo-ball. Assume that $k<\operatorname{deg} F_{B}(z)$. Then

$$
\frac{d^{k}}{d z^{k}} F_{B}(z)=\text { constant } \cdot \prod_{j: y_{j} \in B}\left(z-\operatorname{lc}_{B} \gamma_{j}\right)
$$

Proof. Let $r$ be the height of $B$. Fix the weight $\omega$ such that $\omega(x)=1, \omega(y)=r$ and denote by $\mathrm{in}_{\omega}(h)$ the weighted initial part of $h \in \mathbb{C}\left[\left[x^{1 / N}, y\right]\right]$, where $N \in \mathbb{N}$. First assume that $\lambda_{B}(x)=0$. Then

$$
\operatorname{in}_{\omega} f(x, y)=\text { constant } \cdot x^{A} \prod_{i: \alpha_{i} \in B}\left(y-\operatorname{lc}_{B} \alpha_{i} \cdot x^{r}\right)
$$

and

$$
\operatorname{in}_{\omega} \frac{\partial^{k}}{\partial y^{k}} f(x, y)=\text { constant } \cdot x^{A^{\prime}} \prod_{j: y_{j} \in B}\left(y-\operatorname{lc}_{B} \gamma_{j} \cdot x^{r}\right)
$$

where $A, A^{\prime}$ are rational numbers. If $k \leq \operatorname{deg} F_{B}(z)$, then $\frac{\partial^{k}}{\partial y^{k}} \operatorname{in}_{\omega} f(x, y)$ is nonzero and consequently

$$
\frac{\partial^{k}}{\partial y^{k}} \operatorname{in}_{\omega} f(x, y)=\operatorname{in}_{\omega} \frac{\partial^{k}}{\partial y^{k}} f(x, y)
$$

For $x=1$ we get

$$
\frac{d^{k}}{d y^{k}} \prod_{i: \alpha_{i} \in B}\left(y-\mathrm{lc}_{B} \alpha_{i}\right)=\text { constant } \cdot \prod_{j: y_{j} \in B}\left(y-\operatorname{lc}_{B} y_{j}\right)
$$

If $\lambda_{B}(x) \neq 0$, then taking $g(x, y):=f\left(x, y+\lambda_{B}(x)\right)$ we reduce the proof to the first case.

## 4 Properties of branches

Denote by $\mathbb{U}_{m}$ the multiplicative group of the $m$ th complex roots of unity. This group acts on $\mathbb{C}\left[\left[x^{1 / m}\right]\right]$ in the following way: for $\epsilon \in \mathbb{U}_{m}$ and $\alpha=\sum_{i} a_{i} x^{i / m}$,

$$
\begin{equation*}
\epsilon *_{m} \alpha=\sum_{i} a_{i} \epsilon^{i} x^{i / m} \tag{4.1}
\end{equation*}
$$

The star operation defined in (4.1) preserves the contact, that is, $O\left(\alpha_{1}, \alpha_{2}\right)=O\left(\epsilon *_{m} \alpha_{1}, \epsilon *_{m} \alpha_{2}\right)$.
Let $\alpha$ be a Puiseux series. The smallest natural number $n$ such that $\alpha \in \mathbb{C}\left[\left[x^{1 / n}\right]\right]$ is called the index of $\alpha$. Denote by * the star operation of $\mathbb{U}_{n}$ on $\mathbb{C}\left[\left[x^{1 / n}\right]\right]$ introduced in (4.1). Observe that if the Puiseux series $\alpha$ has index $n$, then $\epsilon_{1} * \alpha \neq \epsilon_{2} * \alpha$, for any two different $n$-th roots of the unity $\epsilon_{1}, \epsilon_{2}$ (see [5, Lemma 3.9]).

For a Puiseux series $\alpha=\sum_{i} a_{i} x^{i / n}$ of positive order and index $n$ we introduce two sequences $\left(e_{i}\right)$ and $\left(b_{i}\right)$ of natural numbers as follows:

- $e_{0}=b_{0}=n$,
- if $e_{k} \neq 1$, then $b_{k+1}:=\min \left\{i: i \not \equiv 0 \bmod e_{k}\right.$ and $\left.a_{i} \neq 0\right\}$,
- $e_{k}=\operatorname{gcd}\left(e_{k-1}, b_{k}\right)$.

The sequence $e_{i}$ is strictly decreasing and for some $h \in \mathbb{N}$ we have $e_{h}=1$. We get

$$
\mathbb{U}_{n}=\mathbb{U}_{e_{0}} \supset \mathbb{U}_{e_{1}} \supset \cdots \supset \mathbb{U}_{e_{h}}=\{1\}
$$

After [5, Lemma 6.8], if $\epsilon \in \mathbb{U}_{e_{k-1} \backslash} \backslash \mathbb{U}_{e_{k}}$, then $\epsilon^{b_{k}} \neq 1$. Consequently,

$$
\begin{equation*}
O(\alpha, \epsilon * \alpha)=\frac{b_{k}}{n} \text { for } \epsilon \in \mathbb{U}_{e_{k-1}} \backslash \mathbb{U}_{e_{k}} \tag{4.2}
\end{equation*}
$$

Let $\alpha$ be a Puiseux series of index $n$ which is a Newton-Puiseux root of an irreducible power series $f(x, y) \in \mathbb{C}[[x, y]]$. Then $\operatorname{Zer} f=\left\{\epsilon * \alpha: \epsilon \in \mathbb{U}_{n}\right\}$ and consequently ord $f(0, y)=n$ (see [5, Theorem 3.10]). The characteristic of an irreducible power series $f(x, y) \in \mathbb{C}[[x, y]]$ is the sequence $\left(b_{0}, b_{1}, \ldots, b_{h}\right)$, associated to any Newton-Puiseux root of $f$. By (4.2) the set Char $f:=\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{h}}{b_{0}}\right\}$ is the set of contacts between the Newton-Puiseux roots of $f$. We call Char $f$ the set of characteristic exponents of $f$.

Let $T_{i}(f)$ be the set of pseudo-balls of height $\frac{b_{i}}{b_{0}}$ having non-empty intersection with Zer $f$.
Property 4.1. For every characteristic exponent $b_{i} / b_{0}$ the set $T_{i}(f)$ consists of $e_{0} / e_{i-1}$ pairwise disjoint pseudoballs. Every $B \in T_{i}(f)$ contains $e_{i-1}$ elements of Zer $f$ and $F_{B}(z)=\left(z^{e_{i-1} / e_{i}}-c_{B}\right)^{e_{i}}$ for some $c_{B} \neq 0$.

Proof. Let $B \in T_{i}(f)$ and $\alpha \in B \cap \operatorname{Zer} f$. By (4.2), $B \cap \operatorname{Zer} f=\left\{\epsilon * \alpha: \epsilon \in \mathbb{U}_{e_{i-1}}\right\}$, which shows that $B$ contains $e_{i-1}$ elements of $\operatorname{Zer} f$. Consequently, $T_{i}(f)$ consists of $e_{0} / e_{i-1}$ pairwise disjoint pseudo-balls. We get

$$
F_{B}(z)=\prod_{\epsilon^{e_{i-1}=1}}\left(z-\operatorname{lc}_{B}(\epsilon * \alpha)\right)=\prod_{\epsilon^{e_{i-1}}=1}\left(z-\epsilon^{b_{i}} a_{b_{i}}\right)=\left(z^{e_{i-1} / e_{i}}-a_{b_{i}}^{e_{i-1} / e_{i}}\right)^{e_{i}},
$$

where $a_{b_{i}}$ is the coefficient of the monomial $x^{b_{i} / b_{0}}$ of $\alpha$. The last equality follows from [2, Lemma 3.4].
Example 4.2. Consider the irreducible complex convergent power series

$$
f(x, y)=\left(\left(y^{3}-x^{4}\right)^{4}+x^{17} y^{3}\right)^{2}+x^{22}\left(y^{3}-x^{4}\right)^{5}
$$

of characteristic $(24,32,62,137)$. Let $\alpha_{i}(x), i=1, \ldots, 24$ be the Newton-Puiseux roots of $f(x, y)=0$.
Take $\epsilon>0$ small enough. A higher contact order between $\alpha_{i}(x), \alpha_{j}(x)$ means a smaller Euclidean distance between $\alpha_{i}(\epsilon), \alpha_{j}(\epsilon)$. Thus the pseudo-balls of $T_{i}(f), i=1,2,3$, correspond to groups of roots of $f(\epsilon, y)=0$. These roots, for $\epsilon=0.75$, are drawn on the left side of Figure 1.

Fix $B \in T_{2}(f)$ and $E \in T_{3}(f)$. Using [4, Lemma 3.3] one can show that there are constants $C_{\epsilon}, D_{\epsilon} \in \mathbb{C}$ such that

$$
\begin{align*}
C_{\epsilon} \cdot f\left(\epsilon, \lambda_{B}(\epsilon)+z \cdot \epsilon^{62 / 24}\right) & \rightarrow F_{B}(z), \\
D_{\epsilon} \cdot f\left(\epsilon, \lambda_{E}(\epsilon)+z \cdot \epsilon^{137 / 24}\right) & \rightarrow F_{E}(z), \tag{4.3}
\end{align*}
$$

when $\epsilon \rightarrow 0$. This asymptotic property is illustrated on the right side of Figure 1. Notice that by Property 4.1 we have $F_{B}(z)=\left(z^{4}-c_{1}\right)^{2}$ and $F_{E}(z)=z^{2}-c_{2}$ for some nonzero constants $c_{1}$ and $c_{2}$.

The convergence in (4.3) is almost uniform. Hence there are similar limits for higher derivatives. This explains why we could detect the position of the roots of the $k$ th derivative of $f$ by the position of the roots of the polynomials $F_{B}^{(k)}(z)$.

### 4.1 Contact of branches

Let $f, g \in \mathbb{C}[[x, y]]$ be irreducible power series coprime with $x$. For every Puiseux series $\gamma$ we define

$$
\operatorname{cont}(f, \gamma)=\max \{O(\alpha, \gamma): \alpha \in \operatorname{Zer} f\}
$$

and call this number the contact between $f$ and $\gamma$. By abuse of notation we put

$$
\operatorname{cont}(f, g)=\max \{O(\alpha, \gamma): \alpha \in \operatorname{Zer} f, \gamma \in \operatorname{Zer} g\}
$$

We say that the branch $g(x, y)=0$ has characteristic contact with $f(x, y)=0$ if $\operatorname{cont}(f, g) \in \operatorname{Char}(f)$.


Figure 1. Behavior of the roots.

In this section we take $m \in \mathbb{N}$ such that Zer $f$, Zer $g \subset \mathbb{C}\left[\left[x^{1 / m}\right]\right]$ and we consider the star operation $*_{m}$ of $\mathbb{U}_{m}$ in $\mathbb{C}\left[\left[x^{1 / m}\right]\right]$ introduced in (4.1). If $\alpha=\sum_{i} a_{i} x^{i / n}$ is a Newton-Puiseux root of $f(x, y)=0$ of index $n$, then $m=q n$ for some $q \in \mathbb{N}$. Then $\alpha=\sum_{i} a_{i} x^{i q / m}$ and $\theta *_{m} \alpha=\theta^{q} *_{n} \alpha$, where $*_{n}$ is the star operation of $\mathbb{U}_{n}$ on $\mathbb{C}\left[\left[x^{1 / n}\right]\right]$. Since $\mathbb{U}_{n}=\left\{\theta^{q}: \theta \in \mathbb{U}_{m}\right\}$, the action of $\mathbb{U}_{m}$ permutes Zer $f$ and for every $\alpha, \alpha^{\prime} \in \operatorname{Zer} f$ there exists $\epsilon \in \mathbb{U}_{m}$ such that $\alpha^{\prime}=\epsilon *_{m} \alpha$. Up to the end of this section we denote $*_{m}$ by $*$.

Property 4.3. For every $y \in \operatorname{Zer} g, \operatorname{cont}(f, \gamma)=\operatorname{cont}(f, g)$.
Proof. It is enough to show that for all $\gamma, \gamma^{\prime} \in \operatorname{Zer} g$ the sets of contact orders $\{O(\alpha, \gamma): \alpha \in \operatorname{Zer} f\}$ and $\left\{O\left(\alpha, \gamma^{\prime}\right): \alpha \in \operatorname{Zer} f\right\}$ are equal. Take $\epsilon \in \mathbb{U}_{m}$ such that $\gamma^{\prime}=\epsilon * \gamma$. Then $O(\alpha, \gamma)=O\left(\epsilon * \alpha, \gamma^{\prime}\right)$ for all $\alpha \in$ Zer $f$. Since the action of $\mathbb{U}_{m}$ permutes $\operatorname{Zer} f$, the sets under consideration are equal.

Property 4.4. For every $q<\operatorname{cont}(f, g) \quad q \in \operatorname{Char} f$ if and only if $q \in \operatorname{Char} g$.
Proof. Let $q<\operatorname{cont}(f, g)$ be a characteristic exponent of $f$. By the definition of the characteristic exponent, $O\left(\alpha, \alpha^{\prime}\right)=q$ for some $\alpha, \alpha^{\prime} \in \operatorname{Zer} f$. Following Property $4.3 \operatorname{cont}(g, \alpha)=\operatorname{cont}\left(g, \alpha^{\prime}\right)=\operatorname{cont}(g, f)$. Hence there exist $\gamma, \gamma^{\prime} \in$ Zer $g$ such that

$$
O(\gamma, \alpha)=O\left(y^{\prime}, \alpha^{\prime}\right)=\operatorname{cont}(g, f)=\operatorname{cont}(f, g)
$$

By STI we get

$$
O\left(\gamma, \gamma^{\prime}\right) \geq \min \left\{O(\gamma, \alpha), O\left(\alpha, \alpha^{\prime}\right), O\left(\alpha^{\prime}, \gamma^{\prime}\right)\right\}=q
$$

Suppose that $O\left(\gamma, \gamma^{\prime}\right)>q$. Then we would have $O\left(\alpha, \alpha^{\prime}\right) \geq \min \left\{O(\alpha, \gamma), O\left(\gamma, \gamma^{\prime}\right), O\left(\gamma^{\prime}, \alpha^{\prime}\right)\right\}>q$ which is absurd. Hence $q=O\left(\gamma, \gamma^{\prime}\right)$ is a characteristic exponent of $g$.
Let $\alpha=\sum_{i} a_{i} x^{i / n} \in \mathbb{C}[[x]]^{*}$ be a Puiseux series. The support of $\alpha$ is the set $\left\{i / n: a_{i} \neq 0\right\}$.
Property 4.5. If $q=\operatorname{cont}(f, g)$ is a characteristic exponent of $f$ and there exists a Puiseux series $\gamma \in \mathrm{Zer} g$ such that $q$ is in the support of $y$ then $q$ is a characteristic exponent of $g$.

Proof. Take $\alpha, \alpha^{\prime} \in \operatorname{Zer} f$ such that $O(\alpha, \gamma)=O\left(\alpha, \alpha^{\prime}\right)=q$ and let $\epsilon \in \mathbb{U}_{m}$ be such that $\alpha^{\prime}=\epsilon * \alpha$. Put $\gamma^{\prime}=\epsilon * \gamma$. By STI we get

$$
O\left(\gamma, \gamma^{\prime}\right) \geq \min \left\{O(\gamma, \alpha), O\left(\alpha, \alpha^{\prime}\right), O\left(\alpha^{\prime}, \gamma^{\prime}\right)\right\}=q
$$

The equality $O\left(\alpha, \alpha^{\prime}\right)=q$ implies that $\epsilon * x^{q} \neq x^{q}$. Thus the monomial $x^{q}$ appears in the difference $\gamma-\epsilon * \gamma$ with a nonzero coefficient which proves that $O\left(\gamma, \gamma^{\prime}\right)=q$. Therefore $q \in$ Char $g$.

## 5 Roots of derivatives of special polynomials

In this section we study the roots of the complex polynomial $\frac{d^{k}}{d z^{k}}\left(z^{n}-c\right)^{e}$.
Property 5.1. Let $F(t)=H\left(t^{n}\right)$ be a complex polynomial. If $t_{0}$ is a nonzero root of $F(t)$ of multiplicity $m$, then $\left(t^{n}-t_{0}^{n}\right)^{m}$ divides $F(t)$.

Proof. It is enough to factorize $F(t)$ in the ring $\mathbb{C}\left[t^{n}\right]$ and notice that $t_{0}$ is a root of a factor $t^{n}-a$ if and only if $a=t_{0}^{n}$.

Lemma 5.2. Let $F(t)$ be a real polynomial of positive degree of the form

$$
F(t)=C \cdot t^{a}\left(t^{n}-1\right)^{b} \prod_{i=1}^{d}\left(t^{n}-c_{i}\right)
$$

where $a, b, d$ are nonnegative integers and $c_{i}$ are pairwise distinct real numbers from the interval $(0,1)$. Then the derivative of $F(t)$ has the form

$$
F^{\prime}(t)=C^{\prime} \cdot t^{a^{\prime}}\left(t^{n}-1\right)^{b^{\prime}} \prod_{i=1}^{d^{\prime}}\left(t^{n}-c_{i}^{\prime}\right),
$$

where $c_{i}^{\prime}$ are pairwise distinct real numbers from the interval ( 0,1 ). Moreover:

- if $a>0$, then $a^{\prime}=a-1$,
- if $a=0$, then $a^{\prime}=n-1$,
- if $b>0$, then $b^{\prime}=b-1$,
- if $b=0$, then $b^{\prime}=0$.

Proof. Let $a^{\prime}$ be the multiplicity of 0 as a root of $F^{\prime}(t)$, let $b^{\prime}$ be the multiplicity of 1 as a root of $F^{\prime}(t)$ and let $d^{\prime}$ be the number of distinct real roots of $F^{\prime}(t)$ in the interval $(0,1)$. We will check that

$$
\begin{equation*}
\left(a^{\prime}-a\right)+\left[\left(b^{\prime}-b\right)+\left(d^{\prime}-d\right)\right] n \geq-1 \tag{5.1}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
a^{\prime}+\left(b^{\prime}+d^{\prime}\right) n \geq \operatorname{deg} F^{\prime}(t) \tag{5.2}
\end{equation*}
$$

Consider several cases depending on the values of $a$ and $b$.
Case (I): $a, b>0$. The polynomial $F(t)$ has $d+2$ distinct real roots in the closed interval [ 0,1 ]. These roots divide $[0,1]$ to $d+1$ sub-intervals. By Rolle's Theorem inside each sub-interval there is at least one root of $F^{\prime}(t)$. Hence $d^{\prime} \geq d+1$. The differentiation decreases the multiplicity of a root of a polynomial by 1 . Thus $a^{\prime}=a-1$ and $b^{\prime}=b-1$.

Case (II): $a>0$ and $b=0$. By similar arguments as before we get $a^{\prime}=a-1, b^{\prime} \geq 0$, and $d^{\prime} \geq d$.
Case (III): $a=0$ and $b>0$. In this case $F(t)$ is a polynomial of $t^{n}$. Taking the derivative we get $a^{\prime} \geq n-1$. Moreover, $b^{\prime}=b-1$ and $d^{\prime} \geq d$.

Case (IV): $a=b=0$. We get $a^{\prime} \geq n-1, b^{\prime} \geq 0$ and $d^{\prime} \geq d-1$.
One easily verifies that inequality (5.1) holds in each case.
Let $t_{1}, \ldots, t_{d^{\prime}}$ be the pairwise distinct real roots of $F^{\prime}(t)$ from the interval ( 0,1 ). Consider the polynomial

$$
P(t)=t^{a^{\prime}}\left(t^{n}-1\right)^{b^{\prime}} \prod_{i=1}^{d^{\prime}}\left(t^{n}-t_{i}^{n}\right)
$$

Inequality (5.2) reads deg $P(t) \geq \operatorname{deg} F^{\prime}(t)$. By Property 5.1, $P(t)$ divides $F^{\prime}(t)$. Hence $P(t)$ and $F^{\prime}(t)$ are equal up to a multiplication by a constant. This proves the first statement of the lemma.

By the form of $F^{\prime}(t)$, we get $\operatorname{deg} F^{\prime}(t)=a^{\prime}+\left(b^{\prime}+d^{\prime}\right) n$. Consequently, the inequality " $\geq$ " in (5.1) can be replaced by " $=$ ". This implies that all weak inequalities obtained in the above case-by-case analysis must be equalities and proves the second statement of the lemma.

Lemma 5.3. Let $F(z)=\left(z^{n}-c\right)^{e}$ be a complex polynomial. Then for $1 \leq k<\operatorname{deg} F(z)$ one has

$$
\frac{d^{k}}{d z^{k}} F(z)=C z^{a}\left(z^{n}-c\right)^{b} \prod_{i=1}^{d}\left(z^{n}-c_{i}\right)
$$

where $C \in \mathbb{C}$ and
(1) $0 \leq a<n$ and $a+k \equiv 0(\bmod n)$,
(2) $b=\max \{e-k, 0\}$,
(3) $d=\min \{e, k\}-\left\lceil\frac{k}{n}\right\rceil$, where $\lceil x\rceil$ denotes the smallest integer number larger than or equal to $x$,
(4) $a+(b+d) n=n e-k$,
(5) $c_{i} \neq c_{j}$ for $1 \leq i<j \leq d$ and $0 \neq c_{i} \neq c$ for $1 \leq i \leq d$.

Proof. Without loss of generality we may assume that $c=1$. The general case reduces to this case by replacing $F(z)$ by the polynomial $c^{-e} F(\sqrt[n]{c} z)$.

The polynomial $F(z)=\left(z^{n}-1\right)^{e}$ satisfies the assumptions of Lemma 5.2. Applying this lemma to subsequent derivatives of $F(z)$ we see that the $k$ th derivative of $F(z)$ has the form

$$
\frac{d^{k}}{d z^{k}} F(z)=C_{k} z^{a_{k}}\left(z^{n}-1\right)^{b_{k}} \prod_{i=1}^{d_{k}}\left(z^{n}-c_{i, k}\right)
$$

and verifies the assumptions of Lemma 5.2, for $0 \leq k<\operatorname{deg} F(z)$. This implies (4) and (5). By Lemma 5.2 we get $0 \leq a_{k} \leq n-1$. The congruence $a_{k}+k \equiv 0(\bmod n)$ is a consequence of (4), and we conclude (1). Remark that $b_{0}=e$ and by Lemma 5.2 we have $b_{i+1}=b_{i}-1$ if $b_{i}>0$ and $b_{i+1}=0$ if $b_{i}=0$. This gives (2). Now we will prove (3). By (2) we get $e-b_{k}=\min \{e, k\}$ and by (1) the quotient $\frac{a_{k}+k}{n}$ is the smallest integer number larger than or equal to $\frac{k}{n}$. Computing $d_{k}$ from (4) we get $d_{k}=e-b_{k}-\frac{a_{k}+k}{n}=\min \{e, k\}-\left\lceil\frac{k}{n}\right\rceil$.
Corollary 5.4. Let $F(z)=\left(z^{n}-c\right)^{e}$ be a complex polynomial. Then every nonzero derivative $\frac{d^{k}}{d z^{k}} F(z)$ has no multiple complex roots except 0 and the roots of $z^{n}=c$.

## 6 Higher order polars of a branch

Let $i_{k}$ be the nonnegative number such that $e_{i_{k}} \leq k<e_{i_{k}-1}$. Remember that the $k$ th partial derivative of $f(x, y)$ admits a decomposition

$$
\frac{\partial^{k}}{\partial y^{k}} f(x, y)=\text { unit } \prod_{j=1}^{n-k}\left(y-\gamma_{j}\right)
$$

where $\gamma_{j}$ are Puiseux series of positive order.
Hereafter the notation $[a \bmod n]$ for an integer $a$ and a natural number $n$ means the remainder of the division of $a$ by $n$.

Lemma 6.1. Let $f(x, y) \in \mathbb{C}[[x, y]]$ be an irreducible power series of characteristic $\left(b_{0}, b_{1}, \ldots, b_{h}\right)$.
(i) For every $\gamma_{j} \in \operatorname{Zer} \frac{\partial^{k}}{\partial y^{k}} f$ we have $\operatorname{cont}\left(f, \gamma_{j}\right) \in\left\{b_{1} / b_{0}, \ldots, b_{i_{k}} / b_{0}\right\}$.
(ii) If $i<i_{k}$, then the number of $\gamma_{j}$ with $\operatorname{cont}\left(f, \gamma_{j}\right)=b_{i} / b_{0}$ equals $\left(b_{0} / e_{i}-b_{0} / e_{i-1}\right) k$.
(iii) If $i=i_{k}$, then the number of $\gamma_{j}$ with $\operatorname{cont}\left(f, \gamma_{j}\right)=b_{i} / b_{0}$ equals $b_{0}-b_{0} / e_{i-1} k$.
(iv) If $i \leq i_{k}$, then the number of $\gamma_{j}$ with $\operatorname{cont}\left(f, \gamma_{j}\right)=b_{i} / b_{0}$ and such that $b_{i} / b_{0}$ is not in the support of $\gamma_{j}$ equals $\left(b_{0} / e_{i-1}\right)\left[-k \bmod n_{i}\right]$, where $n_{i}=e_{i-1} / e_{i}$.
(v) If $i \leq i_{k}$, then the number of $\gamma_{j}$ with $\operatorname{cont}\left(f, \gamma_{j}\right)=b_{i} / b_{0}$ and such that $b_{i} / b_{0}$ is in the support of $\gamma_{j}$ equals $\left(b_{0} / e_{i}\right)\left(\min \left\{e_{i}, k\right\}-\left\lceil k / n_{i}\right\rceil\right)$.

Proof. Let $i \leq i_{k}$. Take a pseudo-ball $B \in T_{i}(f)$. After Property 4.1 the polynomial $F_{B}(z)$ has the form $\left(z^{n_{i}}-c\right)^{e_{i}}$ for some $c \neq 0$, so its degree is $e_{i-1}$ and after the choice of $i$ we get $k<e_{i_{k}-1} \leq e_{i-1}=\operatorname{deg} F_{B}(z)$. By Lemma 5.3 we get

$$
\begin{equation*}
\frac{d^{k}}{d z^{k}} F_{B}(z)=C z^{a}\left(z^{n_{i}}-c\right)^{b} \prod_{j=1}^{d}\left(z^{n_{i}}-c_{j}\right) \tag{6.1}
\end{equation*}
$$

By Lemma 3.1 the number of $\gamma_{j}$ in $B$ such that $\operatorname{cont}\left(f, \gamma_{j}\right)=\frac{b_{i}}{b_{0}}$ equals $i_{B}:=a+d n_{i}$. After Lemma 5.3 we have

$$
i_{B}= \begin{cases}\left(n_{i}-1\right) k & \text { if } i<i_{k} \\ e_{i_{k}-1}-k & \text { if } i=i_{k}\end{cases}
$$

For every $\gamma_{j}$ satisfying $\operatorname{cont}\left(f, \gamma_{j}\right)=\frac{b_{i}}{b_{0}}$ there is a unique pseudo-ball $B \in T_{i}(f)$ containing $\gamma_{j}$. By Property 4.1 the total number of $\gamma_{j}$ with $\operatorname{cont}\left(f, \gamma_{j}\right)=\frac{b_{i}}{b_{0}}$ equals $i_{B} \frac{b_{0}}{e_{i-1}}$. This gives the second and third statements. As a consequence the number of $\gamma_{j}$ with $\operatorname{cont}\left(f, \gamma_{j}\right) \in\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i_{k}}}{b_{0}}\right\}$ equals

$$
\sum_{i=1}^{i_{k}-1}\left(\frac{b_{0}}{e_{i}}-\frac{b_{0}}{e_{i-1}}\right) k+b_{0}-\frac{b_{0}}{e_{i_{k}-1}} k=b_{0}-k
$$

which is the total number of Newton-Puiseux roots of $\frac{\partial^{k}}{\partial y^{k}} f(x, y)=0$. This proves the first statement.
Given $B \in T_{i}(f)$, consider all $\gamma_{j} \in B$ such that $\operatorname{cont}\left(f, \gamma_{j}\right)=b_{i} / b_{0}$. By Lemma 3.1 the number of such $\gamma_{j}$ with $\operatorname{lc}_{B} y_{j}=0$ equals $a$ while the number of such $\gamma_{j}$ with $\operatorname{lc}_{B} \gamma_{j} \neq 0$ equals $n_{i} d$, where $a$ and $d$ are from (6.1).

Recall that there are $\frac{b_{0}}{e_{i-1}}$ pseudo-balls in $T_{i}(f)$. We finish the proof of the last two statements computing the values $\frac{b_{0}}{e_{i-1}} a$ and $\frac{b_{0}}{e_{i-1}} n_{i} d=\frac{b_{0}}{e_{i}} d$ using the first and the third items of Lemma 5.3.

The next theorem is an improvement of [1, Theorem 3.1].
Theorem 6.2. Let $f(x, y) \in \mathbb{C}[[x, y]]$ be an irreducible power series of characteristic $\left(b_{0}, b_{1}, \ldots, b_{h}\right)$. Put $e_{s}=\operatorname{gcd}\left(b_{0}, \ldots, b_{s}\right)$. Fix $k$ with $1 \leq k<\operatorname{ord} f(0, y)$, and let $i_{k}$ be the nonnegative integer number such that $e_{i_{k}} \leq k<e_{i_{k-1}}$. Then $\frac{\partial^{k}}{\partial y^{k}} f(x, y)$ admits a factorization as follows:

$$
\frac{\partial^{k}}{\partial y^{k}} f(x, y)=\Gamma^{(1)} \cdots \Gamma^{\left(i_{k}\right)},
$$

where $\Gamma^{(i)}$ are power series, not necessarily irreducible, verifying:
(i) For each $1 \leq i \leq i_{k}$, all branches of $\Gamma^{(i)}$ have contact $b_{i} / b_{0}$ with $f(x, y)=0$. The order of $\Gamma^{(i)}(0, y)$ equals $\left(b_{0} / e_{i}-b_{0} / e_{i-1}\right) k$, for $i<i_{k}$ and $b_{0}-b_{0} / e_{i-1} k$ for $i=i_{k}$.
(ii) $\Gamma^{(i)}$ can be written as a product $\Gamma_{1}^{(i)} \Gamma_{2}^{(i)}$, where for any irreducible factor $g$ of $\Gamma_{1}^{(i)}$ the first $i-1$ characteristic exponents of $f$ and $g$ are the same and $b_{i} / b_{0} \notin$ Char $g$; and $\left\{b_{1} / b_{0}, \ldots, b_{i} / b_{0}\right\}$ is the set of characteristic exponents of any irreducible factor of $\Gamma_{2}^{(i)}$.
(iii) The order of $\Gamma_{1}^{(i)}(0, y)$ equals $\left(b_{0} / e_{i-1}\right)\left[-k \bmod n_{i}\right]$, where $n_{i}=e_{i-1} / e_{i}$.
(iv) The order of $\Gamma_{2}^{(i)}(0, y)$ equals $\left(b_{0} / e_{i}\right)\left(\min \left\{e_{i}, k\right\}-\left\lceil k / n_{i}\right\rceil\right)$.
(v) The power series $\Gamma_{2}^{(i)}$ has $\min \left\{e_{i}, k\right\}-\left\lceil k / n_{i}\right\rceil$ irreducible factors.

Proof. We factorize $\prod_{j=1}^{n-k}\left(y-y_{j}\right)$ into $\bar{\Gamma}^{(1)} \cdots \bar{\Gamma}^{\left(i_{k}\right)}$, where every $\bar{\Gamma}^{(i)}$ is the product $\Pi\left(y-\gamma_{j}\right)$ running over $\gamma_{j} \in \operatorname{Zer} \frac{\partial^{k}}{\partial y^{k}} f$ with $\operatorname{cont}\left(f, \gamma_{j}\right)=\frac{b_{i}}{b_{0}}$. By Property 4.3 if $g$ is an irreducible factor of $\frac{\partial^{k}}{\partial y^{k}} f$ and $\gamma, \gamma^{\prime} \in \operatorname{Zer} g$, then

$$
\operatorname{cont}(f, \gamma)=\operatorname{cont}\left(f, \gamma^{\prime}\right)
$$

Hence $\bar{\Gamma}^{(i)}$ is the product of irreducible power series, so it is a power series. By (3.1) we have

$$
\frac{\partial^{k}}{\partial y^{k}} f(x, y)=\Gamma^{(1)} \cdots \Gamma^{\left(i_{k}\right)},
$$

where $\Gamma^{(i)}$ equals $\bar{\Gamma}^{(i)}$, up to multiplication by a unit, for $1 \leq i \leq i_{k}$.
The first statement of the theorem is a consequence of the first three statements of Lemma 6.1, since the order of $\Gamma^{(i)}(0, y)$ is the number of $y_{j}$ 's with $\operatorname{cont}\left(f, \gamma_{j}\right)=\frac{b_{i}}{b_{0}}$.

By Property 4.4 for any irreducible factor $g$ of $\Gamma^{(i)}$ the first $i-1$ characteristic exponents of $f$ and $g$ are the same. We define $\Gamma_{1}^{(i)}$ (respectively $\Gamma_{2}^{(i)}$ ) as the product of all irreducible factors $g$ of $\Gamma^{(i)}$ such that $\frac{b_{i}}{b_{0}}$ is not in the support of the Newton-Puiseux roots of $g$ (respectively $\frac{b_{i}}{b_{0}}$ is in the support of the Newton-Puiseux roots of $g$ ). We will prove that $\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i}}{b_{0}}\right\}$ is the set of characteristic exponents of any irreducible factor of $\Gamma_{2}^{(i)}$. By Property 4.5, the first $i$ characteristic exponents of $g$ are $\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i}}{b_{0}}$. Suppose that there exists a rational
number $r>\frac{b_{i}}{b_{0}}$ which is also a characteristic exponent of $g$. Let $\gamma$ be a Newton-Puiseux root of $g$. By (4.2) there is $\gamma^{\prime} \in$ Zer $g, \gamma^{\prime} \neq \gamma$ such that $O\left(\gamma, \gamma^{\prime}\right)=r>\frac{b_{i}}{b_{0}}$. Consider the pseudo-ball $B \in T_{i}(f)$ containing $\gamma$. We get $\operatorname{lc}_{B} \gamma=\operatorname{lc}_{B} \gamma^{\prime}$. Hence $\operatorname{lc}_{B} \gamma$ is a multiple root of $\frac{d^{k}}{d z^{k}} F_{B}(z)$, which contradicts Corollary 5.4.

The third and fourth statements of the theorem are a consequence of the fourth and fifth items of Lemma 6.1.

For any irreducible factor $g$ of $\Gamma_{2}^{(i)}$ the order of $g(0, y)$ is the least common denominator of the elements of Char $g=\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i}}{b_{0}}\right\}$, that is, $\frac{b_{0}}{e_{i}}$. The number of irreducible factors of the power series $\Gamma_{2}^{(i)}$ is the quotient of the order of $\Gamma_{2}^{(i)}(0, y)$ by $\frac{b_{0}}{e_{i}}$, which finishes the proof.
The first part of Theorem 6.2 is [1, Theorem 3.1]. For $k=1$ the power series $\Gamma_{2}^{(i)}$ is a unit and consequently $\Gamma^{(i)}=\Gamma_{1}^{(i)}$ for every factor of Casas-Alvero's decomposition.

Corollary 6.3. With the notations and assumptions of Theorem 6.2:
(i) If $k=e_{i-1}-1$, then $\Gamma^{(i)}$ is irreducible and

$$
\operatorname{Char} \Gamma^{(i)}=\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i-1}}{b_{0}}\right\}
$$

(ii) If $k=e_{i-1}-n_{i}$, then $\Gamma^{(i)}$ is irreducible and

$$
\operatorname{Char} \Gamma^{(i)}=\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i}}{b_{0}}\right\}
$$

Proof. If $k=e_{i-1}-1$ then by the first and the third statements of Theorem 6.2 we get

$$
\operatorname{ord} \Gamma_{1}^{(i)}(0, y)=\operatorname{ord} \Gamma^{(i)}(0, y)=n_{1} \cdots n_{i-1}
$$

On the other hand, by the second statement of Theorem 6.2, the first $i-1$ characteristic exponents of any branch $g$ of $\Gamma_{1}^{(i)}$ are $\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i-1}}{b_{0}}$, and consequently the order of $g(0, y)$ is greater than or equal to $n_{1} \cdots n_{i-1}$. So there exists a unit $u \in \mathbb{C}[[x, y]]$ such that $\Gamma^{(i)}=u g$ and $\Gamma^{(i)}$ is irreducible with

$$
\operatorname{Char} \Gamma^{(i)}=\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i-1}}{b_{0}}\right\} .
$$

If $k=e_{i-1}-n_{i}>0$, then $\operatorname{ord} \Gamma^{(i)}(0, y)=\operatorname{ord} \Gamma_{2}^{(i)}(0, y)=n_{1} \cdots n_{i}$. By the fifth statement of Theorem 6.2 we conclude that $\Gamma^{(i)}$ is irreducible and by the second statement of Theorem 6.2 we get

$$
\operatorname{Char} \Gamma^{(i)}=\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i}}{b_{0}}\right\} .
$$

Remark 6.4. The characteristic sequence or equivalently the set of characteristic exponents determines the equisingularity class (in the sense of Zariski) of an irreducible singular curve. Since the contact orders of the irreducible power series $f$ and the branches of its higher order polars $\frac{\partial^{k}}{\partial y^{k}} f(x, y)$, for $k<e_{h-1}$ are precisely the characteristic exponents of $f$, they determine the equisingularity class of $f(x, y)=0$. The case $k=1$ is well known after [8, p. 110].

Remember that if $f(x, y) \in \mathbb{C}[[x, y]]$ is irreducible with $\operatorname{Char} f=\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{h}}{b_{0}}\right\}$, then an irreducible power series $g$ is called an $(i-1)$-semi-root of $f$ if Char $g=\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i-1}}{b_{0}}\right\}$ and $\operatorname{cont}(f, g)=\frac{b_{i}}{b_{0}}$. In the language of resolution of singularities, a branch with characteristic contact is a semi-root if and only if its strict transform is a curvetta of the divisor corresponding to an end vertex of valency 1 (different to the root) of the dual resolution graph of $f(x, y)=0$.

Assume that $1 \leq i \leq i_{k}$. Proceeding as in the proof of Corollary 6.3 we can show that if $k+1 \equiv 0\left(\bmod n_{i}\right)$, then $\Gamma_{1}^{(i)}$ is an $(i-1)$-semi-root of $f$.

We call an irreducible power series $g$ an $i$-threshold semi-root of $f$ if

$$
\text { Char } g=\left\{\frac{b_{1}}{b_{0}}, \ldots, \frac{b_{i}}{b_{0}}\right\} \quad \text { and } \quad \operatorname{cont}(f, g)=\frac{b_{i}}{b_{0}}
$$

Remark that an $i$-threshold semi-root of $f$ is not an $i$-semi-root since its contact with $f$ is not hight enough.
The irreducible factors of $\Gamma_{2}^{(i)}$ are i-threshold semi-roots.

The figure below is the schematic picture of the dual resolution graph of the curve $f \cdot \Gamma_{2}^{(i)} \cdot f^{(i-1)} \cdot f^{(i)}$, where $f^{(j)}$ is a $j$-semi-root of $f, E_{j}$ denotes the $j$-th rupture point and $\bar{g}$ means the strict transform of $g=0$.


Here we assume that $x=0$ and $f=0$ are transverse.
Example 6.5. Consider $f(x, y)=\left(y^{3}-x^{4}\right)^{2}-x^{9} \in \mathbb{C}[[x, y]]$. The curve $f(x, y)=0$ is irreducible of characteristic $\left(b_{0}, b_{1}, b_{2}\right)=(6,8,11)$. Then for the first partial derivative

$$
\frac{\partial}{\partial y} f(x, y)=\Gamma^{(1)} \Gamma^{(2)},
$$

where $\Gamma^{(1)}=6 y^{2}$ and $\Gamma^{(2)}=y^{3}-x^{4}$. If $2 \leq k \leq 5$, then

$$
\frac{\partial^{k}}{\partial y^{k}} f(x, y)=\Gamma^{(1)}
$$

For $k=2$ we have $\Gamma^{(1)}=\Gamma_{1}^{(1)} \Gamma_{2}^{(1)}$, where $\Gamma_{1}^{(1)}=6 y$ and $\Gamma_{2}^{(1)}=5 y^{3}-2 x^{4}$. For $k=3$ the factor $\Gamma_{1}^{(1)}$ is a unit, while for $k \in\{4,5\}$ the factor $\Gamma_{2}^{(1)}$ is a unit. In the figure below, the dual resolution graph of the curve $y \cdot\left(y^{3}-x^{4}\right) \cdot\left(5 y^{3}-2 x^{4}\right) \cdot f$ is drawn.


Here $E_{j}$ denotes the $j$ th-divisor and $\bar{g}$ means the strict transform of $g=0$.

Acknowledgment: The authors thank Bernard Teissier for suggesting the name threshold semi-root.

Funding: The first-named author was partially supported by the Spanish Project MTM2012-36917-C03-01 and the second author was partially supported by the Plan Propio de Investigación de la Universidad de La Laguna-2014.

## References

[1] E. Casas-Alvero, Higher order polar germs, J. Algebra 240 (2001), 326-337.
[2] E. R. García Barroso, J. Gwoździewicz and A. Lenarcik, Non-degeneracy of the discriminant, Acta Math. Hungar. 147 (2015), no. 1, 220-246.
[3] E. R. García Barroso and B. Teissier, Concentration multi-échelles de courbure dans des fibres de Milnor, Comment. Math. Helv. 74 (1999), 398-418.
[4] J. Gwoździewicz, Ephraim's pencils, Int. Math. Res. Not. IMRN 2013 (2013), no. 15, 3371-3385.
[5] A. Hefez, Irreducible plane curve singularities, in: Real and Complex Singularities (Saõ Carlos 2000), Lecture Notes Pure Appl. Math. 232, Marcel Dekker, New York (2003), 1-120.
[6] S. Koike and A. Parusiński, Equivalence relations for two variable real analytic function germs, J. Math. Soc. Japan 65 (2013), no. 1, 237-276.
[7] D. T. Lê, F. Michel and C. Weber, Sur le comportement des polaires associées aux germes de courbes planes, Compos. Math. 72 (1989), 87-113.
[8] M. Merle, Invariants polaires des courbes planes, Invent. Math. 41 (1977), 103-111.
[9] F. Pham, Déformations equisingulières des idéaux Jacobiens de courbes planes, in: Proceedings of Liverpool Singularities Symposium II, Lecture Notes in Math. 209, Springer, Berlin (1971), 218-233.


[^0]:    *Corresponding author: Evelia R. García Barroso: Departamento de Matemáticas, Estadística e I.O., Sección de Matemáticas, Universidad de La Laguna. Apartado de Correos 456, 38200 La Laguna, Tenerife, Spain, e-mail: ergarcia@ull.es Janusz Gwoździewicz: Institute of Mathematics, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Cracow, Poland, e-mail: gwozdziewicz@up.krakow.pl

