Chapter 1
The Combinatorics of Plane Curve Singularities

How Newton Polygons Blossom into Lotuses

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This paper is dedicated to Bernard Teissier for his 75th birthday.

Contents

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Abstract This survey may be seen as an introduction to the use of toric and tropical geometry in the analysis of *plane curve singularities*, which are germs (C, o) of complex analytic curves contained in a smooth complex analytic surface S. The *embedded topological type* of such a pair (S, C) is usually defined to be that of the oriented link obtained by intersecting C with a sufficiently small oriented Euclidean sphere centered at the point o, defined once a system of local coordinates (x, y) was chosen on the germ (S, o). If one works more generally over an arbitrary algebraically closed field of characteristic zero, one speaks instead of the *combinatorial type* of (S, C). One may define it by looking either at the Newton-Puiseux series associated to C relative to a generic local coordinate system (x, y), or at the set of infinitely near points which have to be blown up in order to get the minimal embedded resolution of the germ (C, o) or, thirdly, at the preimage of this germ by the resolution. Each point of view leads to a different encoding of the combinatorial type by a decorated tree: an *Eggers-Wall tree*, an *Enriques diagram*, or a *weighted dual graph*. The three trees contain the same information, which in the complex setting is equivalent to the knowledge of the embedded topological type. There are known algorithms for transforming one tree into another. In this paper we explain how a special type of two-dimensional simplicial complex called a *lotus* allows to think geometrically about the relations between the three types of trees. Namely, all of them embed in a natural lotus, their numerical decorations appearing as invariants of it. This lotus is constructed from the finite set of Newton polygons created during any process of resolution of (C, o) by successive toric modifications.

1.1 Introduction

The aim of this paper is to unify various combinatorial objects classically used to encode the equisingularity/combinatorial/embedded topological type of a plane curve singularity. Often, a *plane curve singularity* means a germ (C, o) of algebraic or holomorphic curve defined by one equation in a smooth complex algebraic surface. In this paper we will allow the ambient surface to be any germ (S, o) of smooth complex algebraic or analytic surface, and C to be a formal germ of curve. Using a local formal coordinate system (x, y) on the germ (S, o), the global structure of S disappears completely and one may suppose that C is formally embedded in the affine plane $\mathbb{C}^2$. Usually one analyses in the following ways the structure of this embedding:

- By considering the *Newton-Puiseux series* which express one of the variables (x, y) in terms of the other, whenever the equation $f(x, y) = 0$ defining C is satisfied. Their combinatorics may be encoded in two rooted trees, the *Kuo-Lu tree* and a Galois quotient of it, the *Eggers-Wall tree*.
- By blowing up points starting from $o \in S$, until obtaining an embedded resolution of C, that is, a total transform of C which is a divisor with normal crossings. This blow up process may be encoded in an *Enriques diagram*, and the final total transform of C in a *weighted dual graph*.
- When the singularity C is holomorphic, by intersecting a representative of C with a small enough Euclidean sphere centered at the origin, defined using an arbitrary holomorphic local coordinate system (x, y) on (S, o). This leads to an oriented link in an oriented 3-dimensional sphere. This link is an *iterated torus link*, whose structure may be encoded in terms of another tree, called a *splice diagram*.

Unlike the first two procedures, the third one cannot be applied if the formal germ C is not holomorphic or if one works over an arbitrary algebraically closed field of characteristic zero. For this reason, we do not develop it in this paper. Let us mention only that it was initiated in Brauner's pioneering paper [13], whose historical background was described by Epple in [36]. For its developments, one may consult chronologically Reeve [107], Lê [80], A'Campo [5], Eisenbud & Neumann [34, Appendix to Chap. I], Schrauwen [110], Lê [81], Wall [131, Chap. 9], Weber [132] and the present authors [46, Chap. 5]. Similarly, we will not consider the discrete invariants constructed usually using the topology of the Milnor fibration of a holomorphic germ f, as Milnor numbers, Seifert forms, monodromy operators and their Zeta functions. The readers interested in such invariants may consult the textbooks [15] of Brieskorn and Knörrer and [131] of Wall.

There are algorithms allowing to pass between the Eggers-Wall tree, the dual graph and the Enriques diagram of C. However, they do not allow geometric representations of those passages. Our aim is to represent all these relationships using a single geometric object, called a *lotus*, which is a special type of simplicial complex of dimension at most two.

Our approach for associating lotuses to plane curve singularities is done in the spirit of the papers of Lê & Oka [83], A'Campo & Oka [8], Oka [93], González Pérez [52, Section 3.4], and Cassou Noguès & Libgober [21]. Namely, we use the fact that one may obtain an embedded resolution of C by composing a sequence of *toric* modifications determined by the successive Newton polygons of C or of strict transforms of it, relative to suitable local coordinate systems.

One may construct a lotus using the previous Newton polygons (see Definition 1.5.26). Its one dimensional skeleton may be seen as a dual complex representing the space-time of the evolution of the dual graph during the process of blow ups of points which leads to the embedded resolution. Besides the irreducible components of C and the components of the exceptional divisor, one takes also into account the curves defined by the chosen local coordinate systems. If A and B are two such exceptional or coordinate curves, and them or their strict transforms intersect transversally at a point p which is blown up at some moment of the process, then a two dimensional simplex with vertices labeled by A, B and the exceptional divisor of the blow up of p belongs to the lotus. These simplices are called the *petals* of the lotus (see an example of a lotus with 18 petals in Fig. 1.1). The Eggers-Wall tree, the Enriques diagram and the weighted dual graph embed simultaneously inside the lotus, and the geometry of the lotus also captures the numerical decorations of the weighted dual graph and the Eggers-Wall tree (see Theorem 1.5.29). For instance, the self-intersection number of a component of the final exceptional divisor is the opposite of the number of petals containing the associated vertex of the lotus. The previous lotuses associated to C have also valuative interpretations: they embed canonically in the space of semivaluations of the completed local ring of the germ (S, o) (see Remark 1.5.34).

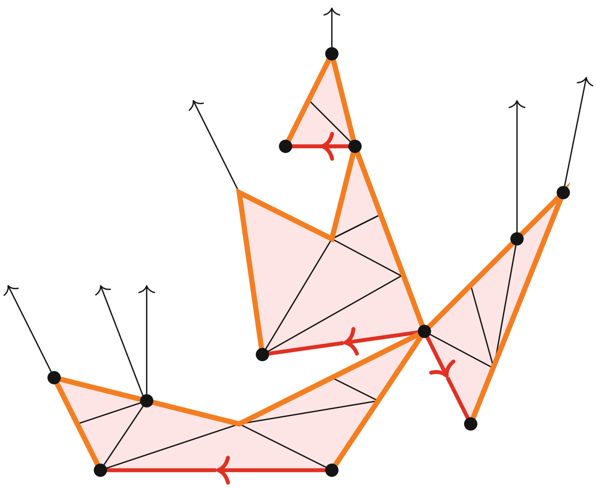


Fig. 1.1 A lotus. It is part of Fig. 1.36, which corresponds to Example 1.5.28

Let us describe the structure of the paper.

In Sect. 1.2 we introduce basic notions about *complex analytic varieties*, *plane curve singularities*, their *multiplicities* and *intersection numbers*, *normalizations*, *Newton-Puiseux series*, *blow ups*, *embedded resolutions of plane curve singularities* and the associated *weighted dual graphs*. The notions of *Newton polygon*, dual *Newton fan* and *lotus* are first presented here on a *Newton non-degenerate* example.

Section 1.3 begins with an explanation of basic notions of *toric geometry*: *fans* and their subdivisions, the associated *toric varieties* and *toric modifications* (see Sects. 1.3.1, 1.3.2 and 1.3.3). In particular, we describe the *toric boundary* of a toric variety—the reduced divisor obtained as the complement of its dense torus—in terms of the associated fan. Then we pass to *toroidal geometry*: we introduce *toroidal varieties*, which are pairs $(\Sigma, \partial\Sigma)$ consisting of a normal complex analytic variety Σ and a reduced divisor $\partial\Sigma$ on it, which are locally analytically isomorphic to a germ of a pair formed by a toric variety and its *boundary divisor*. A basic example of toroidal surface is that of a germ (S, o) of smooth surface, endowed with the divisor $L + L'$, where (L, L') is a *cross*, that is, a pair of smooth transversal germs of curves. A *morphism* $\phi : (\Sigma_2, \partial\Sigma_2) \to (\Sigma_1, \partial\Sigma_1)$ *of toroidal varieties* is a complex analytic morphism such that $\phi^{-1}(\partial\Sigma_1) \subseteq \partial\Sigma_2$ (see Sect. 1.3.4).

In Sect. 1.4 we explain in which way one may associate various morphisms of toroidal surfaces to the plane curve singularity $C \hookrightarrow S$. First, choose a cross (L, L') on (S, o), defined by a local coordinate system (x, y). The *Newton polygon* $\mathcal{N}(f)$ of a defining function $f \in \mathbb{C}[[x, y]]$ of the curve singularity C depends only on C and on the cross (L, L'). Its associated *Newton fan* is obtained by subdividing the first quadrant along the rays orthogonal to the compact edges of the Newton polygon. This fan defines a toric modification of S, the *Newton modification of* S *defined by* C *relative to the cross* (L, L') (see Sect. 1.4.1). The Newton modification becomes a toroidal morphism when we endow its target S with the boundary divisor $\partial S := L + L'$ and we define the boundary divisor of its source to be the preimage of $L + L'$. We emphasize the fact that those notions depend only on the objects $(S, C, (L, L'))$, in order to insist on the underlying geometric structures. The strict transform of C by the previous Newton modification intersects the boundary divisor only at smooth points of it, which belong to the exceptional divisor and are smooth points of the ambient surface. If one completes the germ of exceptional divisor into a cross at each such point o_i, then one gets again a triple of the form (*surface*, *curve*, *cross*), where this time the curve is the germ at o_i of the strict transform of C. Therefore one may perform again a Newton modification at each such point, and continue in this way until the strict transform of C defines everywhere crosses with the exceptional divisor. The total transform of C and of all coordinate curves introduced during previous steps define the toroidal boundary $\partial\Sigma$ on the final surface Σ. This non-deterministic algorithm produces morphisms $\pi : (\Sigma, \partial\Sigma) \to (S, \partial S)$ of toroidal surfaces, which are *toroidal pseudo-resolutions* of the plane curve singularity C (see Sect. 1.4.2). The surface Σ has a finite number of singular points, at which it is locally analytically isomorphic to normal toric surfaces. In Sect. 1.4.3 we show how to pass from the toroidal pseudo-resolution π to a *toroidal embedded resolution* by composing π with the minimal resolution of these toric singularities. Finally, we encode the process of successive Newton modifications in a *fan tree*, in terms of the Newton fans produced by the pseudo-resolution process (see Sect. 1.4.4).

In Sect. 1.5 we explain the notion of lotus. A *Newton lotus* associated to a fan encodes geometrically the continued fraction expansions of the slopes of the rays of the fan, as well as their common parts (see Sect. 1.5.2). It is composed of *petals*, and each petal corresponds to the blow up of the base point of a cross. One may clarify the subtitle of the paper by saying that *the collection of Newton polygons appearing during the toroidal pseudo-resolution process blossomed into the associated lotus, each petal corresponding to a blow up operation*. We explain how to associate to the fan tree of the toroidal pseudo-resolution a *lotus*, which is a 2-dimensional simplicial complex obtained by gluing the *Newton lotuses* associated to the Newton fans of the process (see Sects. 1.5.1 and 1.5.3). The lotus of a toroidal pseudo-resolution depends on the choices of crosses made during the process of pseudo-resolution (see Sect. 1.5.4). We explain then how to embed in the lotus the Enriques diagram and the dual graph of the embedded resolution. We conclude the section by defining a *truncation operation* on lotuses, and we explain how it may be used to understand the part of the embedded resolution which does not depend on the supplementary curves introduced during the pseudo-resolution process (see Sect. 1.5.5).

We begin Sect. 1.6 by introducing the notion of *Eggers-Wall tree* of the curve C relative to the smooth germ L (see Sect. 1.6.1) and by expressing the Newton polygon of C relative to a cross (L, L') in terms of the Eggers-Wall tree of $C + L'$ relative to L (see Sect. 1.6.2). Then we explain that the fan tree of the previous toroidal pseudo-resolution process is canonically isomorphic to the Eggers-Wall tree relative to L of the curve obtained by adding to C the projections to S of all the crosses built during the process and how to pass from the numerical decorations of the fan tree to those of the Eggers-Wall tree (see Sect. 1.6.5). As preliminary results, we prove *renormalization formulae* which describe the Eggers-Wall tree of the strict transform of C by a Newton modification, relative to the exceptional divisor, in terms of the Eggers-Wall tree of C relative to L (see Sects. 1.6.3 and 1.6.4).

The final Sect. 1.7 begins by an overview of the construction of a fan tree and of the associated lotus from the Newton fans of a toroidal pseudo-resolution process (see Sect. 1.7.1). Section 1.7.2 describes perspectives on possible applications of lotuses to problems of singularity theory. The final Sect. 1.7.3 contains a list of the main notations used in the article.

Starting from Sect. 1.3, each section ends with a subsection of historical comments. We apologize for any omission, which may result from our limited knowledge. One may also find historical information about various tools used to study plane curve singularities in Enriques and Chisini's book [35], in the first chapter of Zariski's book [134] and in the final sections of the chapters of Wall's book [131].

We tried to make this paper understandable to PhD students who have only a basic knowledge about singularities. Even if everything in this paper holds over an arbitrary algebraically closed field of characteristic zero, we will stick to the complex setting, in order to make things more concrete for the beginner. We accompany the definitions with examples and many figures. Indeed, one of our objectives is to show that lotuses may be a great visual tool for relating the combinatorial objects used to study plane curve singularities. There is a main example, developed throughout the paper starting from Sect. 1.4 (see Examples 1.4.28, 1.4.34, 1.4.36, 1.5.28, 1.5.31, 1.5.36, 1.6.29 and the overview Fig. 1.58). We recommend to study it carefully in order to get a concrete feeling of the various objects manipulated in this paper. We also recommend to those readers who are learning the subject to refer to the Sect. 1.7.1 from time to time, in order to measure their understanding of the geometrical objects presented here.

1.2 Basic Notions and Examples

In this section we recall basic notions about *complex varieties* and *plane curve singularities* (see Sect. 1.2.1), *normalization morphisms* (see Sect. 1.2.2), the relation between *Newton-Puiseux series* and plane curve singularities (see Sect. 1.2.3) and *resolution of such singularities* by iteration of *blow ups of points* (see Sect. 1.2.4). We describe such a resolution for the *semi-cubical parabola* (see Sect. 1.2.5). We give a flavor of the main construction of this paper in Sect. 1.2.6. We show there how to transform the Newton polygon of a certain *Newton non-degenerate* plane curve singularity with two branches into a *lotus*, and how this lotus contains the dual graph of a resolution by blow ups of points.

From now on, $\boxed{\mathbb{N}}$ denotes the set of non-negative integers and $\boxed{\mathbb{N}^*}$ the set of positive integers.

1.2.1 Basic Facts About Plane Curve Singularities

In this subsection we recall basic vocabulary about *complex analytic spaces* (see Definition 1.2.1) and we explain the notions of *plane curve singularity* (see Definition 1.2.5), of *multiplicity* and of *intersection number* (see Definition 1.2.7) for such singularities. Finally, we recall an important way of computing such intersection numbers (see Proposition 1.2.8).

Briefly speaking, a complex analytic space X is obtained by gluing model spaces, which are zero-loci of systems of analytic equations in some complex affine space $\mathbb{C}^n$. One has to prescribe also the analytic "functions" living on the underlying topological space. Those "functions" are elements of a so-called "structure sheaf" $\mathcal{O}_X$, which may contain nilpotent elements. For this reason, they are not classical functions, as they are not determined by their values. For instance, one may endow the origin of $\mathbb{C}$ with the structure sheaves whose rings of sections are the various rings $\mathbb{C}[x]/(x^m)$, with $m \in \mathbb{N}^*$. They are pairwise non-isomorphic and they contain nilpotent elements whenever $m \geq 2$. Let us state now the formal definitions of *complex analytic spaces* and of some special types of complex analytic spaces.

Definition 1.2.1

- A **model complex analytic space** is a ringed space $(X, \mathcal{O}_X)$, where X is the zero locus of I and $\mathcal{O}_X = \mathcal{O}_U/\mathcal{I}$. Here I is a finitely generated ideal of the ring of holomorphic functions on an open set U of $\mathbb{C}^n$, for some $n \in \mathbb{N}^*$, $\mathcal{O}_U$ is the sheaf of holomorphic functions on U and $\mathcal{I}$ is the sheaf of ideals of $\mathcal{O}_U$ generated by I.
- A **complex analytic space** is a ringed space locally isomorphic to a model complex analytic space.
- A complex analytic space is **reduced** if its structure sheaf $\mathcal{O}_X$ is reduced, that is, without nilpotent elements. In this case, one speaks also about a **complex variety**.
- A **complex manifold** is a complex variety X such that any point $x \in X$ has a neighborhood isomorphic to an open set of $\mathbb{C}^n$, for some $n \in \mathbb{N}$. If the non-negative integer n is independent of x, then the complex manifold X is called **equidimensional** and n is its **complex dimension**.
- The **smooth locus** of a complex variety X is its open subspace whose points have neighborhoods which are complex manifolds. Its **singular locus** $\boxed{\mathrm{Sing}(X)}$ is the complement of its smooth locus.
- A **smooth complex curve** is an equidimensional complex manifold of complex dimension one and a **smooth complex surface** is an equidimensional complex manifold of complex dimension two.
- A **complex curve** is a complex variety whose smooth locus is a smooth complex curve and a **complex surface** is a complex variety whose smooth locus is a smooth complex surface.

By construction, the singular locus $\mathrm{Sing}(X)$ of X is a closed subset of X. It is a deep theorem that this subset is in fact a complex subvariety of X (see [66, Corollary 6.3.4]).

Let S be a smooth complex surface. If o is a point of S and $\phi : U \to V$ is an isomorphism from an open neighborhood U of o in S to an open neighborhood V of the origin in $\mathbb{C}^2_{x,y}$, then the coordinate holomorphic functions $x, y : \mathbb{C}^2_{x,y} \to \mathbb{C}$ may be lifted by ϕ to two holomorphic functions on U, vanishing at o. They form a **local coordinate system on the germ** $\boxed{(S, o)}$ **of** S **at** o. By abuse of notations, we still denote this local coordinate system by (x, y), and we see it as a couple of elements of $\boxed{\mathcal{O}_{S,o}}$, the **local ring of** S **at** o, equal by definition to the $\mathbb{C}$-algebra of germs of holomorphic functions defined on some neighborhood of o in S. The local coordinate system (x, y) establishes an isomorphism $\mathcal{O}_{S,o} \simeq \mathbb{C}\{x, y\}$, where $\boxed{\mathbb{C}\{x, y\}}$ denotes the $\mathbb{C}$-algebra of convergent power series in the variables x, y. Denote by $\boxed{\mathbb{C}[[x, y]]}$ the $\mathbb{C}$-algebra of formal power series in the same variables. It is the completion of $\mathbb{C}\{x, y\}$ relative to its maximal ideal $(x, y)\mathbb{C}\{x, y\}$. One has the following fundamental theorem, valid in fact for any finite number of variables (see [66, Corollary 3.3.17]):

Theorem 1.2.2 *The local rings $\mathbb{C}\{x, y\}$ and $\mathbb{C}[[x, y]]$ are factorial.*

In addition to Definition 1.2.1, we use also the following meaning of the term *curve*:

Definition 1.2.3 A **curve** C **on a smooth complex surface** S is an effective Cartier divisor of S, that is, a complex subspace of S locally definable by the vanishing of a non-zero holomorphic function.

This means that for every point $o \in C$, there exists an open neighborhood U of o in S and a holomorphic function $f : U \to \mathbb{C}$ such that $C \subset U$ is the vanishing locus $\boxed{Z(f)}$ of f and such that the structure sheaf $\boxed{\mathcal{O}_{C|U}}$ of $C \subset U$ is the quotient sheaf $\mathcal{O}_U/(f)\mathcal{O}_U$. In this case, once U is fixed, the defining function f is unique up to multiplication by a holomorphic function on U which vanishes nowhere.

The curve C is called **reduced** if it is a reduced complex analytic space in the sense of Definition 1.2.1. This means that any defining function $f : U \to \mathbb{C}$ as above is square-free in all local rings $\mathcal{O}_{S,o}$, where $o \in U$. For instance, the union C of coordinate axes of $\mathbb{C}^2$ is a reduced curve, being definable by the function xy, which is square-free in all the local rings $\mathcal{O}_{\mathbb{C}^2,o}$, where $o \in C$. By contrast, the curve D defined by the function xy^2 is not reduced.

As results from Definition 1.2.3, a complex subspace C of S is a curve on S if and only if, for any $o \in C$, the ideal of $\mathcal{O}_{S,o}$ consisting of the germs of holomorphic functions vanishing on the germ (C, o) of C at o is *principal*. We would have obtained a more general notion of *curve* if we would have asked C to be a 1-dimensional complex subspace of S in the neighborhood of any of its points. For instance, if $S = \mathbb{C}^2_{x,y}$, and C is defined by the ideal (x^2, xy) of $\mathbb{C}[x, y]$, then set-theoretically C coincides with the y-axis $Z(x)$. But the associated structure sheaf $\mathcal{O}_{\mathbb{C}^2}/(x^2, xy)\mathcal{O}_{\mathbb{C}^2}$ is not the structure sheaf of an effective Cartier divisor. In fact the germ of C at the origin cannot be defined by only one holomorphic function $f(x, y) \in \mathbb{C}\{x, y\}$. Otherwise, we would get that both x^2 and xy are divisible by $f(x, y)$ in the local ring $\mathbb{C}\{x, y\}$. As this ring is factorial by Theorem 1.2.2, we see that f divides x inside this ring, which implies that $(f)\mathbb{C}\{x, y\} = (x)\mathbb{C}\{x, y\}$. Therefore, $(x^2, xy)\mathbb{C}\{x, y\} = (x)\mathbb{C}\{x, y\}$ which is a contradiction, as x is of order 1 and each element of the ideal $(x^2, xy)\mathbb{C}\{x, y\}$ is of order at least 2. The notion of *order* used in the previous sentence is defined by:

Definition 1.2.4 Let $f \in \mathbb{C}[[x, y]]$. Its **order** is the smallest degree of its terms.

For instance, the maximal ideal of $\mathbb{C}[[x, y]]$ consists precisely of the power series of order at least 1. It is a basic exercise to show that the order is invariant by the automorphisms of the $\mathbb{C}$-algebra $\mathbb{C}[[x, y]]$ and by multiplication by the elements of order 0, which are the units of this algebra. Therefore, one gets a well-defined notion of *multiplicity* of a germ of *formal curve* on S:

Definition 1.2.5 A **plane curve singularity** is a germ C of formal curve on a germ of smooth complex surface (S, o), that is, a principal ideal in the completion $\boxed{\hat{\mathcal{O}}_{S,o}}$ of the local ring $\mathcal{O}_{S,o}$. It is called a **branch** if it is irreducible, that is, if its defining functions are irreducible elements of the factorial local ring $\hat{\mathcal{O}}_{S,o}$. The **multiplicity** $\boxed{m_o(C)}$ of C at o is the order of a **defining function** $f \in \hat{\mathcal{O}}_{S,o}$ of C, seen as an element of $\mathbb{C}[[x, y]]$ using any local coordinate system (x, y) of the germ (S, o).

Example 1.2.6 Let $\alpha, \beta \in \mathbb{N}^*$ and $f := x^\alpha - y^\beta \in \mathbb{C}[x, y]$. Denote by C the curve on $\mathbb{C}^2$ defined by f. Its multiplicity at the origin O of $\mathbb{C}^2$ is the minimum of α and β. The curve singularity (C, O) is a branch if and only if α and β are coprime. One implication is easy: if α and β have a common factor $\rho > 1$, then $x^\alpha - y^\beta = \prod_{\omega:\, \omega^\rho=1} \left(x^{\alpha/\rho} - \omega y^{\beta/\rho}\right)$, the product being taken over all the complex ρ-th roots ω of 1, which shows that (C, O) is not a branch. The reverse implication results from the fact that, whenever α and β are coprime, C is the image of the parametrization $N(t) := (t^\beta, t^\alpha)$. The inclusion $N(\mathbb{C}) \subseteq C$ being obvious, let us prove the reverse inclusion. Let $(x, y) \in C$. As $N(0) = O$, it is enough to consider the case where $xy \neq 0$. We want to show that there exists $t \in \mathbb{C}^*$ such that $x = t^\beta,\ y = t^\alpha$. Assume the problem solved and consider also a pair $(a, b) \in \mathbb{Z}^2$ such that $a\alpha + b\beta = 1$, which exists by Bezout's theorem. One gets $t = t^{a\alpha+b\beta} = y^a x^b$. Define therefore $t := y^a x^b$. Then:

$$t^\beta = (y^a x^b)^\beta = (y^\beta)^a x^{b\beta} = (x^\alpha)^a x^{b\beta} = x^{a\alpha+b\beta} = x,$$

and similarly one shows that $t^\alpha = y$. This proves that C is indeed included in the image of N.

Let C be a plane curve singularity on the germ of smooth surface (S, o). If $f \in \hat{\mathcal{O}}_{S,o}$ is a defining function of C, it may be decomposed as a product:

$$f = \prod_{i \in I} f_i^{p_i}, \tag{1.1}$$

in which the functions f_i are pairwise non-associated prime elements of the local ring $\hat{\mathcal{O}}_{S,o}$ and $p_i \in \mathbb{N}^*$ for every $i \in I$. Such a decomposition is unique up to permutation of the factors $f_i^{p_i}$ and up to a replacement of each function f_i by an associated one (recall that two such functions are *associated* if one is the product of another one by a unit of the local ring). If $C_i \subseteq S$ is the plane curve singularity defined by f_i, then the decomposition (1.1) gives a decomposition of C seen as a germ of effective divisor $C = \sum_{i \in I} p_i C_i$, where each curve singularity C_i is a branch. The plane curve singularity C is reduced if and only if $p_i = 1$ for every $i \in I$.

The *intersection number* is the simplest measure of complexity of the way two plane curve singularities interact at a given point:

Definition 1.2.7 Let C and D be two curve singularities on the germ of smooth surface (S, o) defined by functions f and $g \in \hat{O}_{S,o}$ respectively. Their **intersection number** $\boxed{(C \cdot D)_o}$, also denoted $\boxed{C \cdot D}$ if the base point o of the germ is clear from the context, is defined by:

$$C \cdot D := \dim_{\mathbb{C}} \frac{\hat{O}_{S,o}}{(f, g)} \in \mathbb{N} \cup \{\infty\},$$

where (f, g) denotes the ideal of $\hat{O}_{S,o}$ generated by f and g.

If C and D are two curve singularities, then one has that $(C \cdot D)_o \geq m_o(C)m_o(D)$, with equality if and only if the curves C and D are **transversal** (see [131, Lemma 4.4.1]), that is, the tangent plane of (S, o) does not contain lines which are tangent to both C and D.

Seen as a function of two variables, the intersection number is symmetric. It is moreover bilinear, in the sense that if $C = \sum_{i \in I} p_i C_i$, then $C \cdot D = \sum_{i \in I} p_i (C_i \cdot D)$. Therefore, in order to compute $C \cdot D$, it is enough to find $C_i \cdot D$ for all the branches C_i of C.

One has the following useful property (see [66, Lemma 5.1.5]):

Proposition 1.2.8 *Let C be a branch and D be an arbitrary curve singularity on the smooth germ of smooth surface (S, o). Denote by $N : (\mathbb{C}_t, 0) \to (S, o)$ a formal parametrization of degree one of C and $g \in \hat{O}_{S,o}$ be a defining function of D. Then*

$$C \cdot D = \nu_t(g(N(t))),$$

where $\nu_t(h)$ denotes the order of a power series $h \in \mathbb{C}[[t]]$.

Example 1.2.9 Let us consider two curves $C, D \subseteq \mathbb{C}^2_{x,y}$, defined by polynomials $f := x^\alpha - y^\beta$ and $g := x^\gamma - y^\delta$ of the type already considered in Example 1.2.6. Assume that α and β are coprime. This implies, as shown in Example 1.2.6, that the plane curve singularity (C, O) is a branch and that $N(t) := (t^\beta, t^\alpha)$ is a parametrization of degree one of it. By Proposition 1.2.8, if C is not a branch of D, we get:

$$C \cdot D = \nu_t\left((t^\beta)^\gamma - (t^\alpha)^\delta\right) = \nu_t\left(t^{\beta\gamma} - t^{\alpha\delta}\right) = \min\{\beta\gamma, \alpha\delta\}.$$

For more details about intersection numbers of plane curve singularities, one may consult [15, Sect. 6], [113, Vol. 1, Chap. IV.1] and [39, Chap. 8].

The formal parametrizations $N : (\mathbb{C}_t, 0) \to (S, o)$ of degree one of a branch appearing in the statement of Proposition 1.2.8 are exactly the *normalization morphisms of* C whose sources are identified with $(\mathbb{C}, 0)$. Next subsection is dedicated to the general definition of *normal complex variety* and of *normalization morphism* in arbitrary dimension, as we will need them later also for surfaces.

1.2.2 Basic Facts About Normalizations

In this subsection we explain basic facts about *normal rings* (see Definition 1.2.10), *normal complex varieties* (see Definition 1.2.11) and *normalization morphisms* (see Definition 1.2.16) of arbitrary complex varieties. For more details and proofs one may consult [66, Sections 1.5, 4.4] and [58].

The following definition contains *algebraic* notions, concerning extensions of rings:

Definition 1.2.10 Let R be a commutative ring and let $R \subseteq T$ be an extension of R.

1. An element of T is called **integral over** R if it satisfies a monic polynomial relation with coefficients in R.
2. The extension $R \subseteq T$ of R is called **integral** if each element of T is integral over R.
3. The **integral closure** of R is the set of integral elements over R of the total ring of fractions of R.
4. R is called **normal** if it is reduced (without nonzero nilpotent elements) and integrally closed in its total ring of fractions, that is, if it coincides with its integral closure.

The arithmetical notion of *normal ring* allows to define the geometrical notion of *normal variety*:

Definition 1.2.11 Let X be a complex variety in the sense of Definition 1.2.1.

1. If $x \in X$, then the germ (X, x) of X at x is called **normal** if its local ring $\mathcal{O}_{X,x}$ is normal.
2. The complex variety X is **normal** if all its germs are normal.

Normal varieties may be characterized from a more function-theoretical viewpoint as those complex varieties on which holds the following "Riemann extension property": *every bounded holomorphic function defined on the smooth part of an open set extends to a holomorphic function on the whole open set* (see [66, Theorem 4.4.15]).

Recall now the following algebraic regularity condition (see [66, Sect. 4.3]):

Definition 1.2.12 Let $\mathcal{O}$ be a Noetherian local ring, with maximal ideal $\mathfrak{m}$.

1. The **Krull dimension** of $\mathcal{O}$ is the maximal length of its chains of prime ideals.

2. The **embedding dimension** of $\mathcal{O}$ is the dimension of the $\mathcal{O}/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$.
3. The local ring $\mathcal{O}$ is called **regular** if its Krull dimension is equal to its embedding dimension.

The Krull dimension of $\mathcal{O}$ is always less or equal to the embedding dimension. The name *embedding dimension* may be understood by restricting to the case where $\mathcal{O}$ is the local ring of a complex space (see [66, Lemma 4.3.5]):

Proposition 1.2.13 *Let (X, x) be a germ of complex space. Then the embedding dimension of its local ring $\mathcal{O}_{X,x}$ is equal to the smallest $n \in \mathbb{N}$ such that there exists an embedding of germs $(X, x) \hookrightarrow (\mathbb{C}^n, 0)$. In particular, $\mathcal{O}_{X,x}$ is regular if and only if (X, x) is smooth, that is, a germ of complex manifold.*

The normal varieties of dimension one are exactly the smooth complex curves because, more generally (see [66, Thm. 4.4.9, Cor. 4.4.10]):

Theorem 1.2.14 *A Noetherian local ring of Krull dimension one is normal if and only if it is regular.*

There is a canonical way to construct a normal variety $\tilde{X}$ starting from any complex variety X (see [66, Sect. 4.4]):

Theorem 1.2.15 *Let X be a complex variety. Then there exists a finite and generically 1 to 1 morphism $N : \tilde{X} \to X$ such that $\tilde{X}$ is normal. Moreover, such a morphism is unique up to a unique isomorphism over X.*

Recall that a morphism between complex varieties is *finite* if it is proper with finite fibers and that it is *generically* 1 *to* 1 if it is an isomorphism above the complement of a nowhere dense closed subvariety of its target space. The existence of a morphism with the properties stated in Theorem 1.2.15 may be proven algebraically by considering the integral closures of the rings of holomorphic functions on the open sets of X, and showing that they are again rings of holomorphic functions on complex varieties which admit finite and generically 1 to 1 morphisms to the starting open sets. This algebraic proof extends to formal germs, by showing that the integral closure in its total ring of fractions of a complete ring of the form $\mathbb{C}[[x_1, \ldots, x_n]]/I$, where $n \in \mathbb{N}^*$ and I is an ideal of $\mathbb{C}[[x_1, \ldots, x_n]]$, is a direct sum of rings of the same form.

The canonical morphisms characterized in Theorem 1.2.15 received a special name:

Definition 1.2.16 Let X be a complex variety. Then a morphism $N : \tilde{X} \to X$ is called a **normalization morphism** of X if it is finite, generically 1 to 1 and $\tilde{X}$ is a normal complex variety.

Let now (C, o) be a germ of complex variety of Krull dimension one, that is, an **abstract curve singularity**. Its normalization morphisms are of the form: $N : \bigsqcup_{i \in I} (\tilde{C}_i, o_i) \to (C, o)$, where $(C_i, o)_{i \in I}$ is the finite collection of irreducible components of (C, o), and the restriction $N_i : (\tilde{C}_i, o_i) \to (C, o)$ of N to $\tilde{C}_i$ is a normalization of (C_i, o). By Theorem 1.2.14 and Proposition 1.2.13, we see that each germ $(\tilde{C}_i, o_i)$ is smooth, that is, isomorphic to $(\mathbb{C}, 0)$. After precomposing N with such isomorphisms, we see that (C, o) admits a normalization morphism of the form $\bigsqcup_{i \in I} (\mathbb{C}, 0) \to (C, o)$. In particular, if (C, o) is irreducible, its normalization morphism is of the form $N : (\mathbb{C}, 0) \to (C, o)$. The same construction yields a *formal* parametrization when the starting germ (C, o) is formal. This is precisely a *formal parametrization of degree one* as used in the statement of Proposition 1.2.8.

1.2.3 Newton-Puiseux Series and the Newton-Puiseux Theorem

At the end of the previous subsection we explained that normalizations of irreducible germs of complex analytic or formal curves C are holomorphic or formal parametrizations $(\mathbb{C}, 0) \to C$ of degree one. In this subsection we introduce especially nice parametrizations in the case of plane branches, which lead to the notion of *Newton-Puiseux series* (see Definition 1.2.18). The *Newton-Puiseux theorem* (see Theorem 1.2.20) implies that the field of Newton-Puiseux series is algebraically closed. Another consequence of it is stated in Theorem 1.6.1 below.

Let C be a branch on the smooth germ of surface (S, o). Choose an arbitrary system of local coordinates on (S, o). If the branch C is smooth, assume moreover that the germ at o of the y-axis $Z(x)$ is different from C. This means that for any normalization morphism $N : (\mathbb{C}_t, 0) \to (C, o)$ of C, described in this coordinate system as $t \to (\xi(t), \eta(t))$, where $\xi, \eta \in (t)\mathbb{C}[[t]]$, the power series $\xi(t)$ is not identically zero. We have $\xi(t) = t^n \cdot \epsilon(t)$, where $n \in \mathbb{N}^*$ is the order of the power series $\xi(t)$ and $\epsilon(t)$ is a unit in the ring $\mathbb{C}[[t]]$. The series $\epsilon(t)$ has exactly n different n-th roots in $\mathbb{C}[[t]]$, whose constant terms are the n-th roots of $\epsilon(0)$. Pick one of them, denote it by $\epsilon^{1/n}(t)$, and set $\lambda(t) := t\epsilon^{1/n}(t)$. Therefore $\xi(t) = \lambda(t)^n$ and $\nu_t(\lambda(t)) = 1$.

Remark 1.2.17 More generally, if K is an algebraically closed field of characteristic zero, then any unit of $K[[t]]$ has all its n-th roots in $K[[t]]$. This fact is no longer true if K has positive characteristic. For instance, as a direct consequence of the binomial formula, there is no series $\epsilon(t) \in K[[t]]$ such that $\epsilon(t)^p = 1 + t$ when K is of characteristic p. For this reason, the Newton-Puiseux Theorem 1.2.20 below does not always hold in positive characteristic. For more details about the situation in positive characteristic, one may consult [97].

As $\nu_t(\lambda(t)) = 1$, we see that the morphism $(\mathbb{C}_t, 0) \to (\mathbb{C}_u, 0)$, which maps $t \to \lambda(t)$ is an isomorphism of germs of smooth curves. By composing the morphism $N : (\mathbb{C}_t, 0) \to (C, o)$ with its inverse, one gets a new normalization morphism of the form:

$$\begin{array}{ccc} (\mathbb{C}_u, 0) & \to & (C, o) \\ u & \to & (u^n, \zeta(u)) \end{array}$$

where $\zeta(u) \in \mathbb{C}[[u]]$. Therefore, if $f(x, y) \in \mathbb{C}[[x, y]]$ is a defining function of C in the local coordinate system (x, y), we have:

$$f(u^n, \zeta(u)) = 0. \tag{1.2}$$

From the equations $x = u^n$, $y = \zeta(u)$, one may deduce formally that $u = x^{1/n}$, $y = \zeta(x^{1/n})$. Equation (1.2) becomes:

$$f(x, \zeta(x^{1/n})) = 0. \tag{1.3}$$

The composition $\zeta(x^{1/n})$ is a *Newton-Puiseux series* in the following sense:

Definition 1.2.18 The $\mathbb{C}$-algebra $\boxed{\mathbb{C}[[x^{1/\mathbb{N}}]]}$ of **Newton-Puiseux series** consists of all the formal power series of the form $\eta(x^{1/n})$, where $\eta \in \mathbb{C}[[t]]$ and $n \in \mathbb{N}^*$, that is, $\mathbb{C}[[x^{1/\mathbb{N}}]] = \bigcup_{n \in \mathbb{N}^*} \mathbb{C}[[x^{1/n}]]$. Denote by $\boxed{\nu_x} : \mathbb{C}[[x^{1/\mathbb{N}}]] \to [0, \infty]$ the **order function**, which associates to every Newton-Puiseux series the smallest exponent of its terms, where $\nu_x(0) := \infty$.

The function ν_x is a *valuation* of the $\mathbb{C}$-algebra of Newton-Puiseux series, in the following sense:

Definition 1.2.19 A **valuation** on an integral $\mathbb{C}$-algebra A is a function $\nu : A \to \mathbb{R}_+ \cup \{\infty\}$ which satisfies the following conditions:

1. $\nu(fg) = \nu(f) + \nu(g)$, *for all* $f, g \in A$.
2. $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$, *for all* $f, g \in A$.
3. $\nu(\lambda) = 0$, *for all* $\lambda \in \mathbb{C}^*$.
4. $\nu(f) = \infty$ if and only if $f = 0$.

The basic importance of the ring of Newton-Puiseux series comes from the following *Newton-Puiseux theorem* (see Fischer [39, Chapter 7], Teissier [121, Section 1], [123, Sections 3–4], de Jong & Pfister [66, Section 5.1], Cutkosky [28, Section 2.1] or Greuel, Lossen and Shustin [59, Thm. I.3.3]):

Theorem 1.2.20 (Newton-Puiseux Theorem) *Any non-zero monic polynomial* $f \in \mathbb{C}[[x]][y]$ *such that* $f(0, y) = y^d$ *has* d *roots in the ring* $\mathbb{C}[[x^{1/\mathbb{N}}]]$. *As a consequence, the quotient field of the ring* $\mathbb{C}[[x^{1/\mathbb{N}}]]$ *is the algebraic closure of the quotient field of the ring* $\mathbb{C}[[x]]$.

Proof It is immediate to reduce the proof of the first sentence of the theorem to the case where f is irreducible. Assume that this is the case. By Eq. (1.3), there exists a Newton-Puiseux series $\zeta(x^{1/n})$ which is a root of f. One has necessarily $n = d$. Indeed, by the proof of Eq. (1.3), $u \to (u^n, \zeta(u))$ is a normalization of the formal branch $Z(f)$. Therefore, Proposition 1.2.8 shows that:

$$n = \nu_u(u^n) = Z(f) \cdot Z(x) = Z(x) \cdot Z(f) = \nu_y\left(f(0, y)\right) = d.$$

Consider now the product:

$$F(x, y) := \prod_{\omega:\omega^n=1} \left(y - \zeta(\omega x^{1/n})\right) \in \mathbb{C}[[x^{1/n}]][y].$$

It is invariant by the changes of variables $(x^{1/n}, y) \to (\omega x^{1/n}, y)$, where ω varies among the complex n-th roots of 1, which shows that $F(x, y) \in \mathbb{C}[[x]][y]$. As $\zeta(x^{1/n})$ is a root of both $f(x, y)$ and $F(x, y)$ and that $f(x, y)$ is irreducible, we see that f divides F in the ring $\mathbb{C}[[x]][y]$. Both being monic and of the same degree, we get the equality $f = F$. Therefore, all the roots of f belong to $\mathbb{C}[[x^{1/n}]]$.

The second statement of the theorem results from the first statement and from *Hensel's lemma* (see [66, Corollary 3.3.21]), which ensures that a factorisation of $f(0, y) \in \mathbb{C}[y]$ in pairwise coprime factors lifts to an analogous decomposition of $f(x, y) \in \mathbb{C}[[x]][y]$. □

The proof of Theorem 1.2.20 which we have sketched here also shows that the Galois group of the field extension associated to the ring extension $\mathbb{C}[[x]] \subset \mathbb{C}[[x^{1/n}]]$ is isomorphic to the cyclic group of n-th roots of 1, an element ω of this group acting on $\zeta(x^{1/n}) \in \mathbb{C}[[x^{1/n}]]$ replacing it by $\zeta(\omega x^{1/n})$.

Remark 1.2.21 The proof of Theorem 1.2.20 which we have sketched here also shows that the Galois group of the field extension associated to the ring extension $\mathbb{C}[[x]] \subset \mathbb{C}[[x^{1/n}]]$ is isomorphic to the cyclic group of n-th roots of 1, an element ω of this group acting on $\zeta(x^{1/n}) \in \mathbb{C}[[x^{1/n}]]$ replacing it by $\zeta(\omega x^{1/n})$.

Remark 1.2.22 Most proofs of Theorem 1.2.20 use the Newton polygon $\mathcal{N}(f)$ of f (see Definition 1.4.2 below). As explained in Sect. 1.2.5, the restrictions of f to the compact edges of $\mathcal{N}(f)$ allow to find the possible initial terms of the candidate roots $\eta(x)$ of the equation $f(x, y) = 0$ seen as an equation in the single unknown y. Such proofs proceed then by showing that all those terms may be extended to true roots inside $\mathbb{C}[[x^{1/\mathbb{N}}]]$.

Example 1.2.23 Consider coprime integers $\alpha, \beta \in \mathbb{N}^*$ and $f(x, y) := x^\alpha - y^\beta \in \mathbb{C}[[x]][y]$, as in Example 1.2.6. Then the Newton-Puiseux roots of f are the β series $\omega x^{\alpha/\beta}$, where ω varies among the complex β-th roots of 1. If ω' is another such root of 1, it acts on $\omega\, x^{\alpha/\beta}$ by sending it to $(\omega')^\alpha\, \omega\, x^{\alpha/\beta}$.

1.2.4 Blow Ups and Embedded Resolutions of Singularities

In this subsection we explain the notion of *blow up* of $\mathbb{C}^2$ at the origin (see Definition 1.2.24) and more generally of a smooth complex surface at an arbitrary point of it (see Definition 1.2.29), the notion of *embedded resolution* of a curve in a smooth surface (see Definition 1.2.33) and the fact that an embedded resolution may be achieved after a finite number of blow ups of points (see Theorem 1.2.35). We conclude by recalling the notion of *weighted dual graph* of an embedded resolution (see Definition 1.2.36) and the way to compute its weights when this resolution is constructed iteratively by blowing up points.

Look at the complex affine plane $\mathbb{C}^2_{x,y}$ as a complex vector space. Denote by $\boxed{\mathbb{P}(\mathbb{C}^2)_{[u:v]}}$ its **projectivisation**, consisting of its vector subspaces of dimension one, endowed with the projective coordinates $[u : v]$ associated to the cartesian coordinates (x, y) on $\mathbb{C}^2$.

Definition 1.2.24 Consider the **projectivisation map**

$$\begin{array}{rccc} \boxed{\lambda}: & \mathbb{C}^2 & \dashrightarrow & \mathbb{P}(\mathbb{C}^2) \\ & (x, y) & \dashrightarrow & [x : y]. \end{array}$$

associating to each point of $\mathbb{C}^2 \setminus \{O\}$ the line joining it to the origin O of $\mathbb{C}^2$. Let $\boxed{\Sigma}$ be the closure of its graph in the product algebraic variety $\mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2)$. Then the restriction $\boxed{\pi} : \Sigma \to \mathbb{C}^2$ of the first projection $\mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2) \to \mathbb{C}^2$ is called the **blow up of $\mathbb{C}^2$ at the origin**. By abuse of language, the surface Σ is also called in this way. The preimage $\pi^{-1}(O)$ of O in Σ is called the **exceptional divisor** of the blow up. The restriction $\boxed{\tilde{\lambda}} : \Sigma \to \mathbb{P}(\mathbb{C}^2)$ to Σ of the second projection $\mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2) \to \mathbb{P}(\mathbb{C}^2)$ is called the **Hopf morphism**.

The name "*Hopf morphism*" is motivated by the fact that in restriction to the preimage $\pi^{-1}(\mathbb{S}^3)$ of the unit 3-dimensional sphere $\mathbb{S}^3 \subset \mathbb{C}^2$, the morphism $\tilde{\lambda}$ becomes the "Hopf fibration" $\mathbb{S}^3 \to \mathbb{S}^2$, introduced by Hopf in [64, Section 5] (see also [109] for historical details).

The projectivisation *map* restricts to a *morphism* $\lambda : \mathbb{C}^2 \setminus \{O\} \to \mathbb{P}(\mathbb{C}^2)$. This morphism cannot be extended even by continuity to the origin O, because O belongs to the closures of all its level sets, which are the complex lines of $\mathbb{C}^2$ passing through O. Taking the closure of the graph of λ replaces O by the space $\mathbb{P}(\mathbb{C}^2)$ of lines passing through O. This allows the lift of λ to Σ to extend by continuity, and even algebraically, to the whole surface Σ, becoming the Hopf morphism $\tilde{\lambda}$. This morphism is in fact the projection morphism of the total space of a line bundle, as will be shown in Proposition 1.2.25 below. Before proving it, let us explain how to describe using a simple atlas of two charts the blow up surface Σ.

The projective line $\mathbb{P}(\mathbb{C}^2)_{[u:v]}$ is covered by the two affine lines $\mathbb{C}_{u_1}$ and $\mathbb{C}_{v_2}$, where:

$$\boxed{u_1} := \frac{u}{v}, \quad \boxed{v_2} := \frac{v}{u}.$$

Therefore, the product $\mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2)$ is covered by the two affine 3-folds $\mathbb{C}^3_{x,y,u_1}$ and $\mathbb{C}^3_{x,y,v_2}$.

The surface Σ contained in $\mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2)$ is the zero locus $Z(xv - yu)$ of a homogeneous polynomial of degree one in the variables u, v. Its intersections with the two previous 3-folds are therefore:

$$\Sigma \cap \mathbb{C}^3_{x,y,u_1} = Z(x - yu_1), \quad \text{and} \quad \Sigma \cap \mathbb{C}^3_{x,y,v_2} = Z(xv_2 - y).$$

One recognizes in each case the equation of the graph of a function of two variables, those pairs of variables being (u_1, y) and (x, v_2) respectively. Therefore, by projecting on the planes of those two pairs of variables, one gets isomorphisms:

$$\Sigma \cap \mathbb{C}^3_{x,y,u_1} \simeq \mathbb{C}^2_{u_1,y}, \quad \text{and} \quad \Sigma \cap \mathbb{C}^3_{x,y,v_2} \simeq \mathbb{C}^2_{x,v_2},$$

which may be thought as the charts of an algebraic atlas of Σ. Let us replace y by $\boxed{u_2}$ in the first chart $\mathbb{C}^2_{u_1,y}$ and x by $\boxed{v_1}$ in the second chart $\mathbb{C}^2_{x,v_2}$. The blow up morphism $\pi : \Sigma \to \mathbb{C}^2$ gets expressed in the following way in the two charts:

$$\begin{cases} x = u_1 u_2 \\ y = \quad u_2, \end{cases} \quad \text{and} \quad \begin{cases} x = v_1 \\ y = v_1 v_2. \end{cases} \tag{1.4}$$

The previous formulae show that the exceptional divisor $\pi^{-1}(O)$ becomes the u_1-axis in the chart $\mathbb{C}^2_{u_1,u_2}$ and the v_2-axis in the chart $\mathbb{C}^2_{v_1,v_2}$.

By composing one such morphism with the inverse of the second one, we see that Σ may be obtained from the two copies $\mathbb{C}^2_{u_1,u_2}$ and $\mathbb{C}^2_{v_1,v_2}$ of $\mathbb{C}^2$ by gluing their open subsets $\mathbb{C}^*_{u_1} \times \mathbb{C}_{u_2}$ and $\mathbb{C}_{v_1} \times \mathbb{C}^*_{v_2}$ respectively using the following inverse changes of variables:

$$\begin{cases} v_1 = u_1 u_2 \\ v_2 = \quad u_1^{-1} \end{cases} \quad \text{and} \quad \begin{cases} u_1 = v_2^{-1} \\ u_2 = v_1 v_2. \end{cases} \tag{1.5}$$

The Hopf morphism $\tilde{\lambda} : \Sigma \to \mathbb{P}(\mathbb{C}^2)$ becomes the morphisms $\mathbb{C}^2_{u_1,u_2} \to \mathbb{C}^1_{u_1}$ and $\mathbb{C}^2_{v_1,v_2} \to \mathbb{C}^1_{v_2}$ if one uses the charts $\mathbb{C}^2_{u_1,u_2}, \mathbb{C}^2_{v_1,v_2}$ for Σ and $\mathbb{C}^1_{u_1}, \mathbb{C}^1_{v_2}$ for $\mathbb{P}(\mathbb{C}^2)$. The fibers of these two morphisms have natural structures of complex lines if one identifies them with the standard complex line $\mathbb{C}$ using the parameters u_2 and v_1 respectively. As the gluing maps (1.5) respect those structures, we get:

Proposition 1.2.25 *The Hopf morphism $\tilde{\lambda} : \Sigma \to \mathbb{P}(\mathbb{C}^2)$ is the projection morphism from the total space of a line bundle to its base $\mathbb{P}(\mathbb{C}^2)$. Its zero-section is the exceptional divisor $\pi^{-1}(O)$ of the blow up morphism $\pi : \Sigma \to \mathbb{C}^2$.*

The fundamental numerical invariant of a complex line bundle over a projective curve, which characterises it up to topological isomorphisms in general and up to algebraic isomorphisms if the curve is rational, is its **degree**, defined by:

Definition 1.2.26 The **degree** of a line bundle over a smooth connected projective curve C is the degree of the divisor on C defined by any meromorphic section of the line bundle which is neither constantly 0 nor constantly ∞.

In our case, we have:

Proposition 1.2.27 *The degree of the Hopf line bundle $\tilde{\lambda} : \Sigma \to \mathbb{P}(\mathbb{C}^2)$ is equal to -1.*

Proof Let us consider the meromorphic section s of $\tilde{\lambda}$ which appears as the constant function 1 in the charts $\mathbb{C}^2_{u_1,u_2} \to \mathbb{C}^1_{u_1}$. The equation of its graph is $u_2 = 1$. The change of variables (1.5) transform it into $v_1 v_2 = 1$. Therefore, s appears as the rational function v_2^{-1} in the charts $\mathbb{C}^2_{v_1,v_2} \to \mathbb{C}^1_{v_2}$. This shows that the section s has no zeros and a unique pole of multiplicity one. As a consequence, the degree of the divisor defined by s is equal to -1. □

On any smooth complex algebraic or analytic surface S, one may define a notion of *intersection number of two divisors* whenever at least one of them has compact support. This may be done *algebraically*, by considering first the case when one divisor is a reduced compact curve C on S, the intersection number being then the degree of the pullback of the line bundle defined by the second divisor to the normalization of C. Then, one extends this definition by linearity to arbitrary not necessarily reduced or effective divisors. There is also a *topological* definition, obtained by associating a homology class to one divisor, a cohomology class to the second one and then evaluating the cohomology class on the homology class. One may consult [61, Sect. V.1] for the case of algebraic surfaces and [76, Pages 15–20] for the case of analytic surfaces. It turns out that, either by definition or as a theorem, the self-intersection number of the zero-section of a line bundle over a smooth compact complex curve is equal to the degree of the line bundle. Therefore, Proposition 1.2.27 may be also stated as:

Corollary 1.2.28 *The self-intersection number of the zero-section of the Hopf line bundle $\tilde{\lambda} : \Sigma \to \mathbb{P}(\mathbb{C}^2)$ is equal to -1.*

Till now, we have discussed in this subsection only the blow up of the origin O of $\mathbb{C}^2$. This operation may be extended to any point o of a smooth complex surface S, by choosing first local coordinates (x, y) in a neighborhood U of that point. This allows to identify U with an open neighborhood of O in $\mathbb{C}^2$. Denote by $\pi_U : \Sigma_U \to U$ the restriction to U of the blow up morphism of O in $\mathbb{C}^2$. This complex analytic morphism is an isomorphism over $U \setminus \{O\}$. Therefore, it allows to glue Σ_U and S along $U \setminus \{O\}$, getting a surface $\tilde{S}$ endowed with a morphism $\tilde{\pi} : \tilde{S} \to S$.

Definition 1.2.29 The morphism $\tilde{\pi} : \tilde{S} \to S$ constructed above is called a **blow up morphism of S at the point o**.

It may be shown by a direct computation that the blow up morphism of S at o is independent of the choices of local coordinates and open set U. More precisely, given any two morphisms constructed in this way, there exists a unique isomorphism between their sources above S (see [131, Lemma 3.2.1]). Another way to prove this uniqueness is to characterize such morphisms by a universal property (see [61, Chap. II, Prop. 7.14]):

Proposition 1.2.30 *Let S be a smooth complex surface and $\tilde{\pi} : \tilde{S} \to S$ a blow up morphism of S at its point o. Then for any morphism $f : Y \to S$ such that the ideal sheaf defining o on S lifts to a principal ideal sheaf on Y, there exists a unique morphism $g : Y \to \tilde{S}$ such that $f = \tilde{\pi} \circ g$.*

One may define more generally the blow up of any complex space along a closed subspace, and again this morphism may be characterized using an analogous universal property (see [61, Pages 163–169] for a similar study in the case of schemes).

Returning to the model case of the blow up of $\mathbb{C}^2$ at the origin O, relations (1.4) show that the lift by π to Σ of the maximal ideal (x, y) of $\mathbb{C}[x, y]$ defining O is the principal ideal sheaf defining the exceptional divisor of π. This fact is an algebraic manifestation of the fact that on Σ all the lines of $\mathbb{C}^2$ passing through O get separated: they are simply the fibers of the Hopf morphism $\tilde{\lambda}$. Note that in order to separate them indeed, one does not have to lift them by taking their full preimages by π (called their *total transforms by* π), but only by taking their *strict transforms*. Let us define these notions in greater generality:

Definition 1.2.31 Let $\pi : Y \to X$ be a morphism of complex varieties and $Z \subseteq X$ a closed complex subvariety of X.

1. The morphism π is a **modification** of X if it is proper and bimeromorphic, that is, if it is proper and if there exists a closed nowhere dense subvariety X' of X such that $\pi^{-1}(X')$ is a nowhere dense subvariety of Y and the restriction $\pi : Y \setminus \pi^{-1}(X') \to X \setminus X'$ is an isomorphism.
2. If X' is minimal with the previous properties, then X' is called the **indeterminacy locus** of π^{-1} and $\pi^{-1}(X')$ is called the **exceptional locus** of π.
3. The **total transform** $\boxed{\pi^*(Z)}$ of Z by π is the complex subspace of Y defined by the preimage by π of the ideal sheaf defining Z in X.
4. Assume that no irreducible component of Z is included in the indeterminacy locus X' of π^{-1}. Then the **strict transform** of Z by π is the closure inside Y of $\pi^{-1}(Z \setminus X')$.

The blow up morphisms of surfaces at smooth points are examples of modifications. In the case of the blow up $\pi : \Sigma \to \mathbb{C}^2_{x,y}$ at the origin, the Eqs. (1.4) show that the total transform of a line $Z(y - ax) \subseteq \mathbb{C}^2_{x,y}$, for $a \in \mathbb{C}^*$, may be described as $Z(u_2(1 - au_1)) \subseteq \mathbb{C}^2_{u_1,u_2}$ and $Z(v_1(v_2 - a)) \subseteq \mathbb{C}^2_{u_1,u_2}$ in the two charts covering Σ. As $Z(u_2)$ and $Z(v_1)$ describe the exceptional divisor $\pi^{-1}(O)$ in those two charts, we see that the strict transform of $Z(y - ax)$ is the fiber of $\tilde{\lambda}$ whose equations are $u_1 = a^{-1}$ and $v_2 = a$ in those two charts.

Assume now that C is a finite sum $\sum_{i \in I} L_i$ of such lines L_i passing through the origin in $\mathbb{C}^2$. The strict transform of C by π is the sum of the strict transforms $\tilde{L}_i$ of those lines and the total transform $\pi^*(C)$ is the sum $\pi^{-1}(O) + \sum_{i \in I} \tilde{L}_i$ of the exceptional divisor of π and of the strict transform of C. Therefore, $\pi^*(C)$ is a *normal crossings divisor* in the following sense:

Definition 1.2.32 Let S be a smooth complex surface and D a divisor on it. This divisor is said **to have normal crossings** or to be a **normal crossings divisor** if its support is locally either a smooth curve or the union of two transversal smooth curves.

Coming back to the curve $C = \sum_{i \in I} L_i$ in $\mathbb{C}^2$, the fact that its total transform $\pi^*(C)$ is a normal crossings divisor shows that the blow up morphism $\pi : \Sigma \to \mathbb{C}^2$ is an *embedded resolution* of C, in the following sense:

Definition 1.2.33 Let C be a curve on the smooth complex surface S, in the sense of Definition 1.2.3. An **embedded resolution** of C is a modification $\tilde{\pi} : \tilde{S} \to S$ such that:

1. $\tilde{S}$ is smooth;
2. the total transform $\tilde{\pi}^*(C)$ is a normal crossings divisor;
3. the strict transform $\tilde{C}$ of C by $\tilde{\pi}$ is smooth.

The restriction $\tilde{\pi}_C : \tilde{C} \to C$ of an embedded resolution $\tilde{\pi}$ of C to the strict transform $\tilde{C}$ of C is a *resolution* of C in the following sense:

Definition 1.2.34 Let X be a complex variety. A **resolution** of X is a modification $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is smooth and the indeterminacy locus of π^{-1} is equal to the singular locus of X.

If X is a complex curve, then a resolution of it is the same as a normalization morphism. This is no longer true in higher dimensions, as in each dimension at least 2, there are normal non-smooth complex varieties. For instance, a hypersurface X of $\mathbb{C}^n$ whose singular locus has codimension at least 2 in X is normal (see [1, 92]).

Note that the second condition in Definition 1.2.33 does not imply the third one. For instance, if one takes the folium of Descartes $C \subset \mathbb{C}^2_{x,y}$ defined by the equation $x^3 + y^3 = 3xy$, then C is a normal crossings divisor in $\mathbb{C}^2$ (with a single singular point at the origin), therefore the identity morphism from $\mathbb{C}^2$ to itself satisfies the first two conditions of Definition 1.2.33 but not the last one, because the strict transform of C by it is not smooth, being the curve C itself.

In order to get an embedded resolution of the folium of Descartes, it is enough to blow up $\mathbb{C}^2$ at the origin O. More generally, if C is a curve in a smooth complex surface S such that at each point o of C, the branches of C at o are smooth and pairwise transversal, then the morphism obtained by blowing up S at all the singular points of C is an embedded resolution of C. Conversely, as may be seen by working with the description (1.4) of the blow up morphism at a point in terms of local coordinates, this property of achieving an embedded resolution by blowing up distinct points of S characterizes the previous kind of curves. What about curves with more complicated singularities? It turns out that they also have embedded resolutions, which may be obtained by blowing up points *iteratively* (see [61, Thm. 3.9], [15, Pages 496–497], [66, Thm. 5.4.2], [19, Section 3.7] and [131, Thm. 3.4.4]):

Theorem 1.2.35 *Let C be a curve on the smooth complex surface S. Define $S_0 := S$ and $\pi_0 : S_0 \to S$ to be the identity. Assume that for some $k \geq 0$ one has defined a modification $\pi_k : S_k \to S$ which is not an embedded resolution of C. Denote by $B_k \subset S_k$ the set of points at which either the strict transform of C is not smooth or $\pi_k^*(C)$ is not a normal crossings divisor. Define $\psi_k : S_{k+1} \to S_k$ to be the blow up of S_k at the points of B_k and $\pi_{k+1} := \pi_k \circ \psi_k : S_{k+1} \to S$. Then there exists $k \in \mathbb{N}$ such that π_k is an embedded resolution of C.*

If k is chosen minimal such that π_k is an embedded resolution of C, then π_k is called the **minimal embedded resolution** of C. It may be shown that any other embedded resolution of C factors through it.

The combinatorial structure of the total transform of C on a given embedded resolution $\tilde{\pi} : \tilde{S} \to S$ of C is encoded usually by drawing its *weighted dual graph*:

Definition 1.2.36 Let C be a curve on the smooth complex surface S and $\tilde{\pi} : \tilde{S} \to S$ be an embedded resolution of C. Its **weighted dual graph** is a finite connected graph whose vertices are labeled by the irreducible components of the total transform $\tilde{\pi}^*(C)$, two vertices being connected by an edge whenever their associated curves intersect on Σ. The vertices corresponding to the components of the strict transform of C are drawn arrowheaded. The remaining vertices are weighted by the self-intersection numbers on Σ of the associated irreducible components of the exceptional locus of π.

How to compute the weights of the dual graph of the embedded resolution $\tilde{\pi} : \tilde{S} \to S$? If this resolution is obtained iteratively by the process described in Theorem 1.2.35, then one may compute recursively the self-intersection numbers of the components of the exceptional loci of the modifications π_k using Corollary 1.2.28 and (see [131, Lemma 8.1.6]):

Proposition 1.2.37 *Let C be a compact curve in the smooth complex surface S. Let o be a point of C of multiplicity $m \in \mathbb{N}$. If $\pi : \Sigma \to S$ is the blow up of S at o, then the self-intersection $\tilde{C}^2$ in Σ of the strict transform $\tilde{C}$ of C by π is related to the self-intersection C^2 of C in S by the formula $\tilde{C}^2 = C^2 - m$.*

1.2.5 The Minimal Embedded Resolution of the Semicubical Parabola

In this subsection we show how to achieve the minimal embedded resolution of the *semicubical parabola* using the algorithm described in Theorem 1.2.35 and how to compute its weighted dual graph using Proposition 1.2.37. It is an expansion of [61, Example V.3.9.1].

The **semicubical parabola** is the curve $P \hookrightarrow \mathbb{C}^2_{x,y}$ defined as the vanishing locus of the polynomial $p(x, y) := y^2 - x^3$. The germ of P at the origin O is a branch called sometimes the **standard cusp**. Due to the following *Jacobian criterion* (see

[66, Theorem 4.3.6] for a generalization in arbitrary dimension and codimension), the origin is the only singular point of P.

Theorem 1.2.38 (Jacobian criterion) *Let C be a reduced curve in an open set of $\mathbb{C}^2_{x,y}$, defined by a holomorphic function $f : U \to \mathbb{C}$. Then the singular locus* $\mathrm{Sing}(C)$ *is the zero locus $Z(f, \partial_x f, \partial_y f)$.*

We want to construct a sequence of blow ups which leads to an embedded resolution of P by following the algorithm described in Theorem 1.2.35, whose notations we use. Therefore, denote by $\boxed{\pi_1} : S_1 \to \mathbb{C}^2$ the blow up of the origin $\boxed{O_0} := O$ of $\mathbb{C}^2_{x,y}$, instead of $\pi : \Sigma \to \mathbb{C}^2$ as in Definition 1.2.24. We use the standard charts $\mathbb{C}^2_{u_1,u_2}$ and $\mathbb{C}^2_{v_1,v_2}$ for computations on S_1, the blow up morphism π_1 being then described by the changes of variables (1.4). The total transform $\pi_1^*(P)$ of P by π_1 is defined by the composition $p \circ \pi_1$, which is expressed as follows in the two charts:

$$p(u_1u_2, u_2) = u_2^2(1 - u_1^3u_2), \quad p(v_1, v_1v_2) = v_1^2(v_2^2 - v_1). \tag{1.6}$$

As the curve P is smooth outside the origin, its strict transform $\boxed{P_1}$ by π_1 is also smooth outside the exceptional divisor. This strict transform intersects the exceptional divisor $\pi_1^{-1}(O)$ only in the chart $\mathbb{C}^2_{v_1,v_2}$, because its equations in the two charts are $1 - u_1^3u_2 = 0$ and $v_2^2 - v_1 = 0$. The second equation is that of a parabola, therefore it defines a smooth curve. This shows that the strict transform P_1 is everywhere smooth. Therefore, the restriction of the morphism π_1 to the curve P_1 is *a resolution of* P, in the sense of Definition 1.2.34. But it is not an *embedded resolution* in the sense of Definition 1.2.33, because the total transform $\pi_1^*(P)$ is not a normal crossings divisor at the origin $\boxed{O_1}$ of the chart $\mathbb{C}^2_{v_1,v_2}$. Indeed, the strict transform $P_1 \cap \mathbb{C}^2_{v_1,v_2} = Z(v_2^2 - v_1)$ and the exceptional divisor $\pi_1^{-1}(O) \cap \mathbb{C}^2_{v_1,v_2} = Z(v_1)$ are tangent at O_1.

Blow up now the point O_1, getting a new surface $\boxed{S_2}$. Let $\boxed{\psi_1} : S_2 \to S_1$ be this blow up morphism. The preimage $\psi_1^{-1}(\mathbb{C}^2_{v_1,v_2})$ of the chart $\mathbb{C}^2_{v_1,v_2}$ of S_1 may be covered by two charts $\mathbb{C}^2_{w_1,w_2}$ and $\mathbb{C}^2_{z_1,z_2}$, in which the morphism ψ_1 is described by the following analogs of Eqs. (1.4):

$$\begin{cases} v_1 = w_1w_2 \\ v_2 = \quad w_2, \end{cases} \quad \text{and} \quad \begin{cases} v_1 = z_1 \\ v_2 = z_1z_2. \end{cases} \tag{1.7}$$

In order to cover completely the surface S_2, one needs also the chart $\mathbb{C}^2_{u_1,u_2}$ of S_1, which is left unchanged by the blow up morphism ψ_1 because O_1 does not appear in it.

Denote $\boxed{\pi_2} := \pi_1 \circ \psi_1 : S_2 \to \mathbb{C}^2$. Using Eqs. (1.6) we see that:

$$p \circ \pi_2(w_1, w_2) = w_1^2w_2^3(w_2 - w_1), \quad \text{and} \quad p \circ \pi_2(z_1, z_2) = z_1^3(z_1z_2^2 - 1). \tag{1.8}$$

Therefore, the strict transform $\boxed{P_2}$ of P_1 by π_2 intersects again the exceptional divisor only in one of those charts, namely $\mathbb{C}^2_{w_1,w_2}$. The total transform $\pi_2^*(P) \hookrightarrow S_2$ is still not a normal crossings divisor, because its germ at the origin $\boxed{O_2}$ of $\mathbb{C}^2_{w_1,w_2}$ has three branches: $Z(w_1)$, $Z(w_2)$, $Z(w_2 - w_1)$, as shown by Eq. (1.8). One needs to blow up also this point, getting the morphisms $\boxed{\psi_2} : \boxed{S_3} \to S_2$ and $\boxed{\pi_3} := \pi_2 \circ \psi_2 : S_3 \to \mathbb{C}^2$. The blow up ψ_2 may be described using the following analogs of Eqs. (1.4) above the chart $\mathbb{C}^2_{w_1,w_2}$:

$$\begin{cases} w_1 = s_1 s_2 \\ w_2 = s_2, \end{cases} \quad \text{and} \quad \begin{cases} w_1 = t_1 \\ w_2 = t_1 t_2. \end{cases} \tag{1.9}$$

Composing these changes of variables with the second Eq. (1.8), we get:

$$p \circ \pi_3(s_1, s_2) = s_1^2 s_2^6 (1 - s_1), \quad p \circ \pi_3(t_1, t_2) = t_1^6 t_2^3 (t_2 - 1).$$

In both charts of S_3 the total transform $\pi_3^*(P)$ is a normal crossings divisor. This being the case also in the remaining charts $\mathbb{C}^2_{u_1,u_2}$ and $\mathbb{C}^2_{z_1,z_2}$, we see that π_3 is an embedded resolution of singularities of the semicubical parabola P. By Theorem 1.2.35, it is the minimal such resolution.

We illustrated the previous sequence of blow ups in Fig. 1.2. We drew whenever possible the support of the total transform of P in the chart whose origin is contained in the strict transform of P. In the four charts the strict transforms of P are drawn in orange and the defining polynomial is written near it. We have used systematically the same color for a point O_i which is blown up by a morphism ψ_i, for the exceptional divisor E_i created by this blow up and for its strict transforms $E_{i,j}$ by the next blow ups. Notice that the component $E_{0,2}$ appears on the chart $\mathbb{C}^2_{t_1,t_2}$, but it does not appear on the chart $\mathbb{C}^2_{s_1,s_2}$, represented on the right of Fig. 1.2.

Let us compute now the weighted dual graph of π_3. For every $i \in \{0, 1, 2\}$, denote by $\boxed{E_i} \hookrightarrow S_{i+1}$ the exceptional divisor of the blow up of the point $O_i \in S_i$. If $0 \leq i < j \leq 2$, denote by $\boxed{E_{i,j}}$ the strict transform of E_i on the surface S_{j+1} by the modification $\psi_j \circ \cdots \circ \psi_i : S_{j+1} \to S_i$. By Corollary 1.2.28, one has $E_0^2 =$

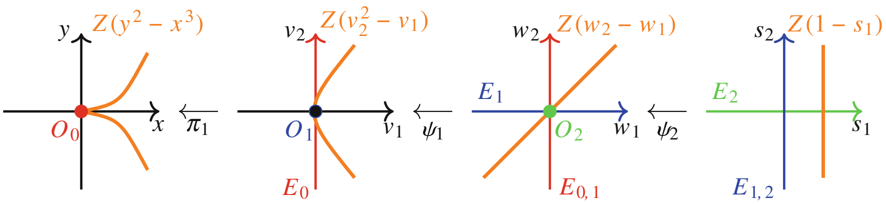


Fig. 1.2 Building iteratively the minimal embedded resolution of the semicubical parabola

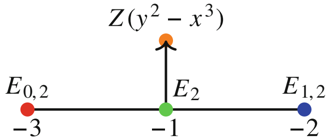


Fig. 1.3 The weighted dual graph of the minimal embedded resolution of the semicubical parabola

$E_1^2 = E_2^2 = -1$. Equations (1.6) and (1.8) imply that $O_1 \in E_0$ and $O_2 \in E_1 \cap E_{0,1}$, because in the chart $\mathbb{C}^2_{v_1,v_2}$ one has $E_{0,1} = Z(v_1)$, $O_1 = (0,0)$ and in the chart $\mathbb{C}^2_{w_1,w_2}$ one has $E_1 = Z(w_2)$, $E_{0,1} = Z(w_1)$, $O_2 = (0,0)$. Using Theorem 1.2.37, we get $E_{0,2}^2 = E_0^2 - 2 = -3$ and $E_{1,2}^2 = E_1^2 - 1 = -2$. Therefore, the weighted dual graph of the minimal embedded resolution $\pi_3 : S_3 \to \mathbb{C}^2$ of the semicubical parabola P is as shown in Fig. 1.3. Near the arrowhead vertex corresponding to the strict transform of P, we have written the defining function of the semicubical parabola.

The previous computations involve many charts, therefore many variables and changes of variables. It is easy to get lost in them. One feels the need of being able to arrive at the final result, the weighted dual graph, without such manipulations. In the next subsection we show how to achieve this goal by a simpler method, without working with charts. We will explain the method using an apparently more complicated example, with two branches. After reading it, we suggest the reader to verify that in the case of the semicubical parabola, the method leads again to the weighted tree of Fig. 1.3.

1.2.6 A Newton Non-degenerate Reducible Example

In this subsection we present on a simple example of *Newton non-degenerate* plane curve singularity the notions of *Newton polygon* and *Newton fan* of a non-zero function $f(x, y) \in \mathbb{C}[[x, y]]$. Then we introduce the associated *lotus* and we show how to construct from it the weighted dual graph of the minimal embedded resolution of the given singularity. These notions are briefly introduced in this section to illustrate our second elementary example and will be revisited formally in Sects. 1.4 and 1.5.

Let $(C, O) \hookrightarrow (\mathbb{C}^2_{x,y}, O)$ be the plane curve singularity defined by the function:

$$f(x, y) := (y^2 - 4x^3)(y^3 - x^7). \tag{1.10}$$

It is the sum of two branches, defined by the equations $y^2 - 4x^3 = 0$ and $y^3 - x^7 = 0$ respectively. Thinking of them as polynomial equations in the unknown y, as explained in Sect. 1.2.3, they have degrees 2 and 3. Their respective sets of roots are $\{\pm 2x^{3/2}\}$ and $\{\omega x^{7/3}\}$, where ω varies among the complex cubic roots of 1. We could express readily in terms of x the roots of the equation $f(x, y) = 0$ seen as a quintic polynomial equation in the variable y, because we knew a factorization of $f(x, y)$ into binomial factors. Is it possible to reach the same objective if one starts instead from the following expanded expression of f?

$$f(x, y) = y^5 - 4x^3y^3 - x^7y^2 + 4x^{10}. \tag{1.11}$$

By the Newton-Puiseux Theorem 1.2.20, we know a priori that the roots of $f(x, y)$ may be expressed as *Newton-Puiseux series*. Newton's fundamental insight was that one may always compute the leading terms of such series only by looking at special terms of f (see the beginning of Sect. 1.4.5). Let us explain this insight in the case of the polynomial (1.11), forgetting its factorization (1.10). Denote by cx^ν the leading term (that is, the term of least degree) of such a series, where $c \in \mathbb{C}^*$ and $\nu > 0$. We have the equality:

$$f(x, cx^\nu + o(x^\nu)) = 0. \tag{1.12}$$

Using formula (1.11), this equality may be rewritten as:

$$(cx^\nu + o(x^\nu))^5 - 4x^3(cx^\nu + o(x^\nu))^3 - x^7(cx^\nu + o(x^\nu))^2 + 4x^{10} = 0,$$

that is, as:

$$\left(c^5x^{5\nu} + o(x^{5\nu})\right) + \left(-4c^3x^{3+3\nu} + o(x^{3+3\nu})\right) + \left(-c^2x^{7+2\nu} + o(x^{7+2\nu})\right) + 4x^{10} = 0. \tag{1.13}$$

The left-hand side of this equation is a sum of four series, whose leading exponents are $5\nu,\ 3 + 3\nu,\ 7 + 2\nu,\ 10$, since $c \neq 0$. The fundamental observation of Newton was that *if the sum* (1.13) *vanishes, then the minimal value of those four exponents is reached at least twice*.

Now, these four exponents may be expressed as the products $(1, \nu) \cdot (a, b) := a + b\nu$, where (a, b) varies among the exponents $(a, b) \in \mathbb{N}^2$ of the monomials $x^a y^b$ appearing in the expanded form (1.11) of $f(x, y)$, that is, as the evaluations of the linear form $l_\nu(a, b) := a + b\nu$ on the *support* $\mathcal{S}(f)$ of the series $f(x, y)$. In our example the support is finite, but it may be infinite if one allows f to be a power series in the variables x, y. It is at this point that *convex geometry* enters into the game, through the following property (which is a consequence of [94, Assertion III.1.5.2]):

Proposition 1.2.39 *Let $\mathcal{S}$ be a subset of $\mathbb{N}^2$. If l is a linear form with non-negative coefficients on $\mathbb{R}^2$, then its restriction to $\mathcal{S}$ achieves its minimum precisely on the subset of $\mathcal{S}$ lying on a face of the convex hull* $\mathrm{Conv}(\mathcal{S} + \mathbb{R}_+^2)$.

Coming back to Eq. (1.13), we see that the linear form $l_\nu(a, b) = a + b\nu$, which computes the leading exponents of the terms appearing in the left-hand side of (1.13), indeed has non-negative coefficients. Therefore,

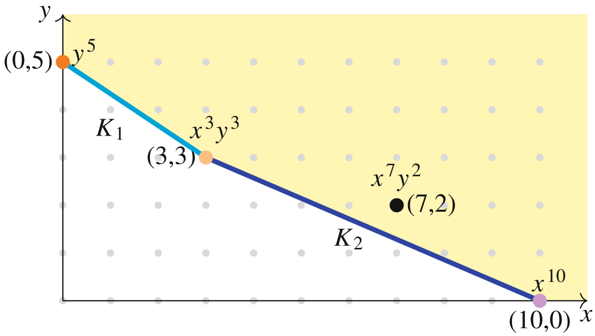


Fig. 1.4 The Newton polygon of the series $f(x, y) = (y^2-2x^3)(y^3-x^7)$

the hypotheses of Proposition 1.2.39 are satisfied. This shows that the minimal value $\min\{5\nu, 3 + 3\nu, 7 + 2\nu, 10\}$ is achieved on a face of the convex hull $\mathrm{Conv}(\mathcal{S}(f) + \mathbb{R}^2_+)$. This convex hull, called the *Newton polygon* $\mathcal{N}(f)$ of $f \in \mathbb{C}[[x, y]]$ (see Definition 1.4.2 below), is represented in Fig. 1.4. It has three vertices, which are $(0, 5)$, $(3, 3)$, $(10, 0)$, corresponding to the terms y^5, $-4x^3y^3$ and $4x^{10}$ of the expansion (1.11). It has two compact edges $K_1 := [(0, 5), (3, 3)]$ and $K_2 := [(3, 3), (10, 0)]$. If the minimum is to be achieved at least twice on $\mathcal{S}(f)$, then it must be achieved on one of those two compact edges, because $\nu > 0$. This means that the linear form l_ν must be orthogonal to one of those compact edges. There are therefore two possibilities:

- Either l_ν achieves its minimum on K_1, which means that $(1, \nu)$ is orthogonal to it. In other words $(1, \nu)\cdot(3-0, 3-5) = 0$, that is, $\nu = 3/2$. Writing that the sum of the terms of the left-hand side of Eq. (1.13) whose leading exponents achieve the minimum vanishes, one gets the equation $c^5 - 4c^3 = 0$. As $c \neq 0$, this is equivalent to the equation $c^2 = 4$, hence $c = \pm 2$.
- Or l_ν achieves its minimum on K_2. In other words $(1, \nu) \cdot (10 - 3, 0 - 3) = 0$, that is, $\nu = 7/3$. One gets then the equation $-4c^3 + 4 = 0$. That is, c varies now among the cubic roots of 1.

It follows that the possible leading terms of a Newton-Puiseux series η in the variable x such that $f(x, \eta) = 0$ belong to the union $\{\pm 2x^{3/2}\} \cup \{\omega x^{7/3} : \omega^3 = 1\}$. One recognizes the roots from the factorization (1.10). Newton's method shows that *those are the leading terms of the roots $y(x)$ of the equation $g(x, y) = 0$, for any $g \in \mathbb{C}[[x, y]]$ whose Newton polygon is the same as $\mathcal{N}(f)$, and whose restrictions to the compact sides of the polygon coincide with the analogous restrictions for f*. Any such function g defines a *Newton non-degenerate* singularity (see Definition 1.4.21 below), because both equations $c^2 = 4$ and $-4c^3 + 4 = 0$ obtained by restricting g to the compact edges of its Newton polygon have simple roots. Variants of Newton's previous line of thought will be followed again in the proofs of Propositions 1.4.11 and 1.4.18 below.

In general, for any series $f(x, y)$, once a first term cx^ν of a potential root of $f(x, y) = 0$ is computed, one may perform a formal change of variables and compute a second term. Newton explained that one could compute as many terms as needed, but it was Puiseux who proved carefully that by pushing this iterative process to its limit, one gets true roots of the equation, which are Newton-Puiseux series. Moreover, he proved that whenever one starts from a convergent function f, one gets only roots of the form $\xi(x^{1/p})$, where $\xi(t) \in \mathbb{C}[[t]]$ is convergent and $p \in \mathbb{N}^*$. This approach leads to a proof of the Newton-Puiseux Theorem 1.2.20, different from the one given above (see Remark 1.2.22).

Let us come back to our example. It turns out that in this *Newton non-degenerate* case, the weighted dual graph of the minimal embedded resolution is determined by the Newton polygon $\mathcal{N}(f)$. In fact, one needs only the *inclinations* of its compact edges. This information is encoded in the associated *Newton fan*, obtained by subdividing the first quadrant along the rays orthogonal to the compact edges of $\mathcal{N}(f)$ (see the left side of Fig. 1.5 and Definition 1.4.9 below). Consider now inside the first quadrant all the triangles with vertices $f_1, f_2, f_1 + f_2$, where (f_1, f_2) is a basis of the ambient lattice $\mathbb{Z}^2$. The edges of those triangles may be drawn recursively by starting from the segment $[e_1, e_2]$ which joins the elements of the canonical basis (e_1, e_2) and, each time a new segment $[f_1, f_2]$ is drawn, by drawing also the segments $[f_1, f_1 + f_2]$ and $[f_2, f_1 + f_2]$. If one performs this construction only whenever the interior of the segment $[f_1, f_2]$ intersects one of the rays of the Newton fan, one gets its associated *lotus*, represented on the right side of Fig. 1.5.

In fact, one needs to attach to it new arrowhead vertices corresponding to the branches of C, as shown in Fig. 1.6. In this figure the lotus was redrawn as an abstract simplicial complex, without representing its precise embedding in the plane $\mathbb{R}^2$. This abstract simplicial structure is sufficient for seeing how it contains the weighted dual graph of the minimal embedded resolution of the curve singularity $Z(xy(y^2 - 4x^3)(y^3 - x^7))$ as part of its boundary. The self-intersection number of an exceptional divisor is simply the opposite of the number of triangles containing the vertex representing this divisor (compare Figs. 1.6 and 1.7).

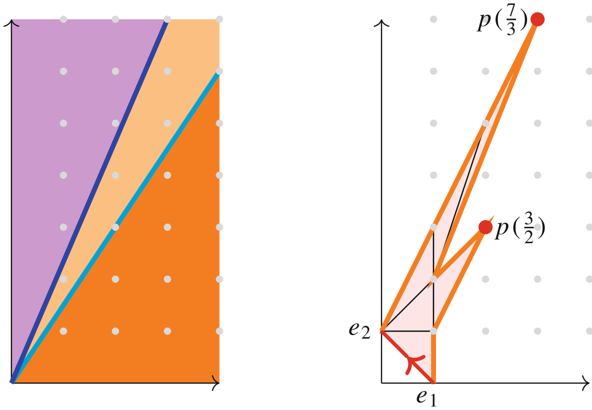


Fig. 1.5 The Newton fan of $f(x, y) = (y^2 - 4x^3)(y^3 - x^7)$ and its associated lotus

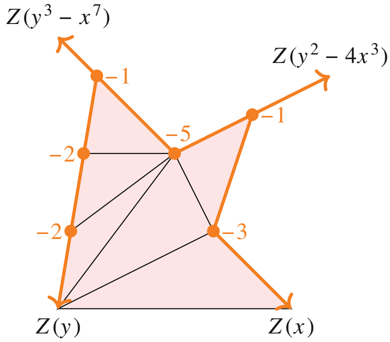


Fig. 1.6 The lotus of $f(x, y) = (y^2 - 4x^3)(y^3 - x^7)$

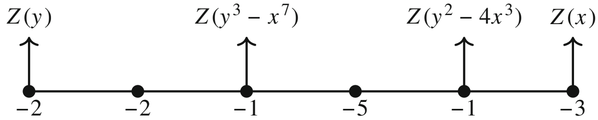


Fig. 1.7 The weighted dual graph of the minimal embedded resolution of $Z(xyf(x, y))$

In the sequel we will associate lotuses to any plane curve singularity C (see Definition 1.5.26). The data needed to construct them will be a finite sequence of Newton polygons generated by a *toroidal pseudo-resolution algorithm* (see Algorithm 1.4.22). We will embed analogously inside them the weighted dual graphs of associated embedded resolutions of *completions* of the curve (see Definition 1.4.15 and Theorem 1.5.29). We will also explain the notions of *fan tree* (see Definition 1.4.33), *Enriques diagram* (see Definition 1.4.31) and *Eggers-Wall tree* (see Definition 1.6.3) of C or of an associated toroidal pseudo-resolution process and we will show that they embed similarly in the corresponding lotus (see Theorem 1.5.29).

1.3 Toric and Toroidal Surfaces and Their Morphisms

In this section we explain basic definitions and intuitions about toric and toroidal varieties and their modifications, which will be used in the subsequent sections in the study of plane curve singularities. Namely, *fans* are introduced in Definition 1.3.3, *affine toric varieties* in Definition 1.3.14, their *boundaries* in Definition 1.3.18, *toric morphisms* in Sect. 1.3.3, in particular the toric description of 2-dimensional blow ups in Example 1.3.27 and the *category of toroidal varieties* in Sect. 1.3.4. Section 1.3.5 contains historical information about the development of toric and toroidal geometry and about its applications to the study of singularities.

1.3.1 Two-Dimensional Fans and Their Regularizations

In this subsection we explain the basic notions of two-dimensional convex geometry needed to define toric varieties in Sect. 1.3.2 and toric morphisms in Sect. 1.3.3: *lattices, rational cones* and *fans*. For more details about toric geometry one may consult the standard textbooks [37, 41, 91] and [26].

A **lattice** is a free $\mathbb{Z}$-module of finite rank. A pair $(a, b) \in \mathbb{Z}^2$ may be seen as an instruction to build two kinds of objects: the Laurent monomial $x^a y^b$ and the parametrized monomial curve $t \to (t^a, t^b)$. The fact that monomials and curves are distinct geometrical objects indicates that it would be good to think also in two ways about the pairs (a, b), that is, as coordinates of vectors relative to bases in two different lattices. Those two lattices are not to be chosen independently of each other. Indeed, given a monomial $x^a y^b$ and a parametrized monomial curve $t \to (t^c, t^d)$, one may substitute the parametrization in the monomial, getting a new monomial, this time in the variable t alone:

$$(x^a y^b) \circ (t^c, t^d) = t^{ac+bd}. \tag{1.14}$$

This indicates that those two lattices should be seen as factors of the domain of definition of the unimodular $\mathbb{Z}$-valued bilinear form $(a, b) \cdot (c, d) := ac + bd$, that is, that they should be *dual* lattices.

In order to distinguish clearly the roles of these two lattices, one denotes them usually by distinct letters, instead of simply writing for instance $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^\vee$. It became traditional after the appearance of Fulton's book [41] to denote by $\boxed{M}$ the lattice whose elements are exponents of monomials in several variables, and by $\boxed{N}$ the dual lattice, whose elements are thought as exponents of parametrized monomial curves in the space of the same variables. It is important to allow for changes of bases of those $\mathbb{Z}$-modules, corresponding to monomial changes of variables of the form $x = u^\alpha v^\gamma$, $y = u^\beta v^\delta$, for which the matrix of exponents is **unimodular**:

$$\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix} = \pm 1. \tag{1.15}$$

This means that one does not have to fix identifications $M = \mathbb{Z}^2$, $N = \mathbb{Z}^2$, but instead to allow those identifications to depend on the context. Note also that the elements of N may be seen as **weights** for the variables x, y. That is, if $(c, d) \in N$, one gives the weight c to x and the weight d to y, which endows the monomial $x^a y^b$ with the weight $ac + bd$ appearing in the equality (1.14). For this reason, N is called sometimes the **weight lattice** associated to the **monomial lattice** M.

We will call **vectors** the elements of a lattice. Those non-zero vectors which cannot be written as non-trivial integral multiples of other lattice vectors are called **primitive**. Any non-zero lattice vector w may be written uniquely in the form $l_{\mathbb{Z}}(w)\, w'$, with $l_{\mathbb{Z}}(w) \in \mathbb{N}^*$ and w' a primitive lattice vector.

Definition 1.3.1 Let N be a lattice and $w \in N \setminus \{0\}$. The positive integer $\boxed{l_{\mathbb{Z}}(w)}$ is the **integral length** of w. We extend this definition to the whole lattice N by setting $l_{\mathbb{Z}}(0) := 0$. For $w_1, w_2 \in N$, the **integral length** $\boxed{l_{\mathbb{Z}}[w_1, w_2]} \in \mathbb{N}$ of the segment $[w_1, w_2]$ is equal to $l_{\mathbb{Z}}(w_2 - w_1) = l_{\mathbb{Z}}(w_1 - w_2)$.

If N is a lattice, denote by $\boxed{N_{\mathbb{R}}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ the real vector space generated by N. We will say that the elements of N are the **integral points** of the real vector space $N_{\mathbb{R}}$. By a **cone of** N we will mean a convex rational polyhedral cone, that is, a subset of $N_{\mathbb{R}}$ of the form:

$$\boxed{\mathbb{R}_+\langle w_1, \dots, w_k\rangle} := \mathbb{R}_+ w_1 + \cdots + \mathbb{R}_+ w_k,$$

where $w_1, \dots, w_k \in N$. If the cone does not contain a positive dimensional vector subspace of $N_{\mathbb{R}}$, it is called **strictly convex**.

If the lattice N is of rank two, then the strictly convex cones are of three sorts, according to their dimensions:

- The 2-dimensional cones are of the form $\mathbb{R}_+\langle w_1, w_2\rangle$, where $w_1, w_2 \in N$ are non-proportional. In classical geometric terminology, they are strictly convex angles with apex at the origin of $N_{\mathbb{R}}$.
- The 1-dimensional cones are the closed half-lines emanating from the origin; we will call them **rays**.
- There is only one 0-dimensional cone: the origin of N.

As a particular case of a terminology used in any dimension, one speaks about the **faces** of a given cone $\sigma \subseteq N_{\mathbb{R}}$: those are the subsets of σ on which the restriction to σ of a linear form $l \in N_{\mathbb{R}}^{\vee} = M_{\mathbb{R}}$ reaches its minimum. The faces of a strictly convex 2-dimensional cone $\mathbb{R}_+\langle w_1, w_2\rangle$ are the cone itself, its **edges** $\mathbb{R}_+ w_1$, $\mathbb{R}_+ w_2$ and the origin. The faces of a ray are the ray itself and the origin. Finally, the origin has only one face, which is the origin itself.

Endowing the 2-dimensional lattice N with a basis (e_1, e_2) allows to speak of the **slope** $d/c \in \mathbb{R} \cup \{\infty\}$ relative to (e_1, e_2) of any vector $w = c\, e_1 + d\, e_2 \in N_{\mathbb{R}} \setminus \{0\}$ or of the associated ray $\mathbb{R}_+ w$. In terms of the coordinates (c, d), the integral length $l_{\mathbb{Z}}(w)$ of w is equal to the greatest common divisor $\gcd(c, d)$.

Notations 1.3.2 If the basis (e_1, e_2) of N is fixed and clear from the context, we denote by:

$$\boxed{\sigma_0} := \mathbb{R}_+\langle e_1, e_2\rangle \subseteq N_{\mathbb{R}}$$

the cone generated by it. If $\lambda \in \mathbb{Q}_+ \cup \{\infty\}$, we denote by $\boxed{p(\lambda)}$ the unique primitive element of the lattice N contained in the cone σ_0, and which has slope λ.

In the sequel it will be important to work with the following special sets of cones, which are fundamental in toric geometry:

Definition 1.3.3 A **fan** of the lattice N is a finite set of strictly convex cones of N which is closed under the operation of taking faces of its cones and such that the intersection of any two of its cones is a face of each of them. The **support** $\boxed{|\mathcal{F}|}$ of a fan $\mathcal{F}$ is the union of its cones. A fan $\mathcal{F}$ **refines** (or **subdivides**) another fan $\mathcal{F}'$ if they have the same support and if each cone of $\mathcal{F}$ is contained in some cone of $\mathcal{F}'$. A fan **subdivides a cone** σ if it subdivides the fan formed by its faces. We often denote again by σ the fan formed by the faces of a cone σ, by a slight abuse of notation.

Let us complete the previous definition, valid in arbitrary rank, with terminology and notations specific to rank two:

Definition 1.3.4 Let (e_1, e_2) be a basis of the lattice N of rank two and σ_0 be the associated cone $\mathbb{R}_+\langle e_1, e_2\rangle$. Any fan $\mathcal{F}$ subdividing σ_0 is determined by the finite set of slopes $\mathcal{E} \subset \mathbb{Q}_+^*$ of its rays contained in the interior of σ_0. In this case we denote the fan by $\boxed{\mathcal{F}(\mathcal{E})}$ and we call it the **fan of the set** $\mathcal{E}$. We extend the definition of $\mathcal{F}(\mathcal{E})$ to the case where $\mathcal{E}$ contains 0 or ∞, by setting in this case $\mathcal{F}(\mathcal{E}) := \mathcal{F}(\mathcal{E} \setminus \{0, \infty\})$. If $\mathcal{E} = \{\lambda_1, \ldots, \lambda_p\}$, we write also $\boxed{\mathcal{F}(\lambda_1, \ldots, \lambda_p)}$ instead of $\mathcal{F}(\mathcal{E})$.

Note that $\mathcal{F}(\emptyset)$ is simply the fan consisting of the cone σ_0 and its faces.

Definition 1.3.5 A cone of a lattice N is called **regular** if it can be generated by elements which form a subset of a basis of N. A fan all of whose cones are regular is called **regular**.

It is convenient to set $\mathbb{R}_+\langle\emptyset\rangle := \{0\}$. This implies that $\{0\}$ is also a regular cone.

Assume that a basis (e_1, e_2) of the lattice N is fixed. If $f_1 = \alpha e_1 + \beta e_2$ and $f_2 = \gamma e_1 + \delta e_2$ are two primitive vectors of N, then the cone $\mathbb{R}_+\langle f_1, f_2\rangle$ generated by them is regular if and only if the matrix of the pair (f_1, f_2) in the basis (e_1, e_2) is unimodular, that is, the equality (1.15) holds.

Example 1.3.6 If $\mathcal{E} = \{3/5, 2/1, 5/2\}$, then the rays of the fan $\mathcal{F}(\mathcal{E})$ are represented in Fig. 1.8. On each ray of the fan which is distinct from the edges of the cone σ_0, we indicated by a small red disc the unique primitive element of the lattice N lying on it. That is, on the ray of slope $\lambda \in \mathcal{E}$ is indicated the point $p(\lambda)$. The fan $\mathcal{F}(\mathcal{E})$ contains also 4 cones of dimension 2, which are $\mathbb{R}_+\langle e_1, p\,(3/5)\rangle$, $\mathbb{R}_+\langle p\,(3/5)\,, p\,(2/1)\rangle$, $\mathbb{R}_+\langle p\,(2/1)\,, p\,(5/2)\rangle$, $\mathbb{R}_+\langle p\,(5/2)\,, e_2\rangle$. Using the unimodularity criterion above, we see that $\mathbb{R}_+\langle p\,(2/1)\,, p\,(5/2)\rangle$ is the only 2-dimensional cone of the fan $\mathcal{F}(\mathcal{E})$ which is regular.

The following result is specific for lattices of rank two (see [91, Prop. 1.19]):

Proposition 1.3.7 *If the lattice N is of rank two, any fan relative to N has a minimal regular subdivision, in the sense that any other regular subdivision refines it.*

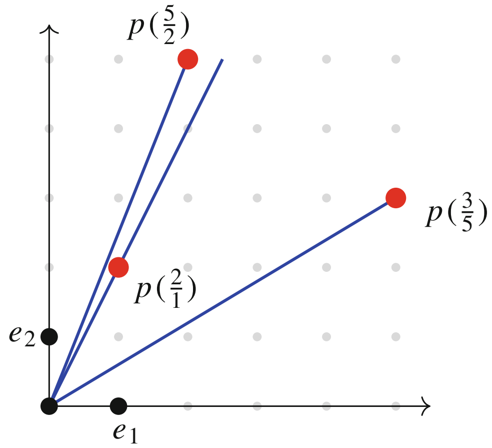


Fig. 1.8 The fan $\mathcal{F}(3/5, 2/1, 5/2)$ and the points $p\,(3/5)$, $p\,(2/1)$, $p\,(5/2)$

Proposition 1.3.7 motivates the following definition:

Definition 1.3.8 If $\mathcal{F}$ is a 2-dimensional fan, we denote by $\boxed{\mathcal{F}^{reg}}$ its minimal regular subdivision, and we call it the **regularization** of $\mathcal{F}$.

The importance of the regularization operation in our context stems from the fact that it allows to describe combinatorially the minimal resolution of a toric surface (see Proposition 1.3.28 below). The regularization of a 2-dimensional cone may be described in the following way (see [91, Proposition 1.19]):

Proposition 1.3.9 *Let N be a lattice of rank two and let σ be a 2-dimensional strictly convex cone of N. Then the regularization σ^{reg} of the fan of its faces is obtained by subdividing σ using the rays directed by the integral points lying on the boundary of the convex hull of the set of non-zero integral points of σ. If $\mathcal{F}$ is a fan of a lattice of rank two, then its regularization is the union of the regularizations of its cones.*

An alternative recursive description of σ^{reg} was given by Mutsuo Oka in [94, Chap. II.2].

Example 1.3.10 Let us consider again the fan $\mathcal{F}(3/5, 2/1, 5/2)$ of Example 1.3.6. The rays of $\mathcal{F}^{reg}\,(3/5, 2/1, 5/2) = \mathcal{F}(1/2, 3/5, 2/3, 1/1, 2/1, 5/2, 3/1)$ are drawn in green in Fig. 1.9. The thick orange polygonal line, on the right side of this figure, is the union of compact edges of the boundaries of the convex hulls of the sets of non-zero integral points of its 2-dimensional cones.

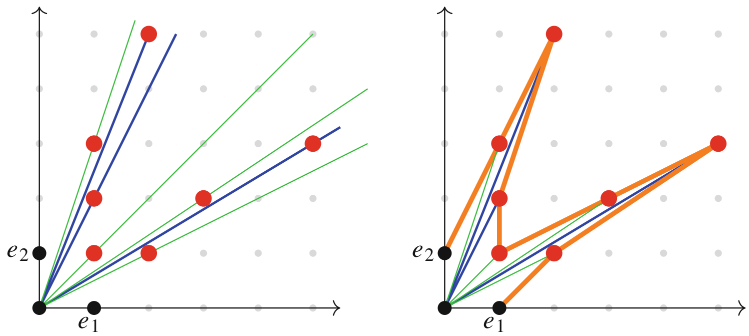


Fig. 1.9 The regularization $\mathcal{F}^{reg}$ $(3/5, 2/1, 5/2)$ of the fan of Fig. 1.8

1.3.2 Toric Varieties and Their Orbits

In this subsection we explain in which way fans determine special kinds of complex algebraic varieties, called *toric varieties*. Namely, every rational polyhedral cone relative to a lattice determines a *monoid algebra* (see Definition 1.3.11), whose maximal spectrum is an *affine toric variety* (see Definition 1.3.14). More generally, every fan determines a toric variety by gluing the affine toric varieties associated to its cones (see Definition 1.3.15).

One associates with a lattice N of rank n the following **complex algebraic torus** of dimension n (that is, an algebraic group isomorphic to $\big((\mathbb{C}^*)^n, \cdot\big)$):

$$\boxed{\mathcal{T}_N} := N \otimes_{\mathbb{Z}} \mathbb{C}^*. \tag{1.16}$$

Here the factors are considered as abelian groups $(N, +)$ and $(\mathbb{C}^*, \cdot)$, therefore they are endowed with canonical structures of $\mathbb{Z}$-modules, relative to which is taken the previous tensor product. This algebraic torus may be also described in terms of the dual lattice M of N, defined by:

$$\boxed{M} := \text{Hom}\,(N, \mathbb{Z}).$$

Namely, one has:

$$\mathcal{T}_N = \text{Hom}(M, \mathbb{C}^*). \tag{1.17}$$

Equations (1.16) and (1.17) allow in turn to give the following interpretations of the lattices N and M in terms of morphisms of algebraic groups:

$$\begin{aligned} N &= \mathrm{Hom}(\mathbb{C}^*, \mathcal{T}_N) = \\ &= \text{the group of } \textbf{one parameter subgroups} \text{ of } \mathcal{T}_N; \\ M &= \mathrm{Hom}(\mathcal{T}_N, \mathbb{C}^*) = \\ &= \text{the group of } \textbf{characters} \text{ of } \mathcal{T}_N. \end{aligned} \tag{1.18}$$

If $w \in N$ is seen as an element of the lattice N, we denote by $\boxed{t^w}$ the same element seen as a morphism of abelian groups from $\mathbb{C}^*$ to $\mathcal{T}_N$.

Let us explain this notation in the case when N has rank 2. If t is viewed as the parameter on the source $\mathbb{C}^*$ and one identifies $\mathcal{T}_N$ with $(\mathbb{C}^*)^2$ using the basis (e_1, e_2) of N, then the morphism becomes the following map from $\mathbb{C}^*$ to $(\mathbb{C}^*)^2$:

$$t \to (t^c, t^d).$$

Here (c, d) denote as before the coordinates of w in the chosen basis (e_1, e_2) of N. One gets therefore a parametrized monomial curve as at the beginning of Sect. 1.3.1. The advantage of seeing it as an element of $\mathrm{Hom}(\mathbb{C}^*, \mathcal{T}_N)$ is that one gets a viewpoint independent of the choice of coordinates for $\mathcal{T}_N$, that is, of bases for M or for N.

It is customary to say that a morphism $t^w \in \mathrm{Hom}\,(\mathbb{C}^*, \mathcal{T}_N)$ is a *one parameter subgroup* of $\mathcal{T}_N$, even when this morphism is not injective. Note that t^w is injective if and only if w is a primitive element of N. In general, when $w \in N \setminus \{0\}$, the map t^w is a cyclic covering of its image, of degree $l_{\mathbb{Z}}(w)$ (see Definition 1.3.1). Note also that t^0 is the constant map with image the unit element 1 of the group $\mathcal{T}_N$.

We introduced the notation t^w in order to be able to distinguish between N seen as an abstract group, and seen as the lattice of one parameter subgroups of $\mathcal{T}_N$. In an analogous way, if $m \in M$, one uses the notation $\boxed{\chi^m} : \mathcal{T}_N \to \mathbb{C}^*$ for its associated character, in order to distinguish between M seen as an abstract group and seen as the lattice of characters of $\mathcal{T}_N$. If one denotes by $\boxed{w \cdot m} \in \mathbb{Z}$ the result of applying the canonical duality pairing $N \times M \to \mathbb{Z}$ to $(w, m) \in N \times M$, then the composite morphism $\chi^m \circ t^w : \mathbb{C}^* \to \mathbb{C}^*$ is simply given by $t \to t^{w \cdot m}$. This is the intrinsic description of the composition performed in formula (1.14).

Let us see more precisely how the choice of basis (e_1, e_2) of N determines an isomorphism $\mathcal{T}_N \simeq (\mathbb{C}^*)^2$. To have such an isomorphism amounts to choosing a special pair (x, y) of regular functions on $\mathcal{T}_N$, which are the pull-backs of the coordinate functions on $(\mathbb{C}^*)^2$. This isomorphism should be not only an isomorphism of algebraic surfaces, but also of groups. As the coordinate functions on $(\mathbb{C}^*)^2$ are characters of $\big((\mathbb{C}^*)^2, \cdot\big)$, that is, elements of $\mathrm{Hom}((\mathbb{C}^*)^2, \mathbb{C}^*)$, we deduce that x, y are also characters, this time of $(\mathcal{T}_N, \cdot)$. It means that they are elements of the lattice M (see the equalities (1.18)). In which way does the basis (e_1, e_2) of N determine a pair of elements of M? Well, this pair is simply the dual basis $\boxed{(\epsilon_1, \epsilon_2)}$ of (e_1, e_2)! Therefore, one has $(x, y) = (\chi^{\epsilon_1}, \chi^{\epsilon_2})$ in terms of the dual basis $(\epsilon_1, \epsilon_2) \in M^2$ of $(e_1, e_2) \in N^2$.

The choice of coordinates (x, y) allows to embed the torus $\mathcal{T}_N$ into the affine plane $\mathbb{C}^2$ with the same coordinates. The coordinate ring of this affine plane is of course $\mathbb{C}[x, y]$. In our context it is important to interpret this ring as the $\mathbb{C}$-algebra of the commutative monoid of monomials with non-negative exponents in the variables x and y. This monoid is isomorphic (using the map $m \to \chi^m$) to the monoid $\mathbb{R}_+\langle\epsilon_1, \epsilon_2\rangle \cap M$. In turn, the cone $\mathbb{R}_+\langle\epsilon_1, \epsilon_2\rangle$ is in the following sense the dual cone of $\sigma_0 := \mathbb{R}_+\langle e_1, e_2\rangle$:

Definition 1.3.11 Let σ be a cone of N. Its **dual** is the cone $\sigma^\vee$ of M defined by:

$$\boxed{\sigma^\vee} := \{m \in M_{\mathbb{R}},\ w \cdot m \geq 0 \text{ for all } w \in \sigma\},$$

and its associated **monoid algebra** is the $\mathbb{C}$-algebra of the abelian monoid $(\sigma^\vee \cap M, +)$:

$$\boxed{\mathbb{C}[\sigma^\vee \cap M]} := \left\{ \sum_{\text{finite}} c_m \chi^m \ ,\ m \in \sigma^\vee \cap M \text{ and } c_m \in \mathbb{C} \right\}.$$

Note that σ is strictly convex if and only if the dimension of $\sigma^\vee$ is equal to the rank of the lattice M. The $\mathbb{C}$-algebra $\mathbb{C}[\sigma^\vee \cap M]$ is finitely generated, since the monoid $(\sigma^\vee \cap M, +)$ is finitely generated by Gordan's Lemma (see [41, Section 1.1, Proposition 1]).

The set $\mathbb{C}^2$ with coordinates (x, y) may now be interpreted in the two following ways:

$$\begin{aligned} \mathbb{C}^2_{x,y} &= \text{the maximal spectrum of the ring } \mathbb{C}[\sigma_0^\vee \cap M] = \\ &= \mathrm{Hom}(\sigma_0^\vee \cap M, \mathbb{C}). \end{aligned} \tag{1.19}$$

The last set of homomorphisms is taken in the category of abelian monoids, where $\mathbb{C}$ is considered as a monoid with respect to multiplication. This interpretation is obtained by looking at the evaluation of the monomials χ^m, with $m \in \sigma_0^\vee \cap M$, at the points of $\mathbb{C}^2$.

The equalities (1.19) may be turned into a general way to associate a complex affine variety to a cone σ of N, in arbitrary dimension:

$$\begin{aligned} \boxed{X_\sigma} &:= \text{the maximal spectrum of } \mathbb{C}[\sigma^\vee \cap M] = \\ &= \mathrm{Hom}(\sigma^\vee \cap M, \mathbb{C}). \end{aligned} \tag{1.20}$$

The equalities (1.19) show that $X_{\sigma_0} = \mathbb{C}^2_{x,y}$, if $x = \chi^{\epsilon_1}$ and $y = \chi^{\epsilon_2}$. Therefore, the affine variety X_{σ_0} is smooth. The following proposition characterizes the cones for which the associated variety is smooth (see [41, Section 2.1, Proposition 1]):

Proposition 1.3.12 *Let σ be a strictly convex cone of the lattice N. Then the affine variety X_σ is smooth if and only if σ is regular in the sense of Definition 1.3.5.*

In the sequel, by a **stratification** of an algebraic variety we mean a finite partition of it into locally closed connected smooth subvarieties, called the **strata** of the stratification, such that the closure of each stratum is a union of strata.

Consider the following stratification of $X_{\sigma_0} = \mathbb{C}^2_{x,y}$:

$$\mathbb{C}^2_{x,y} = \{0\} \sqcup \left(\mathbb{C}^*_x \times \{0\}\right) \sqcup \left(\{0\} \times \mathbb{C}^*_y\right) \sqcup (\mathbb{C}^*)^2_{x,y}. \tag{1.21}$$

One may interpret in the following way its strata in terms of vanishing of monomials whose exponents belong to $\sigma_0^\vee \cap M = \mathbb{N}\langle\epsilon_1, \epsilon_2\rangle$:

- 0 is the only point of $\mathbb{C}^2_{x,y}$ at which vanish exactly the monomials with exponents in $\left(\sigma_0^\vee \setminus \{0\}\right) \cap M$.
- $\mathbb{C}^*_x \times \{0\}$ is the set of points of $\mathbb{C}^2_{x,y}$ at which vanish exactly the monomials with exponents in $\left(\sigma_0^\vee \setminus \mathbb{R}_+\epsilon_1\right) \cap M$.
- $\{0\} \times \mathbb{C}^*_y$ is the set of points of $\mathbb{C}^2_{x,y}$ at which vanish exactly the monomials with exponents in $\left(\sigma_0^\vee \setminus \mathbb{R}_+\epsilon_2\right) \cap M$.
- $(\mathbb{C}^*)^2_{x,y} = \mathcal{T}_N$ is the set of points of $\mathbb{C}^2_{x,y}$ at which vanish no monomials, that is, at which vanish exactly the monomials with exponents in $\left(\sigma_0^\vee \setminus \sigma_0^\vee\right) \cap M$.

Note that the sets of exponents of monomials appearing in the previous list are precisely those of the form $\left(\sigma_0^\vee \setminus \tau\right) \cap M$, where τ varies among the faces of the cone $\sigma_0^\vee$. It is customary in toric geometry to express them in a dual way, using the following bijection between the faces of σ and of $\sigma^\vee$, valid in all dimensions for (not necessarily rational) convex polyhedral cones σ (see [26, Proposition 1.2.10]):

Proposition 1.3.13 *Let σ be a cone of $N_\mathbb{R}$. Then the map $\rho \to \rho^\perp \cap \sigma^\vee$ is an order-reversing bijection from the set of faces of σ to the set of faces of $\sigma^\vee$ (see Fig. 1.10).*

Here $\boxed{\rho^\perp} := \{m \in M_\mathbb{R},\ w \cdot m = 0 \text{ for all } w \in \rho\}$ denotes the orthogonal of the cone ρ of N. It is a real vector subspace of $M_\mathbb{R}$, which may be characterized as the maximal vector subspace of the convex cone $\rho^\vee$.

The stratification (1.21) of $\mathbb{C}^2$ is a particular case of a stratification of any affine variety of the form X_σ. In order to define it, one associates with each point p of X_σ the subset of $\sigma^\vee \cap M$ formed by the exponents of the monomials vanishing at p. This defines a function from X_σ to the power set of $\sigma^\vee \cap M$, whose levels are precisely the strata of the stratification of X_σ. The set of strata is in bijective correspondence

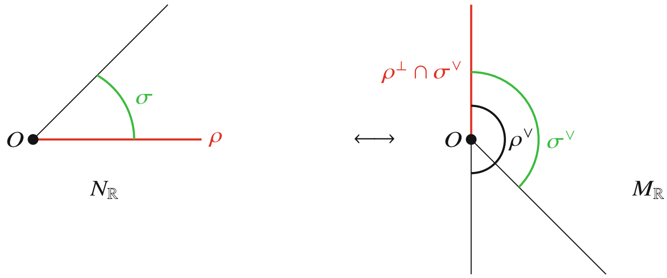


Fig. 1.10 The bijection between the faces of σ and $\sigma^{\vee}$

with the set of faces of σ, the stratum O_{ρ} corresponding to the face ρ of σ being:

$$\boxed{O_{\rho}} := \left\{ p \in \mathrm{Hom}(\sigma^{\vee} \cap M, \mathbb{C}),\ p^{-1}(0) = (\sigma^{\vee} \setminus \rho^{\perp}) \cap M \right\}. \tag{1.22}$$

In particular, $O_{\{0\}} = \mathcal{T}_N$ is the only stratum whose dimension is the same as the dimension of X_{σ}. This shows that the torus $\mathcal{T}_N$ embeds naturally as an affine open set in the affine surface X_{σ}. For this reason, the following vocabulary was introduced:

Definition 1.3.14 If N is a lattice and σ is a strictly convex cone of N, then the variety X_{σ} defined by the equalities (1.20) is called an **affine toric variety**.

Note that for $X_{\sigma_0} = \mathbb{C}^2_{x,y}$, the strata are:

- $O_{\sigma_0} = \{0\}$;
- $O_{\mathbb{R}_+ e_2} = \mathbb{C}^*_x \times \{0\}$;
- $O_{\mathbb{R}_+ e_1} = \{0\} \times \mathbb{C}^*_y$;
- $O_{\{0\}} = (\mathbb{C}^*)^2_{x,y} = \mathcal{T}_N$.

One may feel difficult to remember the second and third equalities, a common error at the time of doing computations being to permute them. A way to remember them is the following: *the orbit corresponding to an edge of a* 2*-dimensional regular cone is the complement of the origin in the axis of coordinates of* $\mathbb{C}^2$ *defined by the vanishing of the dual variable*. In our case, the dual variable of the edge $\mathbb{R}_+ e_1$ is $x = \chi^{\epsilon_1}$, whose 0-locus is the axis of the variable y, and conversely.

The notation O_{ρ} is motivated by the fact that this subset of X_{σ} is an *orbit* of a natural action of the algebraic torus $\mathcal{T}_N$ on X_{σ}. For $X_{\sigma_0} = \mathbb{C}^2_{x,y}$, case in which one may also identify $\mathcal{T}_N$ with $(\mathbb{C}^*)^2_{u,v}$, this action is given by $(u, v) \cdot (x, y) := (ux, vy)$. In general, the action of $\mathcal{T}_N$ on X_{σ} may be described in intrinsic terms by:

$$(M \xrightarrow{\tau} \mathbb{C}^*) \cdot (\sigma^{\vee} \cap M \xrightarrow{p} \mathbb{C}) := (\sigma^{\vee} \cap M \xrightarrow{\tau \cdot p} \mathbb{C}).$$

In the previous equation we used again the interpretations of the points of $\mathcal{T}_N$ and X_σ as morphisms of monoids (see Eqs. (1.17) and (1.20)).

Assume now that $\mathcal{F}$ is a fan of N, in the sense of Definition 1.3.3. Each affine toric variety X_σ, where $\sigma \in \mathcal{F}$, contains the torus $\mathcal{T}_N$ as an affine open set. If σ and τ are two cones of $\mathcal{F}$, then one has a natural identification of their respective tori, and also of their larger Zariski open subsets $X_{\sigma\cap\tau} \subset X_\sigma$ and $X_{\sigma\cap\tau} \subset X_\tau$. If one glues the various affine toric varieties $(X_\sigma)_{\sigma\in\mathcal{F}}$ using the previous identifications, one gets an abstract separated complex algebraic variety $X_{\mathcal{F}}$ which still contains the torus $\mathcal{T}_N$ as an affine open subset (see [91, Theorem 1.4 and 1.5]).

Definition 1.3.15 The **toric variety** associated with a fan $\mathcal{F}$ of a lattice N is the variety $\boxed{X_{\mathcal{F}}}$ constructed above.

Remark 1.3.16 All toric varieties constructed from fans are normal in the sense of Definition 1.2.11 (see [26, Theorem 1.3.5]). One has a more general notion of toric variety, which includes some non-normal varieties as well (see the paper [56] of Teissier and the second author). Those varieties can be described as before by gluing maximal spectra of algebras of not necessarily saturated finite type submonoids of lattices, the normal ones being precisely the toric varieties associated with a fan of Definition 1.3.15.

As a consequence of Proposition 1.3.12, one has a smoothness criterion for toric varieties:

Proposition 1.3.17 *Let $\mathcal{F}$ be a fan of the lattice N. Then the toric variety $X_{\mathcal{F}}$ is smooth if and only if $\mathcal{F}$ is regular in the sense of Definition 1.3.5.*

Let us come back to a fan $\mathcal{F}$ of a weight lattice N. When ρ varies among the cones of $\mathcal{F}$, the actions of the torus $\mathcal{T}_N$ on the affine toric varieties X_ρ glue into an action on $X_{\mathcal{F}}$, whose orbits are still denoted by O_ρ. The conservation of the notation (1.22) is motivated by the fact that in the gluing of X_σ and X_τ, the orbits denoted O_ρ on both sides get identified, for every face ρ of $\sigma \cap \tau$. If ρ is a cone of the fan $\mathcal{F}$, we denote by $\boxed{\overline{O_\rho}}$ the closure in $X_{\mathcal{F}}$ of the orbit O_ρ. The orbit closure $\overline{O_\rho}$ has also a natural structure of normal toric variety (see [41, Chapter 3]).

The torus $\mathcal{T}_N$ is identified canonically with the orbit O_0 corresponding to the origin of $N_{\mathbb{R}}$, seen as a cone of dimension 0. Its complement is the union of all the orbits of codimension at least 1. Let us introduce a special name and notation for this complement:

Definition 1.3.18 Let $X_{\mathcal{F}}$ be a toric variety defined by a fan $\mathcal{F}$. Its **boundary** $\boxed{\partial X_{\mathcal{F}}}$ is the complement of the algebraic torus $\mathcal{T}_N$ inside $X_{\mathcal{F}}$.

The boundary $\partial X_{\mathcal{F}}$ is a reduced Weil divisor inside $X_{\mathcal{F}}$, whose irreducible components are the orbit closures $\overline{O_\rho}$ corresponding to the cones ρ of $\mathcal{F}$ which have dimension 1, that is, to the rays of the fan $\mathcal{F}$.

1.3.3 Toric Morphisms and Toric Modifications

In this subsection we define the notion of *toric morphism* between toric varieties (see Definition 1.3.19) and we explain in which way refining a fan defines a special kind of toric morphism, called a *toric modification* (see Proposition 1.3.21). In Examples 1.3.26 and 1.3.27 we explain how to do concrete computations of toric modifications in dimension two, the second one giving a toric presentation of the blow ups of the origin. Finally, Proposition 1.3.28 explains the combinatorics of the minimal resolution of a normal affine toric surface.

Assume that N_1 and N_2 are two weight lattices, endowed with cones σ_1 and σ_2. Let $\phi : N_1 \to N_2$ be a morphism of lattices which sends the cone σ_1 into the cone σ_2. Using the second interpretation in the equalities (1.20) of the points of affine toric varieties, we see that ϕ induces an algebraic morphism between the associated toric varieties:

$$\begin{array}{rcl} \boxed{\psi^{\sigma_1}_{\sigma_2,\phi}} : X_{\sigma_1} & \to & X_{\sigma_2} \\ p_1 & \to & p_1 \circ \phi^{\vee} \end{array} . \tag{1.23}$$

One sees immediately from the definitions that the adjoint $\phi^{\vee} : M_2 \to M_1$ of ϕ maps $\sigma_2^{\vee}$ into $\sigma_1^{\vee}$, which shows that the composition $p_1 \circ \phi^{\vee}$ belongs indeed to $X_{\sigma_2} = \mathrm{Hom}(\sigma_2^{\vee} \cap M_2, \mathbb{C})$ whenever $p_1 \in X_{\sigma_1} = \mathrm{Hom}(\sigma_1^{\vee} \cap M_1, \mathbb{C})$. The morphism $\psi^{\sigma_1}_{\sigma_2,\phi}$ may be also described using the first interpretation in the equalities (1.20), as the morphism of affine schemes induced by the morphism of $\mathbb{C}$-algebras $\mathbb{C}[\sigma_2^{\vee} \cap M_2] \to \mathbb{C}[\sigma_1^{\vee} \cap M_1]$ which sends each monomial $\chi^{m_2} \in \sigma_2^{\vee} \cap M_2$ to the monomial $\chi^{\phi^{\vee}(m_2)} \in \sigma_1^{\vee} \cap M_1$.

Assume now that N_1 and N_2 are endowed with fans $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, such that ϕ sends each cone of $\mathcal{F}_1$ into some cone of $\mathcal{F}_2$. We say that ϕ **is compatible with the two fans**. It may be checked formally that the previous morphisms $\psi^{\sigma_1}_{\sigma_2,\phi} : X_{\sigma_1} \to X_{\sigma_2}$, for all the pairs $(\sigma_1, \sigma_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ which verify that $\phi(\sigma_1) \subseteq \sigma_2$, glue into an algebraic morphism: $\boxed{\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi}} : X_{\mathcal{F}_1} \to X_{\mathcal{F}_2}$. This morphism is moreover *equivariant* with respect to the actions of $\mathcal{T}_{N_1}$ and $\mathcal{T}_{N_2}$ on $X_{\mathcal{F}_1}$ and $X_{\mathcal{F}_2}$ respectively. For this reason, one uses the following terminology:

Definition 1.3.19 If the morphism of lattices $\phi : N_1 \to N_2$ sends every cone of $\mathcal{F}_1$ into some cone of $\mathcal{F}_2$, then the morphism of algebraic varieties $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi} : X_{\mathcal{F}_1} \to X_{\mathcal{F}_2}$ described above is called the **toric morphism associated with ϕ and the fans $\mathcal{F}_1, \mathcal{F}_2$**.

The toric morphism $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi}$ sends the torus $\mathcal{T}_{N_1} = X_{\mathcal{F}_1} \setminus \partial X_{\mathcal{F}_1}$ into $\mathcal{T}_{N_2} = X_{\mathcal{F}_2} \setminus \partial X_{\mathcal{F}_2}$. This fact implies the following property of toric morphisms relative to the boundaries of their sources and targets, in the sense of Definition 1.3.18:

Proposition 1.3.20 *Let $\psi : X_{\mathcal{F}_1} \to X_{\mathcal{F}_2}$ be the toric morphism associated with ϕ and the fans $\mathcal{F}_1$ and $\mathcal{F}_2$. Then $\psi^{-1}(\partial X_{\mathcal{F}_2}) \subseteq \partial X_{\mathcal{F}_1}$.*

Toric morphisms have the following properties (see [91, Theorems 1.13, 1.15]):

Proposition 1.3.21 *Let N_1, N_2 be two lattices and $\mathcal{F}_1, \mathcal{F}_2$ be fans of N_1 and N_2 respectively. Let $\phi : N_1 \to N_2$ be a lattice morphism compatible with the two fans. Then:*

1. *The morphism $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi}$ is birational if and only if ϕ is an isomorphism of lattices.*
2. *The morphism $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi}$ is proper if and only if the $\mathbb{R}$-linear map $\phi_{\mathbb{R}} : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$ sends the support of $\mathcal{F}_1$* onto *the support of $\mathcal{F}_2$.*

In particular, $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi}$ is a modification in the sense of Definition 1.2.31 if and only if ϕ is an isomorphism and, after identifying N_1 and N_2 using it, the fan $\mathcal{F}_1$ refines the fan $\mathcal{F}_2$ in the sense of Definition 1.3.3.

We will consider most of the time the particular case in which $N_1 = N_2 = N$ is a lattice of rank 2 and ϕ is the identity. Then, if $\sigma \subset \sigma_0$ is a subcone of σ_0, we denote by $\psi^{\sigma}_{\sigma_0}$ the birational toric morphism induced by the identity:

$$\boxed{\psi^{\sigma}_{\sigma_0}} : X_{\sigma} \to X_{\sigma_0} = \mathbb{C}^2_{x,y}. \tag{1.24}$$

When σ varies among all the cones of a fan $\mathcal{F}$ which subdivides the cone σ_0, the morphisms $\psi^{\sigma}_{\sigma_0}$ glue into a single equivariant birational morphism:

$$\boxed{\psi^{\mathcal{F}}_{\sigma_0}} : X_{\mathcal{F}} \to X_{\sigma_0} = \mathbb{C}^2_{x,y}. \tag{1.25}$$

By Proposition 1.3.21, this morphism is also proper, because $\mathcal{F}$ and σ_0 have the same support. Therefore, $\psi^{\mathcal{F}}_{\sigma_0}$ is a modification of $\mathbb{C}^2_{x,y}$.

The strict transform of $\boxed{L} := Z(x)$ (resp. of $\boxed{L'} := Z(y)$) by the modification $\psi^{\mathcal{F}}_{\sigma_0}$ is the orbit closure $\overline{O}_{\mathbb{R}_+ e_1}$ (resp. $\overline{O}_{\mathbb{R}_+ e_2}$) in $X_{\mathcal{F}}$. The preimage of $0 \in \mathbb{C}^2_{x,y}$, called the **exceptional divisor** of $\psi^{\mathcal{F}}_{\sigma_0}$, and the preimage of the sum $L + L'$ of the coordinate axes, which is the total transform of $L + L'$ are:

$$\begin{aligned} (\psi^{\mathcal{F}}_{\sigma_0})^{-1}(0) \quad &= \overline{O}_{\rho_1} + \cdots + \overline{O}_{\rho_k}, \\ (\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L') &= \overline{O}_{\mathbb{R}_+ e_1} + \overline{O}_{\rho_1} + \cdots + \overline{O}_{\rho_k} + \overline{O}_{\mathbb{R}_+ e_2}, \end{aligned} \tag{1.26}$$

where $\rho_1, \ldots, \rho_k$ denote the rays of $\mathcal{F}$ contained in the interior of σ_0, labeled as in Fig. 1.11. Note that $L + L' = \partial X_{\sigma_0}$ and $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L') = \partial X_{\mathcal{F}}$, which is a particular case of Proposition 1.3.21.

Recall now the following classical notion of (unweighted) *dual graph*, which extends that of Definition 1.2.36 and whose historical evolution was sketched by the third author in [104]:

Definition 1.3.22 A **simple normal crossings curve** is a reduced abstract complex curve whose irreducible components are smooth and whose singularities are normal crossings, that is, analytically isomorphic to the germ at the origin of the union of coordinate axes of $\mathbb{C}^2$. The **dual graph** of a simple normal crossings curve D is the abstract graph whose set of vertices is associated bijectively with the set of irreducible components of D, the edges between two vertices corresponding bijectively with the intersection points of the associated components of D. Each vertex or edge is labeled by the corresponding irreducible component or point of D.

Remark 1.3.23 Let $\sigma = \mathbb{R}_+\langle f_1, f_2\rangle \subset N_{\mathbb{R}}$ be a strictly convex cone of dimension two, not necessarily regular. One may check that the boundary $\partial X_\sigma = \overline{O}_{\mathbb{R}_+ f_1} + \overline{O}_{\mathbb{R}_+ f_2}$ of the affine toric surface X_σ is an abstract simple normal crossings curve, according to Definition 1.3.22.

The dual graph of the total transform $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L')$ may be embedded in the cone $\sigma_0 \subseteq N_{\mathbb{R}}$:

Proposition 1.3.24 *Let $\mathcal{F}$ be a fan which subdivides the regular cone σ_0. Then the dual graph of the divisor $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L')$ is a segment with extremities L and L' and with k intermediate points labeled in order by $\overline{O}_{\rho_1}, \dots, \overline{O}_{\rho_k}$ from L to L'. That is, it is isomorphic to the segment $[e_1, e_2] \subset N_{\mathbb{R}}$, marked with its intersection points with the rays of $\mathcal{F}$, the point $[e_1, e_2] \cap \rho_i$ being labeled by the orbit closure $\overline{O}_{\rho_i}$.*

Therefore, the rays of the fan $\mathcal{F}$ correspond bijectively to the irreducible components of the total transform $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L')$ of $L + L'$. The 2-dimensional cones of $\mathcal{F}$ correspond to the fixed points of the torus action, which are the only possible singular points of the surface $X_{\mathcal{F}}$. The orbit closures $\overline{O}_\rho$ and $\overline{O}_{\rho'}$ intersect at a point $q \in X_{\mathcal{F}}$ if and only if the cone $\rho+\rho'$ is a 2-dimensional cone of $\mathcal{F}$ and then q is the unique orbit $O_{\rho+\rho'}$ of dimension 0 of the affine toric surface $X_{\rho+\rho'} \subset X_{\mathcal{F}}$. The point q is singular on the surface $X_{\mathcal{F}}$ if and only if the cone $\rho+\rho'$ is not regular.

Example 1.3.25 For the fan $\mathcal{F}(3/5, 2/1, 5/2)$ of Fig. 1.8 discussed in Example 1.3.6, the total transform $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L')$ and its dual graph are represented in Fig. 1.11. The 4 singular points of the total transform are also singular on the surface $X_{\mathcal{F}}$, with the exception of $\overline{O}_{\rho_2} \cap \overline{O}_{\rho_3}$. Indeed, the cone $\rho_2 + \rho_3$ is the only regular 2-dimensional cone of the fan $\mathcal{F}$, as may be seen in Fig. 1.9.

Example 1.3.26 Let us explain how to describe in coordinates the morphism $\psi^{\sigma}_{\sigma_0}$ of (1.24), when σ is a *regular* subcone of σ_0. Denote by f_1, f_2 the primitive generators of the edges of σ, ordered in such a way that the bases (e_1, e_2) and (f_1, f_2) define the same orientation of $N_{\mathbb{R}}$ (see Fig. 1.12). Decompose (f_1, f_2) in the basis (e_1, e_2), writing $f_1 = \alpha e_1 + \beta e_2$ and $f_2 = \gamma e_1 + \delta e_2$. This means that the

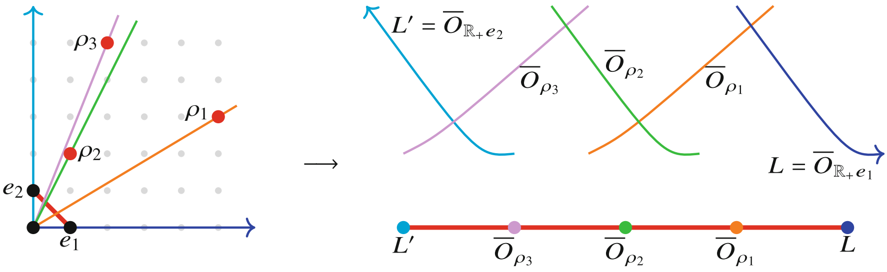


Fig. 1.11 The dual graph of the total transform $(\psi^{\mathcal{F}}_{\sigma_0})^{-1}(L + L')$

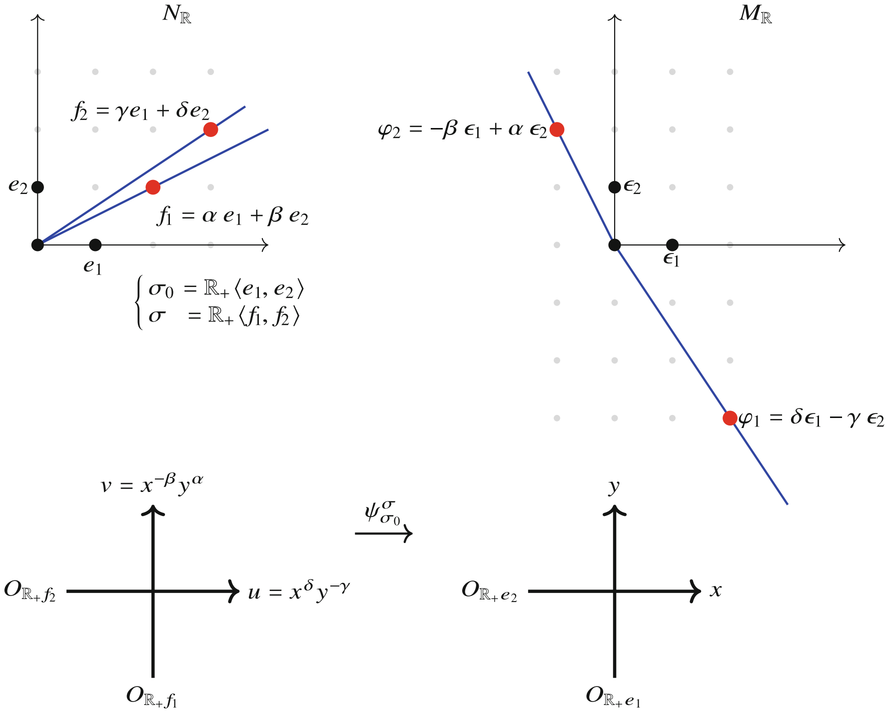


Fig. 1.12 The toric morphism defined by the two regular cones of Example 1.3.26

unimodular matrix of change of bases from (f_1, f_2) to (e_1, e_2) is:

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}. \tag{1.27}$$

Denote by $(\varphi_1, \varphi_2) \in M^2$ the dual basis of (f_1, f_2) and by

$$\begin{cases} u := \chi^{\varphi_1} = x^{\delta} y^{-\gamma} \\ v := \chi^{\varphi_2} = x^{-\beta} y^{\alpha}, \end{cases} \tag{1.28}$$

the associated coordinates. Then, in terms of the identifications $X_\sigma = \mathbb{C}^2_{u,v}$ and $X_{\sigma_0} = \mathbb{C}^2_{x,y}$, the morphism $\psi^{\sigma}_{\sigma_0}$ is given by the following monomial change of coordinates (compare the disposal of exponents with the matrix (1.27)):

$$\begin{cases} x = u^\alpha v^\gamma \\ y = u^\beta v^\delta. \end{cases} \tag{1.29}$$

Note that the system (1.28) implies that the expression of $v = \chi^{\varphi_2}$ as a monomial in x and y is determined only by f_1, being independent of the choice of f_2. This may be explained geometrically. Indeed, as $f_1 \cdot \varphi_2 = 0$, we see that φ_2 belongs to the line $f_1^\perp$ orthogonal to f_1. As φ_2 may be completed into a basis of M, it is primitive, which determines it up to sign. This sign ambiguity is lifted by the constraint that the basis (f_1, f_2) determines the open half-plane bounded by the line $\mathbb{R} f_1$ on which φ_2 has to be positive. Note also that v is a coordinate on the orbit $O_{\mathbb{R}_+ f_1}$ determined by the edge $\mathbb{R}_+ f_1$ of σ. This coordinate determines an isomorphism $O_{\mathbb{R}_+ f_1} \simeq \mathbb{C}^*_v$ of complex tori, and depends only on $\mathbb{R}_+ f_1$, since the orbit $O_{\mathbb{R}_+ f_1}$ can be realized as a subspace of the surface $X_{\mathbb{R}_+ f_1}$ by formula (1.22) above.

Example 1.3.27 In this example we use the explanations given in Example 1.3.26. Let $\mathcal{F}$ be the fan obtained by subdividing $\sigma_0 = \mathbb{R}_+\langle e_1, e_2\rangle$ using the half-line ρ generated by $e_1 + e_2$. It has two cones of dimension 2, denoted $\sigma_1 := \mathbb{R}_+\langle e_1, e_1 + e_2\rangle$ and $\sigma_2 := \mathbb{R}_+\langle e_1 + e_2, e_2\rangle$ (see Fig. 1.13). Then the toric morphism $\psi^{\mathcal{F}}_{\sigma_0}$ may be described by its two restrictions $\psi^{\sigma_1}_{\sigma_0}$ and $\psi^{\sigma_2}_{\sigma_0}$. The matrices of change of bases from $(e_1, e_1 + e_2)$ and $(e_1 + e_2, e_2)$ to (e_1, e_2) respectively are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Denoting by (u_1, u_2) and (v_1, v_2) the coordinates corresponding to the dual bases of $(e_1, e_1 + e_2)$ and $(e_1 + e_2, e_2)$, the general formulas (1.27) and (1.29) show that

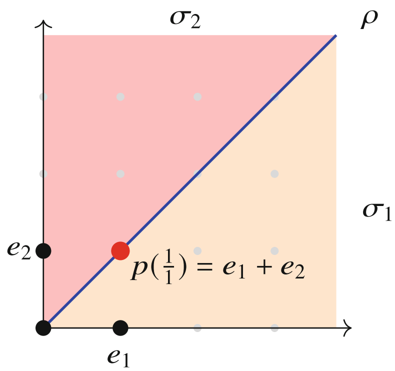


Fig. 1.13 The subdivision of Example 1.3.27, defining the toric blow up of the origin of $\mathbb{C}^2$

the morphisms $\psi_{\sigma_0}^{\sigma_1}$ and $\psi_{\sigma_0}^{\sigma_2}$ are given by the following changes of variables:

$$\begin{cases} x = u_1 u_2 \\ y = \quad u_2, \end{cases} \quad \text{and} \quad \begin{cases} x = v_1 \\ y = v_1 v_2. \end{cases} \tag{1.30}$$

We get the same expressions as in Eqs. (1.4). This shows that $\psi_{\sigma_0}^{\mathcal{F}}$ *is a toric representative of the blow up morphism of* $\mathbb{C}^2_{x,y}$ *at the origin*!

Let σ be a non-regular cone of the weight lattice N of rank two. By Proposition 1.3.12, the affine toric surface X_σ is not smooth. In fact, it has only one singular point, the orbit O_σ of dimension 0. Being of dimension 2, X_σ admits a **minimal resolution**, that is, a resolution through which factors any other resolution (recall that this notion was explained in Definition 1.2.34). It turns out that this minimal resolution may be given by a toric morphism, defined by the regularization of σ in the sense of Definition 1.3.8 (see [91, Proposition 1.19]):

Proposition 1.3.28 *Let σ be a non-regular cone of the weight lattice N of rank two. Denote by σ^{reg} the regularization of the fan formed by the faces of σ. Then the toric modification $\psi_\sigma^{\sigma^{reg}} : X_{\sigma^{reg}} \to X_\sigma$ is the minimal resolution of X_σ. As a consequence, for any fan $\mathcal{F}$ of N, the toric modification $\psi_{\mathcal{F}}^{\mathcal{F}^{reg}} : X_{\mathcal{F}^{reg}} \to X_{\mathcal{F}}$ is the minimal resolution of $X_{\mathcal{F}}$.*

1.3.4 Toroidal Varieties and Modifications in the Toroidal Category

In this subsection we explain analytic generalizations of toric varieties and toric morphisms: the notions of *toroidal variety* and *morphism of toroidal varieties* (see Definition 1.3.29). Then we introduce the notion of *cross* on a smooth germ of surface (see Definition 1.3.31), and we explain how to attach to a cross a canonical oriented regular cone in a two-dimensional lattice (see Definition 1.3.32) and how each subdivision of this cone determines a canonical modification in the toroidal category (see Definition 1.3.33). The toroidal pseudo-resolutions of plane curve singularities introduced in Sect. 1.4.2 below will be constructed as compositions of such toroidal modifications.

Toric surfaces and morphisms are not sufficient for the study of plane curve singularities for the following reasons. One starts often from a germ of curve on a smooth complex surface which does not have a preferred coordinate system. It may be impossible to choose a coordinate system such that the germ of curve gets resolved by only one toric modification relative to the chosen coordinates (if the curve singularity is reduced and such a resolution is possible, then one says that the singularity is *Newton non-degenerate*, see Definition 1.4.21 below). Instead, what may be always achieved is a *morphism of toroidal surfaces*, in the following sense:

Definition 1.3.29 A **toroidal variety** is a pair $(\Sigma, \partial\Sigma)$ consisting of a normal complex variety Σ and a reduced divisor $\partial\Sigma$ on Σ such that the germ of $(\Sigma, \partial\Sigma)$ at any point $p \in \Sigma$ is analytically isomorphic to the germ of a pair $(X_\sigma, \partial X_\sigma)$ at a point of X_σ, where ∂X_σ denotes the boundary of the affine toric variety X_σ in the sense of Definition 1.3.18. Such an isomorphism is called a **toric chart centered at** p of the toroidal variety $(\Sigma, \partial\Sigma)$. The divisor $\partial\Sigma$ is **the boundary of the toroidal variety**.

A **morphism** $\psi : (\Sigma_2, \partial\Sigma_2) \to (\Sigma_1, \partial\Sigma_1)$ between toroidal varieties is a complex analytic morphism $\psi : \Sigma_2 \to \Sigma_1$ such that $\psi^{-1}(\partial\Sigma_1) \subseteq \partial\Sigma_2$. The morphism ψ is a **modification** if the underlying morphism of complex varieties is a modification in the sense of Definition 1.2.31.

Toroidal varieties with their morphisms define a category, called the **toroidal category**.

The previous definition implies that if $(\Sigma, \partial\Sigma)$ is toroidal, then the complement $\Sigma \setminus \partial\Sigma$ is smooth. Indeed, the point p is allowed to be taken outside the boundary $\partial\Sigma$, and the definition shows then that the germ of Σ at p is analytically isomorphic to the germ of a toric variety at a point of the associated torus, which is smooth.

If Σ is of dimension two and if p is a smooth point of $\partial\Sigma$, then p is a smooth point of Σ, since the germ of Σ at p is analytically isomorphic to the germ of a normal toric surface at a point belonging to a 1-dimensional orbit, which is necessarily smooth.

Proposition 1.3.20 implies that a toric morphism $\psi^{\mathcal{F}_1}_{\mathcal{F}_2,\phi} : X_{\mathcal{F}_1} \to X_{\mathcal{F}_2}$ becomes an element of the toroidal category if one looks at it as a complex analytic morphism from the pair $(X_{\mathcal{F}_1}, \partial X_{\mathcal{F}_1})$ to the pair $(X_{\mathcal{F}_2}, \partial X_{\mathcal{F}_2})$, the boundaries being taken in the sense of Definition 1.3.18.

Remark 1.3.30 There exists also a more restrictive notion of *toroidal morphism* $\psi : (\Sigma_2, \partial\Sigma_2) \to (\Sigma_1, \partial\Sigma_1)$ between toroidal varieties. By definition, such a morphism becomes monomial in the neighborhood of any point p of Σ_2, after some choice of toric charts at the source and the target, centered at p and $\psi(p)$ respectively. Toroidal morphisms belong to the toroidal category, but the converse is not true. For instance, take two copies $\mathbb{C}^2_{u,v}$ and $\mathbb{C}^2_{x,y}$ of the complex affine plane and the affine morphism $\psi : \mathbb{C}^2_{u,v} \to \mathbb{C}^2_{x,y}$ defined by $x = u, y = u(1 + v)$. Consider the plane $\mathbb{C}^2_{u,v}$ as a toroidal surface with boundary equal to the union of its coordinate axes, and $\mathbb{C}^2_{x,y}$ as a toroidal surface with boundary equal to the y-axis. As $\psi^{-1}(\partial\mathbb{C}^2_{x,y}) \subseteq \partial\mathbb{C}^2_{u,v}$, ψ is a morphism of toroidal varieties. But it is not a toroidal morphism. Otherwise, it would become the morphism $(u, v) \to (u, u)$ after analytic changes of coordinates in the neighborhoods of the origins of the two planes, which is impossible, because ψ is birational, therefore dominant.

Let us come back to the case of a smooth germ of surface (S, o).

Definition 1.3.31 A **cross** on the smooth germ of surface (S, o) is a pair (L, L') of transversal smooth branches on (S, o). A local coordinate system (x, y) on (S, o) is said **to define the cross** (L, L') if $L = Z(x)$ and $L' = Z(y)$.

We chose the name *cross* by analogy with the denomination *normal crossings divisor* (see Definition 1.2.32). Note the subtle difference between the two notions: the pair (L, L') is a cross if and only if $L + L'$ is a normal crossings divisor, but the knowledge of the divisor does not allow to remember the order of its branches.

Definition 1.3.32 Let (L, L') be a cross on (S, o). We associate with it the two-dimensional lattice $\boxed{M_{L,L'}}$ of integral divisors supported by $L \cup L'$, called the **monomial lattice of the cross** (L, L'). The **weight lattice of the cross** (L, L') is the dual lattice $\boxed{N_{L,L'}}$ of $M_{L,L'}$. Denote by $\boxed{(\epsilon_L, \epsilon_{L'})}$ the basis $\epsilon_L := L, \epsilon_{L'} := L'$ of $M_{L,L'}$, by $\boxed{(e_L, e_{L'})}$ the dual basis of $N_{L,L'}$, and by $\boxed{\sigma_0^{L,L'}}$ the cone $\mathbb{R}_+\langle e_L, e_{L'}\rangle$. When the cross (L, L') is clear from the context, we often write simply $\boxed{(\epsilon_1, \epsilon_2)}$, $\boxed{(e_1, e_2)}$ and $\boxed{\sigma_0}$ instead of $(\epsilon_L, \epsilon_{L'})$, $(e_L, e_{L'})$ and $\sigma_0^{L,L'}$ respectively.

Each time we choose local coordinates (x, y) defining the cross (L, L'), we identify $M_{L,L'}$ with the lattice of exponents of monomials in those coordinates. That is, $a\epsilon_1 + b\epsilon_2$ corresponds to $x^a y^b$. Such a choice of coordinates also identifies holomorphically a neighborhood of o in S with a neighborhood of the origin in $\mathbb{C}^2$ and the cross (L, L') with the coordinate cross in $\mathbb{C}^2$ at the origin. Therefore, any subdivision $\mathcal{F}$ of σ_0 defines an analytic modification $\psi^{\mathcal{F}}_{L,L'} : S_{\mathcal{F}} \to S$ of S. As these modifications are isomorphisms over $S \setminus \{o\}$, it is easy to see that they are independent of the chosen coordinate system (x, y) defining (L, L'), up to canonical analytical isomorphisms above S. Moreover, if we define $\partial S := L + L'$ and $\partial S_{\mathcal{F}} := (\psi^{\mathcal{F}}_{L,L'})^{-1}(L + L')$, the morphism $\psi^{\mathcal{F}}_{L,L'}$ becomes a morphism from the toroidal surface $(S_{\mathcal{F}}, \partial S_{\mathcal{F}})$ to the toroidal surface $(S, \partial S)$.

Definition 1.3.33 If $\mathcal{F}$ is a fan subdividing the cone $\sigma_0 \subset N_{L,L'}$, then the morphism of the toroidal category

$$\boxed{\psi^{\mathcal{F}}_{L,L'} : (S_{\mathcal{F}}, \partial S_{\mathcal{F}}) \to (S, L + L')}$$

associated with $\mathcal{F}$ is the **modification of** S **associated with** $\mathcal{F}$ **relative to the cross** (L, L').

When the fan $\mathcal{F}$ is regular, the morphism $\psi^{\mathcal{F}}_{L,L'}$ between the underlying complex surfaces (forgetting the toroidal structures) is a composition of blow ups of points (see Definition 1.2.29). We will explain the structure of this decomposition of $\psi^{\mathcal{F}}_{L,L'}$ in Sect. 1.5 (see Propositions 1.5.10, 1.5.11).

1.3.5 Historical Comments

Toric varieties were called *torus embeddings* at the beginning of the development of toric geometry in the 1970s, following the terminology of Kempf, Knudsen, Mumford and Saint-Donat's 1973 book [71], as these are varieties into which an algebraic torus embeds as an affine Zariski open subset. The introduction of the book [71] contains information about sources of toric geometry in papers by Demazure, Hochster, Bergman, Sumihiro and Miyake & Oda. Details about the development of toric geometry may be found in Cox, Little and Schenck's 2011 book [26, Appendix A].

The first applications of toric geometry to the study of singularities were done by Kouchnirenko, Varchenko and Khovanskii in their 1976–77 papers [74, 128] and [73] respectively. But one may see in retrospect toric techniques in Puiseux's 1850 paper [106, Sections 20, 23], in Jung's 1908 paper [68], in Dumas' 1911–12 papers [31, 32], in Hodge's 1930 paper [63], in Hirzebruch's 1953 paper [62] and in Teissier's 1973 paper [119]. Indeed, in all those papers, monomial changes of variables more general than those describing blow ups are used in an essential way. For instance, in his paper [62], Hirzebruch described the minimal resolution of an affine toric surface by gluing the toric charts of the resolved surface by explicit monomial birational maps. Toric surfaces appeared in Hirzebruch's paper as normalizations of the affine surfaces in $\mathbb{C}^3$ defined by equations of the form $z^m = x^p y^q$, with $(m, p, q) \in (\mathbb{N}^*)^3$ globally coprime. Interesting details about Hirzebruch's work [62] are contained in Brieskorn's paper [14].

The notion of *toroidal variety* of arbitrary dimension was introduced in a slightly different form in the same book [71] of Kempf, Knudson, Mumford and Saint-Donat. The emphasis was put there on a given complex manifold V, and one looked for partial compactifications of it which were locally analytically isomorphic to embeddings of an algebraic torus into a toric variety. Such partial compactifications $\overline{V}$ were called *toroidal embeddings* of V. Therefore, a toroidal embedding was a pair $(\overline{V}, V)$ such that $(\overline{V}, \overline{V} \setminus V)$ is a toroidal variety in our sense. For more remarks about the toroidal category see [4, Section 1.5].

1.4 Toroidal Pseudo-Resolutions of Plane Curve Singularities

In Sect. 1.4.1 we introduce the notions of *Newton polygon* $\mathcal{N}_{L,L'}(C)$, *tropical function* $\text{trop}^C_{L,L'}$, *Newton fan* $\mathcal{F}^C_{L,L'}$ and *Newton modification* $\psi^C_{L,L'}$ (see Definition 1.4.14) determined by a curve singularity C on the smooth germ of surface (S, o), relative to a cross (L, L'). The strict transform of C by its Newton modification is a finite set of germs. If one completes for each one of them the corresponding germ of exceptional divisor into a cross, one gets again a Newton polygon, a fan and a modification. This produces an *algorithm of toroidal pseudo-resolution* of C (see Algorithm 1.4.22). It leads only to a *pseudo-resolution* morphism, because its source is a possibly singular surface (with toric singularities). In Sect. 1.4.3 we explain how to modify Algorithm 1.4.22 in order to get an algorithm of *embedded resolution* of C. In Sect. 1.4.4 we encode the combinatorics of this algorithm into a *fan tree* (see Definition 1.4.33), which is a rooted tree endowed with a *slope function*, constructed by gluing *trunks* associated with the Newton fans generated by the process. The final Sect. 1.4.5 contains historical information about Newton's and Puiseux's work on plane curve singularities, the resolution of such singularities by iteration of morphisms which are toric in suitable coordinates, and the relations with tropical geometry.

1.4.1 Newton Polygons, Their Tropicalizations, Fans and Modifications

This subsection begins with the definitions of the *Newton polygon* $\mathcal{N}(f)$ (see Definition 1.4.2), the *tropicalization* (see Definition 1.4.4) and the *Newton fan* $\mathcal{F}(f)$ (see Definition 1.4.9) associated with a non-zero germ $f \in \mathbb{C}[[x, y]]$. It turns out that they only depend on the germs L, L', C defined by x, y and f respectively (see Proposition 1.4.13). Therefore, given a cross (L, L') and a plane curve singularity C on the smooth germ (S, o), one has associated Newton polygon, tropicalization and fan. This fan allows to introduce the *Newton modification* of the toroidal germ $(S, L + L')$ determined by C (see Definition 1.4.14).

Assume that a cross (L, L') is fixed on (S, o) (see Definition 1.3.31) and that (x, y) is a local coordinate system defining it. This system allows to see any $f \in \hat{\mathcal{O}}_{S,o}$ as a series in the variables (x, y), that is, in toric terms, as a possibly infinite sum of terms of the form $\boxed{c_m(f)\, \chi^m}$, for $c_m(f) \in \mathbb{C}$ and $m \in \sigma_0^\vee \cap M$, where $\boxed{M} := M_{L,L'}$ and $\boxed{\sigma_0} := \sigma_0^{L,L'}$ (see Definition 1.3.32). Denote also $\boxed{N} := N_{L,L'}$. One has canonical identifications $M \simeq \mathbb{Z}^2$, $N \simeq \mathbb{Z}^2$, $\sigma_0 \simeq (\mathbb{R}_+)^2$, and $\sigma_0^\vee \simeq (\mathbb{R}_+)^2$.

Definition 1.4.1 Let $f \in \mathbb{C}[[x, y]]$ be a nonzero series. The **support** $\mathcal{S}(f) \subseteq \sigma_0^\vee \cap M \simeq \mathbb{N}^2$ of f is the set of exponents of monomials with non-zero coefficients in f. That is, if

$$f = \sum_{m \in \sigma_0^\vee \cap M} c_m(f) \chi^m, \tag{1.31}$$

then $\boxed{\mathcal{S}(f)} := \{m \in \sigma_0^\vee \cap M,\ c_m(f) \neq 0\}$.

If Y is a subset of a real affine space, then $\boxed{\text{Conv}(Y)}$ denotes its **convex hull**.

Definition 1.4.2 Let $f \in \mathbb{C}[[x, y]]$. Its **Newton polygon** $\mathcal{N}(f)$ is the following convex subset of $\sigma_0^{\vee} \simeq (\mathbb{R}_+)^2$:

$$\boxed{\mathcal{N}(f)} := \mathrm{Conv}(\mathcal{S}(f) + (\sigma_0^{\vee} \cap M)).$$

Its **faces** are its vertices, its edges and the whole polygon itself. If K is a compact edge of the boundary $\boxed{\partial\mathcal{N}(f)}$ of $\mathcal{N}(f)$, then the **restriction** $\boxed{f_K}$ **of** f **to** K is the sum of the terms of f whose exponents belong to K.

Remark 1.4.3 In general, the Newton polygon of an element of $\hat{O}_{S,o}$ depends on the choice of local coordinates. For instance, let us consider the change of coordinates $(x, y) = (u, u + v)$. The function $f(x, y) := y^2 - x^3$ becomes $g(u, v) := f(u, u + v) = (u+v)^2 - u^3$. The corresponding Newton polygons are represented in Fig. 1.14. In contrast, if the local coordinate change preserves the coordinate curves, then the Newton polygon remains unchanged (see Proposition 1.4.13 below).

Suppose now that the variables x and y are weighted by non-negative real numbers. Denote by $c \in \mathbb{R}_+$ the weight of x and by $d \in \mathbb{R}_+$ the weight of y. Therefore the pair $w := (c, d)$ may be seen as an element of the weight vector space $N_{\mathbb{R}} = (N_{L,L'})_{\mathbb{R}}$. More precisely, one has $w \in (\mathbb{R}_+)^2 \simeq \sigma_0$. Assuming that the non-zero complex constants have weight 0, we see that the weight $w(c_m(f)\chi^m)$ of a non-zero term of f is simply $w \cdot m \in \mathbb{R}_+$. Define then the w**-weight of the series** $f \in \mathbb{C}[[x, y]]$ as the minimal weight of its terms. One gets the function:

$$\boxed{\nu_w} : \begin{array}{ccc} \mathbb{C}[[x, y]] & \to & \mathbb{R}_+ \cup \{\infty\} \\ f & \to & \min\{w \cdot m,\ m \in \mathcal{S}(f)\} \end{array}. \tag{1.32}$$

It is an exercise to show that ν_w is a valuation on the $\mathbb{C}$-algebra $\mathbb{C}[[x, y]]$, in the sense of Definition 1.2.19.

Instead of fixing w and letting f vary, let us fix now a non-zero series $f \in \mathbb{C}[[x, y]]$. Considering the w-weight of f for every $w \in \sigma_0$ leads to the following function:

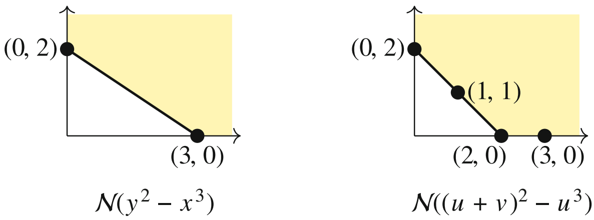


Fig. 1.14 Illustration of Remark 1.4.3

Definition 1.4.4 The **tropicalization** trop^f of $f \in \mathbb{C}[[x, y]] \setminus \{0\}$ is the function:

$$\boxed{\mathrm{trop}^f} : \begin{array}{rcl} \sigma_0 & \to & \mathbb{R}_+ \\ w & \to & \min\{w \cdot m,\ m \in \mathcal{S}(f)\} \end{array}. \tag{1.33}$$

Remark 1.4.5 Let us explain the name of *tropicalization* used in the previous definition (see also Sect. 1.4.5). Consider the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, endowed with the operations $\oplus := \min$ and $\odot := +$. Under both operations, $\overline{\mathbb{R}}$ is a commutative monoid, the product $\odot$ is distributive with respect to addition and the addition $\oplus$ is idempotent, that is, $a \oplus a = a$, for all $a \in \overline{\mathbb{R}}$. One says then that $(\overline{\mathbb{R}}, \oplus, \odot)$ is a *tropical semiring*. Consider now the expression defining trop^f, and compare it with the expansion (1.31) of f as a power series. One sees that one gets formally trop^f from (1.31) by replacing each constant or variable x, y by its weight, and by replacing the usual operations of sum and product by their tropical analogs. For further references see the textbook [84] on *tropical geometry*. Foundations for the tropical study of singularities were written by Stepanov and the third author in the paper [105].

Remark 1.4.6 If A is a subset of a real vector space V, then its *support function* is the function defined on the dual vector space $V^\vee$ and taking values in $\mathbb{R} \cup \{-\infty\}$, which associates to every element of $V^\vee$ seen as a linear form on V, the infimum of its restriction to A. The tropicalization trop^f is the restriction of the support function of the subset $\mathcal{S}(f)$ of the real vector space $M_\mathbb{R}$ to the subset of $M_\mathbb{R}^\vee \simeq N_\mathbb{R}$ on which it does not take the value $-\infty$. The notion of support function is an essential tool in the study of convex polyhedra (see for instance Ewald's book [37]).

For every ray $\rho = \mathbb{R}_+ w$ included in the cone σ_0, consider the following closed half-plane of $M_\mathbb{R}$:

$$\boxed{H_{f,\rho}} := \{m \in M_\mathbb{R},\ w \cdot m \geq \mathrm{trop}^f(w)\}. \tag{1.34}$$

This definition is independent of the choice a generator w of the ray ρ.

The basic reason of the importance of the Newton polygon $\mathcal{N}(f)$ of f in our context is the following strengthening of Proposition 1.2.39:

Proposition 1.4.7 *Let the ray $\rho \subset \sigma_0$ be fixed. Then the closed half-plane $H_{f,\rho}$ of $M_\mathbb{R}$ is a* **supporting half-plane** *of $\mathcal{N}(f)$, in the sense that it contains $\mathcal{N}(f)$ and its boundary $\{m \in M_\mathbb{R},\ w \cdot m = \mathrm{trop}^f(w)\}$ has a non-empty intersection with the boundary $\partial\mathcal{N}(f)$ of $\mathcal{N}(f)$.*

Proof Let w be a generating vector of the ray ρ. The inclusion $\mathcal{N}(f) \subseteq H_{f,\rho}$ is equivalent to the property $w \cdot n \geq \mathrm{trop}^f(w)$, for all $n \in \mathcal{N}(f)$. These inequalities result from Definition 1.4.4 of the tropicalization function $\mathrm{trop}^f(w)$ and from the following basic equality, implied by the hypothesis that $w \in \sigma_0$ (see Proposition 1.2.39):

$$\min\{w \cdot m,\ m \in \mathcal{S}(f)\} = \min\{w \cdot m,\ m \in \mathcal{N}(f)\}.$$

The boundary of the half-plane $H_{f,\rho}$ intersects $\mathcal{N}(f)$ at its points at which the restriction of the linear form $w : M_{\mathbb{R}} \to \mathbb{R}$ to $\mathcal{N}(f)$ achieves its minimum, that is, along its face $\mathcal{N}(f) \cap \{m \in M_{\mathbb{R}},\ w \cdot m = \mathrm{trop}^f(w)\}$. □

As every closed convex subset of a real plane is the intersection of its supporting half-planes, one deduces that the tropicalization trop^f determines the Newton polygon $\mathcal{N}(f)$ in the following way:

$$\mathcal{N}(f) = \{m \in M_{\mathbb{R}},\ w \cdot m \geq \mathrm{trop}^f(w),\ \text{for all } w \in \sigma_0\}. \tag{1.35}$$

Formula (1.35) presents $\mathcal{N}(f)$ as the intersection of an infinite set of closed half-planes. In fact, as a consequence of the previous discussion, a finite number of them suffices:

Proposition 1.4.8 *Let $\mathcal{F}(f)$ be the fan of N obtained by subdividing the cone σ_0 using the rays orthogonal to the compact edges of $\mathcal{N}(f)$. Then:*

1. *The tropicalization* trop^f *is continuous and its restriction to any cone in $\mathcal{F}(f)$ is linear.*
2. *The relative interiors of the cones of $\mathcal{F}(f)$ may be characterized as the levels of the following map from σ_0 to the set of faces of $\mathcal{N}(f)$, in the sense of Definition 1.4.2:*

$$w \to \mathcal{N}(f) \cap \{m \in M_{\mathbb{R}},\ w \cdot m = \mathrm{trop}^f(w)\}.$$

3. *This map realizes an inclusion-reversing bijection between $\mathcal{F}(f)$ and the set of faces of $\mathcal{N}(f)$. If K_σ is the face of $\mathcal{N}(f)$ corresponding to the cone σ of $\mathcal{F}(f)$, then:*

$$\mathrm{trop}^f(w) = w \cdot m,\ \text{for all } w \in \sigma,\ \text{and for all } m \in K_\sigma.$$

4. *The Newton polygon $\mathcal{N}(f)$ is the intersection of the closed half-planes $H_{f,\rho}$ defined by relation (1.34), where ρ varies among the rays of the fan $\mathcal{F}(f)$.*

The fans $\mathcal{F}(f)$ appearing in the previous proposition are particularly important for the sequel, that is why they deserve a name:

Definition 1.4.9 The **Newton fan** $\boxed{\mathcal{F}(f)}$ of $f \in \mathbb{C}[[x, y]] \setminus \{0\}$ is the fan of N obtained by subdividing the cone σ_0 using the rays orthogonal to the compact edges of the Newton polygon $\mathcal{N}(f) \subseteq \sigma_0^\vee$ of f, that is, by the interior normals of the compact edges of $\mathcal{N}(f)$. A **Newton fan** in a weight lattice N and relative to a basis (e_1, e_2) is any fan subdividing the regular cone $\sigma_0 = \mathbb{R}_+\langle e_1, e_2\rangle$.

Example 1.4.10 Consider the series $f \in \mathbb{C}[[x, y]]$ defined by:

$$f(x, y) := -x^{12} + x^{14} + x^7y^2 + 2x^5y^3 - x^{10}y^3 + x^3y^4 + 3x^7y^4 + y^9.$$

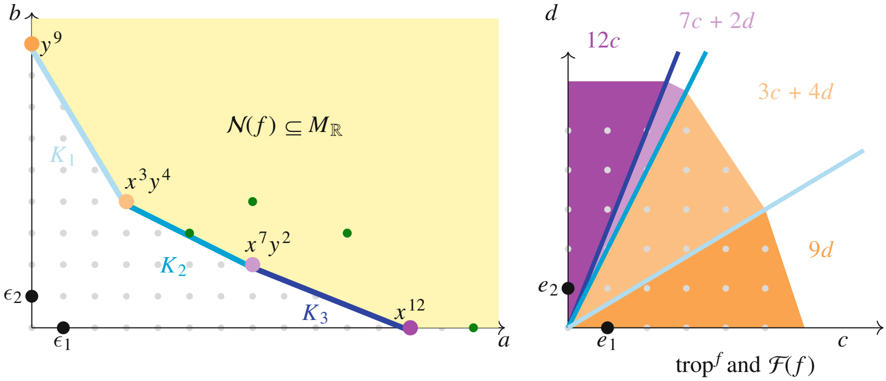


Fig. 1.15 The Newton polygon, the tropicalization and the Newton fan of Example 1.4.10

On the left side of Fig. 1.15 is represented its Newton polygon $\mathcal{N}(f)$, and on the right side are represented its tropicalization trop^f and its Newton fan $\mathcal{F}(f)$. The support of the series f is:

$$\mathcal{S}(f) = \{(12, 0), (14, 0), (7, 2), (5, 3), (10, 3), (3, 4), (7, 4), (0, 9)\}.$$

Among its elements, the vertices of $\mathcal{N}(f)$ are $(12, 0), (7, 2), (3, 4), (0, 9)$. The corresponding monomials are marked on the left of the figure, near the associated vertices. The other elements of $\mathcal{S}(f)$ are marked as green dots. Now, each vertex (a, b) of $\mathcal{N}(f)$ may be seen as the linear function $w = (c, d) \to ac + bd$ on $N_{\mathbb{R}}$. The tropicalization trop^f computes the minimal value of those 4 linear functions at the points of σ_0. The regular cone σ_0 gets decomposed into 4 smaller 2-dimensional subcones, according to the vertex which gives this minimum. On the right side of Fig. 1.15 those subcones are represented in different colors. Each such subcone has the same color as the expression of the associated linear function and the vertex of $\mathcal{N}(f)$ defining it. Each ray separating two successive subcones is orthogonal to a compact edge of $\mathcal{N}(f)$ and both are drawn with the same color. Denoting the compact edges by $K_1 := [(0, 9), (3, 4)]$, $K_2 := [(3, 4), (7, 2)]$, $K_3 := [(7, 2), (12, 0)]$, the associated restrictions of f (see Definition 1.4.2) are:

$$f_{K_1} = x^3y^4 + y^9, \quad f_{K_2} = x^7y^2 + 2x^5y^3 + x^3y^4, \quad \text{and } f_{K_3} = -x^{12} + x^7y^2.$$

The Newton fan of f is $\mathcal{F}(f) = \mathcal{F}(3/5, 2/1, 5/2)$ (see Definition 1.3.4 for this last notation).

If $\alpha \in \mathbb{C}[[t]] \setminus \{0\}$, we denote by $c_{\nu_t(\alpha)}(\alpha)$ the coefficient of $t^{\nu_t(\alpha)}$ in the series α, and we call it the **leading coefficient** of α.

The following proposition shows why it is important to introduce trop^f when studying the germ C defined by f:

Proposition 1.4.11 *Let $f \in \mathbb{C}[[x, y]]$ be a non-zero series. Let $t \to (\alpha(t), \beta(t))$ be a germ of formal morphism from $(\mathbb{C}, 0)$ to $(\mathbb{C}^2, 0)$, whose image is not contained in the union $L \cup L'$ of the coordinate axes. Then one has the inequality:*

$$\nu_t(f(\alpha(t), \beta(t))) \geq \operatorname{trop}^f(\nu_t(\alpha), \nu_t(\beta)),$$

with equality if and only if $f_K(c_{\nu_t(\alpha)}(\alpha), c_{\nu_t(\beta)}(\beta)) \neq 0$, where K is the compact face of $\mathcal{N}(f)$ orthogonal to $(\nu_t(\alpha), \nu_t(\beta)) \in N$, in the sense that its restriction to $\mathcal{N}(f)$ achieves its minimum on this face.

Proof The basic idea of the proof goes back to Newton's method of computing the leading term of a Newton-Puiseux series $\eta(x)$ such that $f(x, \eta(x)) = 0$, which we explained on the example of Sect. 1.2.5, starting from Eq. (1.12).

The hypothesis that the image of $t \to (\alpha(t), \beta(t))$ is not contained in the union of coordinate axes means that both α and β are non-zero series. Therefore, they admit non-vanishing leading coefficients $c_{\nu_t(\alpha)}(\alpha)$ and $c_{\nu_t(\beta)}(\beta)$ (see Definition 1.2.18).

Using the expansion (1.31), we get that $f(\alpha(t), \beta(t))$ is equal to:

$$\begin{aligned}
&\sum_{(a,b)\in\mathcal{S}(f)} c_{(a,b)}(f) \left(c_{\nu_t(\alpha)}(\alpha)t^{\nu_t(\alpha)} + o(t^{\nu_t(\alpha)})\right)^a \left(c_{\nu_t(\beta)}(\beta)t^{\nu_t(\beta)} + o(t^{\nu_t(\beta)})\right)^b = \\
&= \sum_{(a,b)\in\mathcal{S}(f)} c_{(a,b)}(f) \left(c_{\nu_t(\alpha)}(\alpha)\right)^a \left(c_{\nu_t(\beta)}(\beta)\right)^b \left(t^{a\nu_t(\alpha)+b\nu_t(\beta)} + o(t^{a\nu_t(\alpha)+b\nu_t(\beta)})\right).
\end{aligned} \tag{1.36}$$

As a consequence:

$$\nu_t\left(f(\alpha(t), \beta(t))\right) \geq \min_{(a,b)\in\mathcal{S}(f)} \{a\nu_t(\alpha) + b\nu_t(\beta)\} = \operatorname{trop}^f(\nu_t(\alpha), \nu_t(\beta)),$$

where the last equality follows from Definition 1.4.4. This proves the inequality stated in the proposition.

The case of equality follows from the fact, implied by the computation (1.36), that the coefficient of the term with exponent $\operatorname{trop}^f(\nu_t(\alpha), \nu_t(\beta))$ of the series $f(\alpha(t), \beta(t))$ is $f_K(c_{\nu_t(\alpha)}(\alpha), c_{\nu_t(\beta)}(\beta))$. $\square$

In Proposition 1.4.11, K may be either an edge or a vertex of $\mathcal{N}(f)$. Note that this statement plays with the two dual ways of defining a curve singularity on $(\mathbb{C}^2, 0)$, either as the vanishing locus of a function or by a parametrization.

Consider now the reduced image of the morphism $t \to (\alpha(t), \beta(t))$. The hypothesis that it is not contained in $L \cup L'$ shows that it is a branch on (S, o), different from L and L'. Endow it with a multiplicity equal to the degree of the morphism onto its image, seeing it therefore as a divisor A on (S, o). By Proposition 1.2.8, the orders $\nu_t(\alpha(t))$, $\nu_t(\beta(t))$ which appear in Proposition 1.4.11 may be interpreted as $\nu_t(\alpha(t)) = L \cdot A$, and $\nu_t(\beta(t)) = L' \cdot A$. We get the following corollary of Proposition 1.4.11:

Proposition 1.4.12 *Let (L, L') be a cross on (S, o) and C be a curve singularity on (S, o). Assume that the local coordinate system (x, y) defines the cross (L, L') and that $f \in \hat{O}_{S,o}$ defines C. Then, for every effective divisor A on (S, o) supported on a branch distinct from L and L', one has the inequality:*

$$C \cdot A \geq \text{trop}^f((L \cdot A)e_1 + (L' \cdot A)e_2).$$

Moreover, one has equality when A is generic for fixed values of $L \cdot A$ and $L' \cdot A$.

One may describe the genericity condition involved in the last sentence of Proposition 1.4.12 as follows. As a consequence of the proof of Proposition 1.4.18 below, one has $f_K(c_{\nu_t(\alpha)}(\alpha), c_{\nu_t(\beta)}(\beta)) \neq 0$ (which is equivalent to the equality $C \cdot A = \text{trop}^f((L \cdot A)e_1 + (L' \cdot A)e_2)$) if and only if the strict transforms of A and C by the Newton modification $\psi^C_{L,L'}$ of S defined by C (see Definition 1.4.14 below) are disjoint.

As a consequence of Propositions 1.4.8 (3) and 1.4.12 we have:

Proposition 1.4.13 *Let (L, L') be a cross on (S, o) and C be a curve singularity on (S, o). Assume that the local coordinate system (x, y) defines the cross (L, L') and that $f \in \hat{O}_{S,o}$ defines C. Then the Newton polygon $\mathcal{N}(f)$, the tropicalization* trop^f *and the Newton fan $\mathcal{F}(f)$ do not depend on the choice of the defining functions x, y, f of the curve germs L, L', C.*

By contrast, the support of f depends on the choice of the local coordinate system (x, y) defining a fixed cross, even if $f \in \hat{O}_{S,o}$ is fixed. For instance, the monomial xy becomes a series with infinite support if one replaces simply x by $x(1 + x + x^2 + \cdots)$.

Proposition 1.4.13 implies that the following notions are well-defined:

Definition 1.4.14 Let (L, L') be a cross on (S, o), and let (x, y) be a local coordinate system defining it. Let C be a curve singularity on (S, o), defined by a function $f \in \hat{O}_{S,o}$, seen as a series in $\mathbb{C}[[x, y]]$ using the coordinate system (x, y). Then:

- The **Newton polygon** $\boxed{\mathcal{N}_{L,L'}(C)} \subseteq M_{L,L'}$ **of** C **relative to the cross** (L, L') is the Newton polygon $\mathcal{N}(f)$.
- The **tropical function** $\boxed{\text{trop}^C_{L,L'}} : \sigma_0 \to \mathbb{R}_+$ of C **relative to the cross** (L, L') is the tropicalization trop^f of the series f.
- The **Newton fan** $\boxed{\mathcal{F}_{L,L'}(C)}$ **of** C **relative to the cross** (L, L') is the fan $\mathcal{F}(f)$.
- The **Newton modification** $\boxed{\psi^C_{L,L'}} : (S_{\mathcal{F}_{L,L'}(C)}, \partial S_{\mathcal{F}_{L,L'}(C)}) \to (S, L + L')$ **of** S **defined by** C **relative to the cross** (L, L') is the modification of S associated with $\mathcal{F}_{L,L'}(C)$ relative to the cross (L, L'), that is, $\psi^C_{L,L'} := \psi^{\mathcal{F}_{L,L'}(C)}_{L,L'}$ (see Definition 1.3.33). The strict transform of C by $\psi^C_{L,L'}$ is denoted $\boxed{C_{L,L'}}$.

Note that we consider the Newton modification $\psi^C_{L,L'}$ as a morphism in the toroidal category, by endowing S with the boundary $L + L'$ and the modified surface $S_{\mathcal{F}_{L,L'}(C)}$ with a boundary equal to the reduced total transform of $L + L'$.

1.4.2 An Algorithm of Toroidal Pseudo-Resolution

In this subsection we assume for simplicity that the plane curve singularity C is reduced (see Remark 1.4.27). We explain that, once a smooth branch L is fixed on the germ of smooth surface (S, o), one may obtain a so-called *toroidal pseudo-resolution* of C on (S, o) (see Definition 1.4.15) by completing the smooth branch into a cross (L, L'), by performing the associated Newton modification, and by iterating these steps at every point at which the strict transform of C intersects the exceptional divisor of the Newton modification (see Theorem 1.4.23). The algorithm stops after the first step if and only if C is *Newton non-degenerate* relative to the cross (L, L') (see Definition 1.4.21).

The following definition formulates two notions of possibly partial *resolution of C in the toroidal category*, relative to the ambient smooth germ of surface S:

Definition 1.4.15 Let (L, L') be a cross in the sense of Definition 1.3.31 on the smooth germ of surface (S, o) and let C be a curve singularity on S. Consider a modification $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ of $(S, L + L')$ in the toroidal category, in the sense of Definition 1.3.29. It is called, in decreasing generality:

- A **toroidal pseudo-resolution of** C if the following conditions are satisfied:
  1. the boundary $\partial\Sigma$ of Σ contains the reduction of the total transform $\pi^*(C)$ of C by π;
  2. the strict transform of C by π (see Definition 1.2.31) does not contain singular points of Σ.
- A **toroidal embedded resolution of** C if, moreover, the surface Σ is smooth.

  If $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ is a toroidal pseudo-resolution of C, then the reduction of the image $\pi(\partial\Sigma)$ of $\partial\Sigma$ in S is called the **completion** $\boxed{\hat{C}_\pi}$ **of** C **relative to** π.

Remark 1.4.16 Note that if $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ is a toroidal pseudo-resolution of C, then the strict transform of C by π is smooth and $\hat{C}_\pi \supseteq C + L + L'$. If moreover π is an embedded resolution, then the total transform $\pi^*(C)$ is a normal crossings divisor in Σ (see Definition 1.2.32). Note also that if $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ is a toroidal embedded resolution of C, then $\pi : \Sigma \to S$ is an embedded resolution of C according to Definition 1.2.33. From now on, we will keep track carefully of the toroidal structures, considering only toroidal embedded resolutions in the sense of Definition 1.4.15.

Remark 1.4.17 If $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ is a toroidal pseudo-resolution of C, then the strict transform of C is transversal to the critical locus of π. Our choice of terminology in Definition 1.4.15 is inspired by Goldin and Teissier's paper [51], where an analogous notion of (embedded) *toric pseudo-resolution* of a subvariety of the affine space is considered.

Let us look now at the strict transform $C_{L,L'}$ of C by the Newton modification $\psi^C_{L,L'}$ defined by C relative to the cross (L, L') (see Definition 1.4.14). The following proposition describes its intersection with the boundary $\partial S_{\mathcal{F}_{L,L'}(C)}$:

Proposition 1.4.18 *Assume that neither L nor L' is a branch of C. Then the strict transform $C_{L,L'}$ of C by the Newton modification $\psi^C_{L,L'}$ intersects the boundary $\partial S_{\mathcal{F}_{L,L'}(C)}$ of the toroidal surface $(S_{\mathcal{F}_{L,L'}(C)}, \partial S_{\mathcal{F}_{L,L'}(C)})$ only at smooth points of it. Moreover, if ρ is a ray of the Newton fan $\mathcal{F}_{L,L'}(C)$ different from the edges of σ_0, then $C_{L,L'}$ intersects the corresponding component $\overline{O_\rho}$ of the exceptional divisor of $\psi^C_{L,L'}$ only inside the orbit O_ρ. The number of intersection points counted with multiplicities is equal to the integral length of the edge of the Newton polygon $\mathcal{N}_{L,L'}(C)$ which is orthogonal to the ray ρ.*

Proof We give a detailed proof of this proposition in geometric language, in order to emphasize the roles played by the fundamental combinatorial objects $\mathcal{N}_{L,L'}(C)$, $\text{trop}^C_{L,L'}$ and $\mathcal{F}_{L,L'}(C)$ associated with C relative to the cross (L, L') (see Definition 1.4.14).

The orbit O_ρ is independent of the toric surface containing it, because any two such surfaces contain the affine toric surface $X_\rho \supset O_\rho$ as Zariski open sets. Therefore, in order to compute the intersection of the strict transform of C with O_ρ, we may choose another surface than $X_{\mathcal{F}_{L,L'}(C)}$.

Choose local coordinates (x, y) defining the cross (L, L'). In this way $M_{L,L'}$ gets identified with the lattice of exponents of Laurent monomials in (x, y). Assume that $f_1 := \alpha e_1 + \beta e_2$ is the unique primitive generator of the ray ρ. Let us complete it in a basis (f_1, f_2) of the lattice $N_{L,L'}$, such that the cone $\sigma := \mathbb{R}_+\langle f_1, f_2\rangle$ is contained in one of the two cones of dimension 2 of $\mathcal{F}_{L,L'}(C)$ adjacent to ρ. We are now in the setting of Example 1.3.26. As explained there, if (φ_1, φ_2) is the basis of $M_{L,L'}$ dual to the basis (f_1, f_2) of $N_{L,L'}$ and $u := \chi^{\varphi_1}$, $v := \chi^{\varphi_2}$, then $v = x^{-\beta} y^{\alpha}$ is a coordinate of the orbit O_ρ. Moreover, it realises an isomorphism of its closure in the affine toric surface $X_\sigma = \mathbb{C}^2_{u,v}$ with the affine line $\mathbb{C}_v$.

Let K_ρ be the edge of the Newton polygon $\mathcal{N}_{L,L'}(C)$ which is orthogonal to the ray ρ. It is parallel to the line $\mathbb{R}\varphi_2$, because by definition $f_1 \cdot \varphi_2 = 0$. Orient K_ρ by the vector φ_2 and denote then its vertices by m_0 and m_1, such that K_ρ is oriented from m_0 to m_1. This means that $m_1 - m_0 = L_\rho\, \varphi_2$, where L_ρ denotes the integral length of the segment K_ρ, in the sense of Definition 1.3.1. Moreover, the points of $K_\rho \cap M$ are precisely those of the form:

$$m := m_0 + k\, \varphi_2, \text{ for } k \in \{0, 1, \dots, L_\rho\}. \tag{1.37}$$

Consider the smooth toric surface $X_\sigma = \mathbb{C}^2_{u,v}$. The orbit O_ρ is its pointed v-axis $\mathbb{C}^*_v$. Therefore, one may compute the intersection of the strict transform of C with this orbit by taking the lift $(\psi^\sigma_{\sigma_0})^* f$ of a defining function f of C to $\mathbb{C}^2_{u,v}$, by simplifying by the greatest monomial in $\sigma^\vee \cap M$ which divides it, and then by setting $u = 0$. Let therefore

$$f := \sum_{m \in \mathcal{S}(f)} c_m(f) \chi^m \in \mathbb{C}[[x, y]]$$

be a defining function of C. As the bases (f_1, f_2) and (φ_1, φ_2) are dual of each other, we have the relation $m = (f_1 \cdot m)\varphi_1 + (f_2 \cdot m)\varphi_2$. This implies that $\chi^m = u^{f_1 \cdot m}\, v^{f_2 \cdot m}$. As the ray $\rho = \mathbb{R}_+ f_1$ is orthogonal to the edge K_ρ of the Newton polygon $\mathcal{N}_{L,L'}(C) = \mathcal{N}(f)$, we know that:

$$f_1 \cdot m \geq h_\rho \ \text{ for all } \ m \in \mathcal{S}(f),$$

where $h_\rho := \mathrm{trop}^f(f_1)$, with equality if and only if $m \in K_\rho$. Therefore, the highest power of u which divides

$$(\psi^\sigma_{\sigma_0})^* f = \sum_{m \in \mathcal{S}(f)} c_m(f)\, u^{f_1 \cdot m}\, v^{f_2 \cdot m}$$

is u^{h_ρ}, and it is achieved only on the edge K_ρ of $\mathcal{N}(f)$. Moreover, the linear form $m \to f_2 \cdot m$ achieves its minimum (at least) at the vertex m_0 of $\mathcal{N}(f)$, by the hypothesis that σ is contained in one of the two 2-dimensional cones of $\mathcal{F}(f) = \mathcal{F}_{L,L'}(C)$ which are adjacent to ρ. This shows that the maximal monomial in (u, v) which divides $(\psi^\sigma_{\sigma_0})^* f$ is $u^{h_\rho} v^{f_2 \cdot m_0}$. After simplifying by it and setting $u = 0$, one gets the following polynomial equation in the variable v, describing the intersection of the strict transform of C with the v-axis:

$$\sum_{m \in K_\rho \cap M} c_m(f)\, v^{f_2 \cdot (m - m_0)} = 0. \tag{1.38}$$

We recognize here the equation obtained from $f_{K_\rho} = 0$ after the change of variables from (x, y) to (u, v) and the simplification of the highest dividing monomial. This illustrates the importance in our context of the operation of restriction of f to a compact edge of its Newton polygon, introduced in Definition 1.4.2. Using Eq. (1.37), we see that Eq. (1.38) becomes:

$$\sum_{k=0}^{L_\rho} c_{m_0 + k\,\varphi_2}(f)\, v^k = 0. \tag{1.39}$$

The two extreme coefficients $c_{m_0}(f)$ and $c_{m_1}(f)$ of the previous polynomial equation being non-zero, we see that the strict transform of C does not pass through the origin of $\mathbb{C}^2_{u,v}$ and that it intersects the orbit O_ρ in $L_\rho = l_{\mathbb{Z}} K_\rho$ points, counted with multiplicities. The solutions of Eq. (1.39) are the v-coordinates of the intersection points of the strict transform of C with the orbit O_ρ.

By using the same kind of argument for all the cones of the regularization of $\mathcal{F}_{L,L'}(C)$, we may show also that the strict transform of C misses all the singular points of the boundary divisor of $X_{\mathcal{F}_{L,L'}(C)}$. □

Example 1.4.19 Let us give an example of the objects manipulated in the proof of Proposition 1.4.18. Consider the function $f \in \mathbb{C}[[x, y]]$ of Example 1.4.10. Let ρ be the ray of slope $2/1$ of $\mathcal{F}(f)$. Then K_ρ is the side $K_2 := [(3, 4), (7, 2)]$ of $\partial\mathcal{N}(f)$ (see Fig. 1.15). One has $f_1 = e_1 + 2e_2$. A possible choice of the vector φ_2 is $\varphi_2 = -2\epsilon_1 + \epsilon_2$. Therefore $v = x^{-2}y$. Orienting K_ρ by this vector φ_2 one gets $m_0 = (7, 2)$ and $m_1 = (3, 4)$. We saw in Example 1.4.10 that $f_{K_\rho} = x^7y^2 + 2x^5y^3 + x^3y^4 = x^3y^2(x^4 + 2x^2y + y^2)$. As $v = x^{-2}y$, Eq. (1.39) is in this case $1 + 2v + v^2 = 0$. We see that its degree is indeed the integral length L_ρ of the side K_ρ. As it has a double root, the series f is not Newton non-degenerate (see Definition 1.4.21 below). The strict transform of C intersects O_ρ at the single point $v = -1$.

The proof of Proposition 1.4.18 yields easily also a proof of the following proposition :

Proposition 1.4.20 *Let (L, L') be a cross and C a curve singularity on S. Let $f \in \mathbb{C}[[x, y]]$ be a defining function of C relative to any coordinate system (x, y) defining the chosen cross. Then the following conditions are equivalent:*

1. *the curve C is reduced and the Newton modification $\psi^C_{L,L'}$ becomes a toroidal pseudo-resolution of C if one replaces the boundary $\partial S_{\mathcal{F}_{L,L'}(C)}$ by the total transform of the divisor $(\psi^C_{L,L'})^*(C + L + L')$;*
2. *for any ray ρ of the Newton fan $\mathcal{F}_{L,L'}(C)$ which is orthogonal to a compact edge of $\mathcal{N}_{L,L'}(C)$, the polynomial equation (1.39) has only simple roots;*
3. *the defining function f of C has the property that all the restrictions f_K of f to the compact edges K of the Newton polygon $\mathcal{N}(f) = \mathcal{N}_{L,L'}(C)$ define smooth curves in the torus $(\mathbb{C}^*)^2_{x,y}$.*

The plane curve singularities which satisfy the equivalent conditions of Proposition 1.4.20 received a special name:

Definition 1.4.21 Let (L, L') be a cross and C a curve singularity on S. Let $f \in \mathbb{C}[[x, y]]$ be a defining function of C relative to any coordinate system associated to the chosen cross. The function f is called **Newton non-degenerate** and the curve C is called **Newton non-degenerate relative to the cross** (L, L') if the equivalent conditions listed in Proposition 1.4.20 are satisfied.

Usually one speaks about Newton non-degenerate germs of holomorphic functions of several variables. We introduce here the notion of *Newton non-degenerate plane curve singularity relative to a cross* in order to emphasize the underlying geometric phenomena.

Let us come back to Proposition 1.4.18. At each point of intersection o_i of the strict transform $C_{L,L'}$ with the exceptional divisor of $\psi^C_{L,L'}$, one has the following dichotomy:

- Either only one branch of $C_{L,L'}$ passes through o_i, where it is moreover smooth and transversal to the exceptional divisor. The germ A_i at o_i of the exceptional divisor and this branch form a canonical cross on $S_{\mathcal{F}_{L,L'}(C)}$. Then, one reaches locally a toroidal pseudo-resolution of C in the neighborhood of that point.
- Or one does not have a canonical cross, but only a canonical smooth branch: the germ A_i at o_i of the exceptional divisor $(\psi^C_{L,L'})^{-1}(o)$ itself.

In the second case, one may complete A_i into a cross (A_i, L_i) by the choice of a germ L_i of smooth branch transversal to it. Then one is again in the presence of a germ of effective divisor (the germ of the strict transform $C_{L,L'}$ of C by $\psi^C_{L,L'}$) on a germ of smooth surface endowed with a cross (the surface $S_{\mathcal{F}_{L,L'}(C)}$ endowed with the cross (A_i, L_i)). One gets again a Newton polygon, a tropical function, a Newton fan and a Newton modification, and the previous construction may be iterated. This iterative process may be formulated as the following *algorithm of toroidal pseudo-resolution* of the germ C:

Algorithm 1.4.22 Let (S, o) be a smooth germ of surface, L a smooth branch on (S, o) and C a reduced germ of curve on (S, o), which does not contain the branch L in its support.

STEP 1. If (L, C) is a cross, then STOP.
STEP 2. Choose a smooth branch L' on (S, o), possibly included in C, such that (L, L') is a cross.
STEP 3. Let $\mathcal{F}_{L,L'}(C)$ be the Newton fan of C relative to the cross (L, L'). Consider the associated Newton modification $\psi^C_{L,L'} : (S_{\mathcal{F}_{L,L'}(C)}, \partial S_{\mathcal{F}_{L,L'}(C)}) \to (S, L + L')$ and the strict transform $C_{L,L'}$ of C by $\psi^C_{L,L'}$ (see Definition 1.4.14).
STEP 4. For each point $\tilde{o}$ belonging to $C_{L,L'} \cap \partial S_{\mathcal{F}_{L,L'}(C)}$, denote:

- $L :=$ the germ of $\partial S_{\mathcal{F}_{L,L'}(C)}$ at $\tilde{o}$;
- $C :=$ the germ of $C_{L,L'}$ at $\tilde{o}$;
- $o := \tilde{o}$;
- $S := S_{\mathcal{F}_{L,L'}(C)}$.

STEP 5. GO TO STEP 1.

Note that one considers that only the smooth branch L is given at the beginning, and that the second branch L' of the cross (L, L') is chosen when one executes STEP 2 for the first time. Note also that the algorithm is non-deterministic, as it involves choices of supplementary branches.

A variant of this algorithm, obtained by replacing Step 3 by a Step 3^{reg}, will be studied in Sect. 1.4.3. It produces a *toroidal embedded resolution* of C instead of a *pseudo-resolution* (see Definition 1.4.15).

Proposition 1.4.20 means that if C is Newton non-degenerate relative to the cross (L, L') chosen at Step 2 of Algorithm 1.4.22, then this algorithm stops after performing only one Newton modification. More generally, a fundamental property of Algorithm 1.4.22 is:

Theorem 1.4.23 *Algorithm 1.4.22 stops after a finite number of iterations.*

Proof Assume that A is a curve singularity on the smooth germ of surface (S, o), obtained after a finite number of steps of the algorithm, and that $(L \cdot A)_o = 1$. Then (L, A) is a cross and the algorithm stops. Therefore, in order to show that the algorithm stops, it is enough to show that after a finite number of steps all the local intersection numbers of the strict transform $C_{L,L'}$ of C with the exceptional divisor are equal to 1.

By the end statement of Proposition 1.4.18, a sequence of such intersection numbers at *infinitely near points of* o (see Definition 1.4.31) which dominate each other is necessarily decreasing:

$$(C \cdot L)_o \geq (C_1 \cdot E_1)_{o_1} \geq \cdots \geq (C_k \cdot E_k)_{o_k} \geq \cdots . \tag{1.40}$$

At the k-th iteration of the algorithm we are considering the strict transform C_k of C at a point o_k, which belongs to the component E_k of the exceptional divisor.

The sequence (1.40) being composed of positive integers, it necessarily stabilizes. If the stable value is 1 for all choices of sequence $o, o_1, o_2, \ldots,$ then the algorithm stops after a finite number of steps.

Let us reason by contradiction, assuming the contrary. Therefore, one may find a sequence as before for which the stable intersection number is $n > 1$. Let us assume without loss of generality, by starting our analysis after the stabilization took place, that:

$$(C \cdot L)_o = (C_1 \cdot E_1)_{o_1} = \cdots = (C_k \cdot E_k)_{o_k} = \cdots = n > 1. \tag{1.41}$$

Therefore, for every $k \geq 1$, (E_k, C_k) is not a cross at o_k. By STEP 2 of the algorithm, a smooth germ L_k was chosen at o_k such that (E_k, L_k) is a cross at o_k.

Let us reformulate the first equality

$$(C_1 \cdot E_1)_{o_1} = (C \cdot L)_o \tag{1.42}$$

of the sequence (1.41) in terms of Newton polygons. By applying again the end statement of Proposition 1.4.18, we see that $(C_1 \cdot E_1)_{o_1}$ is less or equal to the integral length $l_{\mathbb{Z}} K$ of the compact edge K of $\mathcal{N}_{L,L'}(C)$ whose orthogonal ray corresponds to the prime exceptional curve E_1. One has equality if and only if the strict transform of C intersects E_1 at a single point. In turn, the integral length $l_{\mathbb{Z}} K$ is less or equal to the height $(C \cdot L)_o = n$ of $\mathcal{N}_{L,L'}(C)$ (the ordinate of its lowest point on the vertical axis), with equality if and only if K is the only compact edge of $\mathcal{N}_{L,L'}(C)$ and $K = [(0, n), (m_1 n, 0)]$, with $m_1 \in \mathbb{N}^*$.

As a consequence, *one has the equality (1.42) if and only if $\mathcal{N}_{L,L'}(C)$ has a single compact edge, of the form $[(0, n), (m_1 n, 0)]$, with $m_1 \in \mathbb{N}^*$, and the associated polynomial in one variable has only one root in $\mathbb{C}^*$*. In terms of local coordinates (x, y) on (S, o) defining the cross (L, L') and a defining unitary polynomial $f \in \mathbb{C}[[x]][y]$ of the plane curve singularity C (see Theorem 1.6.1 below), equality holds in (1.42) if and only if f is of the form $f = (y - c_1 x^{m_1})^n + \cdots$, with $c_1 \in \mathbb{C}^*$, $m_1 \in \mathbb{N}^*$ and where we wrote only the restriction f_K of f to the compact edge K of the Newton polygon $\mathcal{N}_{L,L'}(C)$, in the sense of Definition 1.4.2. Then, STEP 3 is performed simply by considering the morphism:

$$\begin{cases} x = x_1, \\ y = x_1^{m_1} (w_1 + c_1), \end{cases} \tag{1.43}$$

where (x_1, w_1) are local coordinates at o_1 and $Z(x_1) = (E_1, o_1)$. The hypothesis (1.41) implies that (E_1, C_1) is not a cross. Denote by L_1' the smooth branch at o_1 obtained by applying again STEP 2. Therefore, (E_1, L_1') is a cross at o_1. By the formal version of the implicit function theorem, we can choose local coordinates (x_1, u_1) defining the cross (E_1, L_1') in such a way that $u_1 = w_1 - \phi_1(x_1)$, for some $\phi_1 \in \mathbb{C}[[t]]$ with $\phi_1(0) = 0$.

Let us define $y_1 := y - x^{m_1}(c_1 + \phi_1(x))$ and denote $L_1 := Z(y_1)$. Notice that the strict transform of L_1 by the modification (1.43) is equal to L_1' and that (1.43) can be rewritten

$$\begin{cases} x \ = x_1, \\ y_1 = x_1^{m_1} u_1 \end{cases} \tag{1.44}$$

with respect to the local coordinates (x, y_1) and (x_1, u_1). Let us denote by $f_1 \in \mathbb{C}[[x_1]][u_1]$ the monic polynomial defining C_1 relative to the coordinates (x_1, u_1) (see again Theorem 1.6.1). Reasoning as before, the hypothesis (1.41) implies that the polynomial f_1 is of the form $f_1 = (u_1 - c_2 x_1^{m_2})^n + \cdots$, where $c_2 \in \mathbb{C}^*$, $m_2 \in \mathbb{N}^*$ and the exponents of the monomials $x_1^i u_1^j$ which were omitted verify that $i + m_2 j > m_2 n$ and $0 \leq j < n$. Notice that the order of vanishing of f along E_1 is equal to $n m_1$. We recover a defining function of C with respect to the coordinates (x, y_1) by expressing, using the relation (1.44), the monomials appearing in the product $x_1^{m_1 n} \cdot f_1(x_1, u_1)$ as monomials in (x, y_1). We get a defining function of C of the form $(y_1 - c_2 x_1^{m_1+m_2})^n + \cdots$, where the exponents of the monomials $x_1^i y_1^j$ which are not written above verify that $i + (m_1 + m_2) j > (m_1 + m_2) n$ and $0 \leq j < n$.

By induction on $k \geq 1$, one may show similarly that:

- The branch $L_k' = Z(u_k)$ is the strict transform of a smooth branch $L_k = Z(y_k)$ at S, where (x, y_k) is a local coordinate system defining a cross at o and

$$y_k = y_{k-1} - x^{m_1 + \cdots + m_k}(c_k + \phi_k(x)), \tag{1.45}$$

where $\phi_k \in \mathbb{C}[[t]]$ satisfies $\phi_k(0) = 0$.

- The composition of the maps in the algorithm expresses as

$$\begin{cases} x &= x_1, \\ y_k &= x_1^{m_1+\cdots+m_k}\ u_k, \end{cases} \tag{1.46}$$

  with respect to the local coordinates (x_k, u_k) at o_k and the coordinates (x, y_k) at o.
- There exists a defining function of C of the form:

$$(y_k - c_k x^{m_1+\cdots+m_k})^n + \cdots$$

  where the exponents of monomials $x^i y_k^j$ which are not written above verify that $i + (m_1 + \cdots + m_k)j > (m_1 + \cdots + m_k)n$ and $0 \leq j < n$.

In particular, we have shown that the Newton polygon $\mathcal{N}_{L,L_k}(C)$ has only one compact edge with vertices $(0, n)$ and $(m_1 + \cdots + m_k, 0)$, where $L_k \cdot C = m_1 + \cdots + m_k$. When we look at the polygons $\mathcal{N}_{L,L_k}(C)$ as subsets of $\mathbb{R}^2$, we get a nested sequence:

$$\mathcal{N}_{L,L'}(C) \supset \mathcal{N}_{L,L_1}(C) \supset \cdots \supset \mathcal{N}_{L,L_{k-1}}(C) \supset \mathcal{N}_{L,L_k}(C). \tag{1.47}$$

By (1.45), one has that $y_k = y - \xi_k(x)$ with $\xi_k(x) \in \mathbb{C}[[x]]$. One may check, using the shape of relation (1.45), that the sequence $(\xi_k(x))_{k\geq 1}$ converges to a series $\xi_\infty(x)$ in the complete ring $\mathbb{C}[[x]]$. Set $y_\infty := y - \xi_\infty(x)$ and $L_\infty := Z(y_\infty)$. Then (L, L_∞) is a cross at o. We deduce that $L_\infty \cdot C = \nu_x f(x, \xi_\infty(x)) = +\infty$ and by (1.47) one gets the inclusion $\mathcal{N}_{L,L_\infty}(C) \subset \mathcal{N}_{L,L_k}(C)$, for every $k \geq 1$. These two facts together imply that the Newton polygon $\mathcal{N}_{L,L_\infty}(C)$ has only one vertex $(0, n)$. Therefore, a local defining series for C is of the form $(y_\infty)^n$. Since $n > 1$, C would not be a reduced germ, contrary to the hypothesis. □

Remark 1.4.24 The argument used in the proof of Theorem 1.4.23 coincides basically with one step of the proof of the Newton-Puiseux theorem (see Theorems 1.2.20 and 1.6.1), as presented in Teissier's survey [123]. Unlike the rest of the proof of this theorem, this particular step holds without making any assumption on the characteristic of the base field.

Algorithm 1.4.22 involves a finite number of choices, those of the smooth branches introduced in order to get crosses each time one executes STEP 2. Let us introduce the following notations:

Notations 1.4.25 Assume that one executes Algorithm 1.4.22 on (S, o), starting from the curve singularity C and the smooth branch L. Then:

1. $\{\boxed{o_i}, i \in \boxed{I}\}$ is the set of points at which one applies STEP 1 or STEP 2. One assumes that $\{1\} \subseteq I$ and that $o_1 := o$.
2. $\{\boxed{(A_i, B_i)}, i \in I\}$ is the set of crosses considered each time one applies STEP 1 or STEP 2. Therefore $A_1 = L$ and for $i \in I \setminus \{1\}$, the branch A_i is included

in the exceptional divisor of the Newton modification performed at the previous iteration.

3. $\boxed{J} \subseteq I$ consists of those $j \in I$ for which one performs STEP 2 at o_j. Denote by $\boxed{L_j}$ the projection on S of the branch B_j, for every $j \in J$. Therefore, B_i is a strict transform of a branch of C whenever $i \in I \setminus J$ and B_j is the strict transform of L_j whenever $j \in J$.
4. $\boxed{S^{(1)}} := S$. For $k \geq 1$, the surface $S^{(k+1)}$ is obtained from $\boxed{S^{(k)}}$ by performing simultaneously the Newton modification of STEP 3 at all the points o_j of $S^{(k)}$ at which one executes STEP 2. At such a point, denote by $\boxed{\mathcal{F}_{A_j, B_j}(C)}$ the corresponding fan. It is the Newton fan of the germ of strict transform of C at o_j, relative to the cross (A_j, B_j).
5. The previous simultaneous Newton modification is denoted $\boxed{\pi^{(k)}} : S^{(k+1)} \to S^{(k)}$. We call it the **$k$-th level of Newton modifications**.
6. The toroidal boundary $\boxed{\partial S^{(k)}}$ is by definition the total transform on $S^{(k)}$ of all the crosses which appeared in the algorithm until performing STEP 2 at all the points of $S^{(k)}$. In particular, $\partial S = L + L_1$. Each morphism $\pi^{(k)} : (S^{(k+1)}, \partial S^{(k+1)}) \to (S^{(k)}, \partial S^{(k)})$ belongs to the toroidal category, as $(\pi^{(k)})^{-1}(\partial S^{(k)}) \subseteq \partial S^{(k+1)}$.
7. $\boxed{\pi} := \pi^{(1)} \circ \cdots \circ \pi^{(h)}$, where $\boxed{h}$ is the number of modifications $\pi^{(k)}$ produced by the algorithm. We denote by $\boxed{\Sigma}$ the source of π. Therefore, $\pi : \Sigma \to S$ is a modification of the initial germ S.
8. $\boxed{\partial \Sigma}$ denotes $\partial S^{(h)}$. It is the underlying reduced divisor of the total transform $\pi^*(\hat{C}_\pi)$ of the completion $\hat{C}_\pi = C + \sum_{j \in J} L_j$, in the sense of Definition 1.4.15.

There are a lot of notations here! The only way to get used to them, to understand how those objects are related, and why they are important, is to look at examples. That is why we made a detailed one below (see Example 1.4.28). In fact, all the works which deal in a detailed way with processes of resolution of singularities introduce analogously plenty of notations (see for instance Zariski [136], Zariski [137], Lejeune-Jalabert [78], A'Campo and Oka [8], Casas [19], Wall [131] or Greuel, Lossen and Shustin [59]). This is one of the main advantages we see for the notion of *lotus* attached below to such a resolution process (see Definition 1.5.26): it allows to get a simultaneous global view of the previous objects.

We can state in the following way the output of Algorithm 1.4.22 in terms of Definition 1.4.15:

Proposition 1.4.26 *The morphism $\pi : (\Sigma, \partial\Sigma) \to (S, L+L')$ is a toroidal pseudo-resolution of C.*

Remark 1.4.27 We formulated Algorithm 1.4.22 only for *reduced* curve singularities C. It extends readily to an algorithm applicable to any C, by agreeing that one runs it on the reduction of C. One may agree also to define the fan tree of an arbitrary curve singularity C as the fan tree of its reduction (see Definition 1.4.33), each leaf being decorated with the multiplicity of the corresponding branch inside the divisor C. Similar conventions may be chosen in order to associate a lotus to an arbitrary curve singularity C. As we do not use those more general notions in this text, we will not introduce them formally.

Let us give now an example of application of Algorithm 1.4.22. Instead of starting from a particular equation, we will assume that the algorithm involves three levels of toroidal modifications with prescribed Newton fans and we will describe from them the toroidal boundary of the final surface. We will see in Example 1.6.29 below how to write concrete equations for branches C_i and L_j appearing in a toroidal resolution process structured as in Example 1.4.28. The idea is to associate to the Newton polygons of the process a *fan tree* (see Definition 1.4.33), which may be transformed into an *Eggers-Wall tree* (see Definition 1.6.28), which in turn allows to write Newton-Puiseux series defining the branches C_i and L_j. One may take as their defining functions in $\mathbb{C}[[x, y]]$ the minimal polynomials of those Newton-Puiseux series.

Example 1.4.28 We will use Notations 1.4.25, but we will denote in the same way a branch and its various strict transforms by the modifications produced by the algorithm. In particular, we will write L_j instead of B_j, for any $j \in J$.

Assume that, relative to the first cross (L, L_1), which lives on $S^{(1)} = S$, the Newton fan $\mathcal{F}_{L,L_1}(C)$ of the curve singularity C is as represented on the top of Fig. 1.16. Therefore it is the same fan $\mathcal{F}(3/5, 2/1, 5/2)$ as in Fig. 1.8. The associated Newton modification $\pi^{(1)}$ is represented on the bottom of Fig. 1.16.

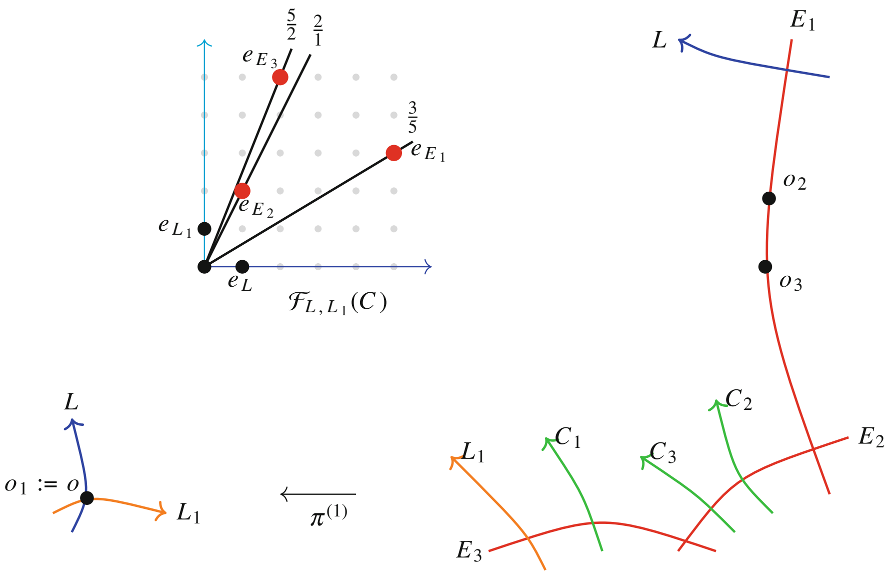


Fig. 1.16 First level of Newton modifications in Example 1.4.28

We have drawn schematically the two boundaries $\partial S^{(1)} = L + L_1$ and $\partial S^{(2)} = L + E_1 + E_2 + E_3 + L_1 + C_1 + C_2 + C_3$. The components E_i of the exceptional divisor of $\pi^{(1)}$ correspond to the rays $\mathbb{R}_+ e_{E_i}$ of the Newton fan $\mathcal{F}_{L,L_1}(C)$. We assume that there are three intersection points of the strict transform C_{L,L_1} of C by $\pi^{(1)}$ at which the algorithm stops at STEP 1. The corresponding components of C are denoted C_1, C_2, C_3. By contrast, at the points o_2 and o_3, one has to apply STEP 2 of Algorithm 1.4.22 (which implies that $\{2, 3\} \subseteq J$).

One introduces two new smooth branches L_2 and L_3 passing through o_2 and o_3 respectively, transversally to the exceptional divisor $E_1 + E_2 + E_3$ of $\pi^{(1)}$. Both points o_2 and o_3 belong to the component E_1. Now one may get the second level of Newton modifications, by looking at the Newton fans $\mathcal{F}_{E_1,L_2}(C)$ and $\mathcal{F}_{E_1,L_3}(C)$ (note that we have written (E_1, L_j) instead of (A_j, L_j), because for $j \in \{2, 3\}$, A_j is the germ of E_1 at o_j). We assume that those Newton fans are as represented on the top of Fig. 1.17. The corresponding composition $\pi^{(2)}$ of Newton modifications at o_2 and o_3 is represented on the bottom of the figure, through a schematic drawing

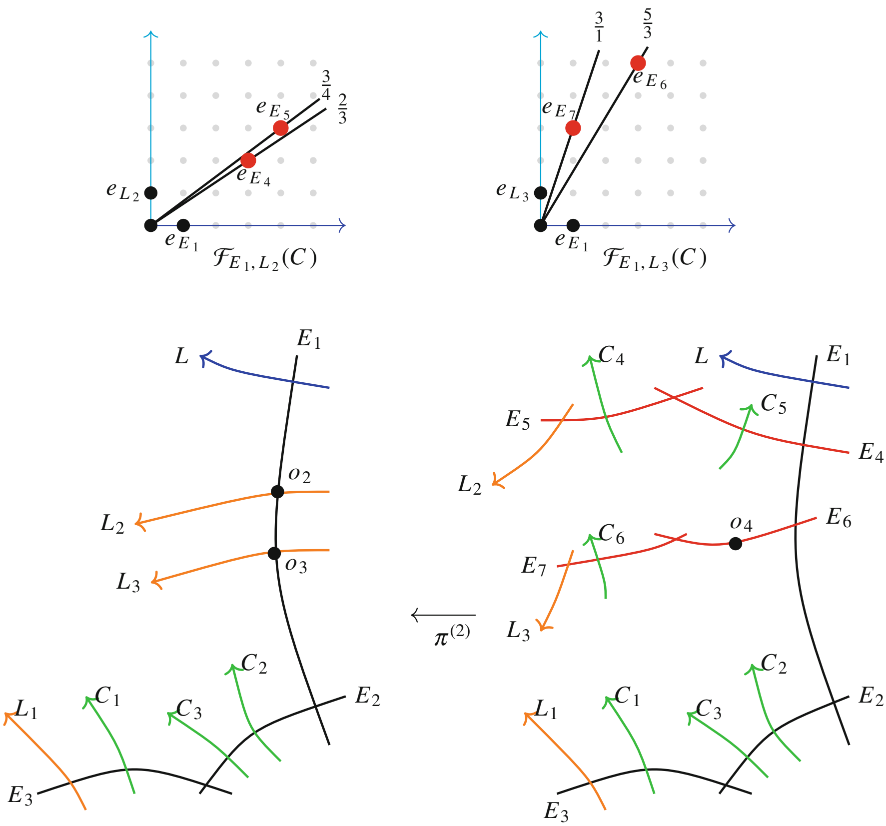


Fig. 1.17 Second level of Newton modifications in Example 1.4.28

of $\partial S^{(2)} + L_2 + L_3$ and of $\partial S^{(3)} = \partial \Sigma$. We assume that the process stops at STEP 1 at three more points, through which pass the strict transforms of the branches C_4, C_5, C_6 of C (see the right bottom part of Fig. 1.17). There remains one point o_4, lying on the component E_6 of the exceptional divisor $E_4 + E_5 + E_6 + E_7$ of $\pi^{(2)}$, at which one has to perform STEP 2.

One completes then the germ A_4 of E_6 at o_4 into a cross (E_6, L_4), represented on the left bottom part of Fig. 1.18. We assume now that the Newton fan $\mathcal{F}_{E_6,L_4}(C)$ is as drawn on the top of the figure. It has only one ray distinct from the edges of the cone $\mathbb{R}_+\langle e_{E_6}, e_{L_4}\rangle$. Therefore, the corresponding Newton modification, which alone gives the third level of Newton modifications $\pi^{(3)}$, introduces only one more irreducible component of exceptional divisor, denoted E_8. It is cut by the strict transform of one more branch of C, denoted C_7 and represented on the bottom right part of Fig. 1.18. The whole curve schematically represented here is the boundary $\partial\Sigma$. On the bottom left is represented the divisor $\partial S^{(3)} + L_4$. The toroidal pseudo-resolution of C produced by the algorithm is the composition $\pi^{(1)} \circ \pi^{(2)} \circ \pi^{(3)} : (\Sigma, \partial\Sigma) \to (S, L + L_1)$. The singular points of the total surface $\Sigma := S^{(3)}$ correspond bijectively to the non-regular 2-dimensional cones of the Newton fans $\mathcal{F}_{L,L_1}(C)$, $\mathcal{F}_{E_1,L_2}(C)$, $\mathcal{F}_{E_1,L_3}(C)$ and $\mathcal{F}_{E_6,L_4}(C)$ produced by the algorithm. We represented them as small blue discs on the bottom right of Fig. 1.18.

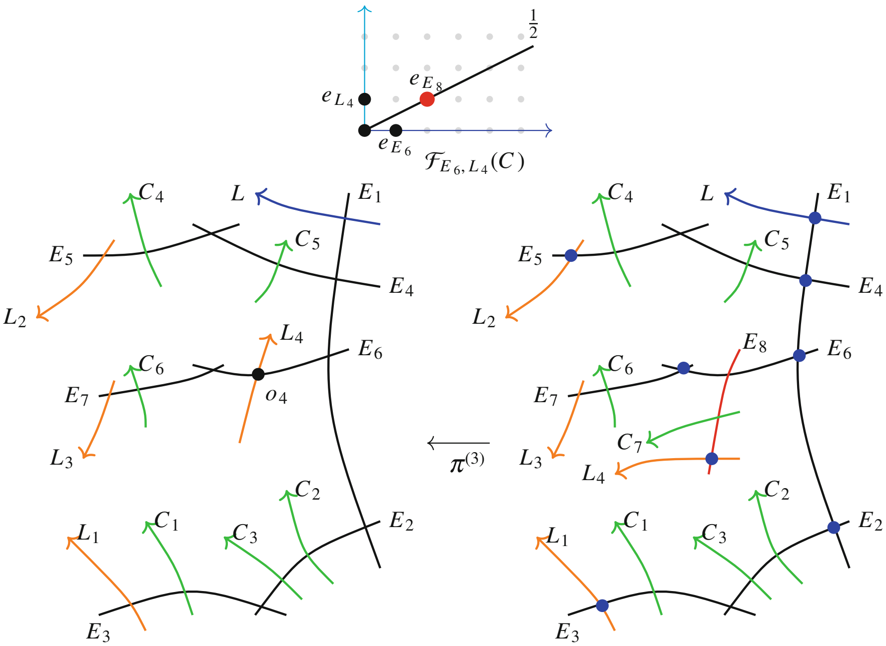


Fig. 1.18 Third level of Newton modifications in Example 1.4.28

1.4.3 From Toroidal Pseudo-Resolutions to Embedded Resolutions

In this subsection, we explain how to get an embedded resolution of $C \hookrightarrow S$ from one of the toroidal pseudo-resolutions produced by Algorithm 1.4.22. Recall first from Definition 1.4.15 the difference between toroidal pseudo-resolutions and embedded ones: in the first ones the source of the modification may have toric singularities, while in the second ones the source is required to be smooth.

Consider a toroidal pseudo-resolution morphism $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ of C produced by Algorithm 1.4.22 (we speak about "a morphism" instead of "the morphism", because of the choices of smooth branches $(L_j)_{j\in J}$ involved in its construction, see Definition 1.4.25). The surface Σ has a finite number of singular points. As explained in Example 1.4.36, they correspond to the 2-dimensional non-regular cones of the Newton fans which appeared during the process. Proposition 1.3.28 shows that one may resolve minimally those singular points by taking the regularization of each such cone. In fact, those regularizations glue into the regularizations of the Newton fans.

A way to regularize all the Newton fans produced by Algorithm 1.4.22 is to run a variant of it, obtained by always replacing STEP 3 with the following "regularized" version of it:

STEP 3^{reg}. *Let* $\mathcal{F}^{reg}_{L,L'}(C)$ *be the regularized Newton fan of* C *relative to the cross* (L, L') *and let* $\psi^{C,reg}_{L,L'} : (S_{\mathcal{F}^{reg}_{L,L'}(C)}, \partial S_{\mathcal{F}^{reg}_{L,L'}(C)}) \to (S, L + L')$ *be the associated Newton modification. Consider the strict transform* $C_{L,L'}$ *of* C *by* $\psi^{C,reg}_{L,L'}$.

We did not change the notations for the successive strict transforms of C from STEP 3 to STEP 3^{reg}, because this variant of the algorithm does never modify the surfaces produced by the first algorithm in the neighborhood of those strict transforms. Indeed, the strict transforms never pass through the singular points of the modified surfaces $S^{(k)}$ (see Proposition 1.4.18 and Notations 1.4.25).

One has the following description of the result of running the "regularized" algorithm:

Proposition 1.4.29 *Let* $\pi : (\Sigma, \partial\Sigma) \to (S, L+L')$ *be a toroidal pseudo-resolution obtained by running Algorithm 1.4.22. Assume that one replaces always STEP 3 with STEP* 3^{reg} *above, choosing the same smooth germs* $(L_j)_{j\in J}$ *as in the construction of* π. *Then one gets a morphism in the toroidal category* $\boxed{\pi^{reg}}$: $(\Sigma^{reg}, \partial\Sigma^{reg}) \to (S, L + L')$, *which is moreover an embedded resolution of* C *and which factors as* $\pi^{reg} = \pi \circ \eta$, *where* $\eta : (\Sigma^{reg}, \partial\Sigma^{reg}) \to (\Sigma, \partial\Sigma)$ *is a modification in the toroidal category whose underlying modification of complex surfaces is the minimal resolution of the complex surface* Σ.

Let us look at the underlying morphism of complex surfaces $\pi^{reg} : \Sigma^{reg} \to S$. Both surfaces are smooth, therefore this morphism is a composition of blow ups of points, by the following theorem of Zariski (see [61, Corollary 5.4] or [113, Vol.1, Ch. IV.3.4, Thm.5]):

Theorem 1.4.30 *Let $\psi : S_2 \to S_1$ be a modification of a smooth complex surface S_1, with S_2 also smooth. Then ψ may be written as a composition of blow ups of points.*

In Sect. 1.5 we will describe explicitly the combinatorics of the decomposition of $\pi^{reg} : \Sigma^{reg} \to S$ into blow ups of points.

Let us recall the following classical terminology about objects associated to a process of blow ups of points, starting from $o \in S$ (see [78], [19, Chap. 3], [102] and [96]):

Definition 1.4.31 Let (S, o) be a smooth germ of surface.

- An **infinitely near point** of o is either o or a point of the exceptional divisor of a smooth modification of (S, o). Two such points, on two modifications, are considered to be the same, if the associated bimeromorphic map between the two modifications is an isomorphism in their neighborhoods.
- If o_1 and o_2 are two infinitely near points of o, then one says that o_2 **is proximate to** o_1, written $\boxed{o_2 \to o_1}$, if o_2 belongs to the strict transform of the irreducible rational curve created by blowing up o_1. If moreover there is no point o_3 such that $o_2 \to o_3 \to o_1$, one says that o_1 **is the parent of** o_2.
- A **finite constellation (above** o**)** is a finite set C of infinitely near points of o, closed under the operation of taking the parent.
- The **Enriques diagram** $\boxed{\Gamma(C)}$ of the finite constellation C is the rooted tree with vertex set C, rooted at o, and such that there is an edge joining each point of C with its parent.

Note that the proximity binary relation on the set of all the infinitely near points of o is not a partial order, as it is neither reflexive, nor transitive. For instance, if o_1 belongs to the exceptional divisor E_0 of the blow up of o and o_2 belongs to the exceptional divisor of the blow up of o_1 but not to the strict transform of E_0 by this blow up, then $o_2 \to o_1 \to o$, but $o_2 \nrightarrow o$. Therefore, the Enriques diagram of a finite constellation encodes only part of the proximity binary relation on it. For this reason, Enriques introduced in [35] supplementary rules for the drawing of his diagrams, allowing to reconstruct completely the proximity relation. Namely, the edges of the Enriques diagram are moreover either *straight* or *curved* and there are *breaking points* between some pairs of successive straight edges. As we do not insist on those aspects, we do not give the precise definitions, sending the interested reader to the literature cited above.

1.4.4 The Fan Tree of a Toroidal Pseudo-Resolution Process

In this subsection we explain how to associate a *fan tree* to each process of toroidal pseudo-resolution of a curve singularity C on the smooth germ of surface (S, o) (see Definition 1.4.33). It is a couple formed by a rooted tree and a $[0, \infty]$-valued function constructed from the Newton fans created by the process. It turns out that it is isomorphic to the dual graph of the boundary $\partial \Sigma$ of the source surface Σ of the toroidal pseudo-resolution morphism $\pi : (\Sigma, \partial\Sigma) \to (S, \partial S)$ (see Proposition 1.4.35).

Fan trees are constructed from *trunks* associated with Newton fans. Let us define first those trunks:

Definition 1.4.32 Let N be a 2-dimensional lattice endowed with a basis (e_1, e_2) and let $\mathcal{F}$ be a Newton fan of N relative to this basis, in the sense of Definition 1.4.9. Its **trunk** $\boxed{\theta(\mathcal{F})}$ is the segment $[e_1, e_2] \subseteq \sigma_0$ endowed with the **slope function** $\boxed{\mathbf{S}_{\mathcal{F}}}$: $[e_1, e_2] \to [0, \infty]$ which associates with each point $w \in [e_1, e_2]$ the slope in the basis (e_1, e_2) of the ray $\mathbb{R}_+ w$ generated by it. Its **marked points** are the points of intersection of $[e_1, e_2]$ with the rays of $\mathcal{F}$. If $\mathcal{E} \subseteq \mathbb{Q}_+ \cup \{\infty\}$, we denote by $\boxed{\theta(\mathcal{E})}$ the trunk of the fan $\mathcal{F}(\mathcal{E})$ introduced in Definition 1.3.4.

Note that the slope function of a trunk is a homeomorphism. Several examples of trunks are represented in Fig. 1.19.

Assume now that we apply Algorithm 1.4.22 to the curve singularity C living on the smooth germ of surface (S, o). Consider the set $\{(A_i, B_i), i \in I\}$ of crosses produced by the algorithm, as explained in Notations 1.4.25. Note that we consider also the crosses at which the algorithm stops at an iteration of STEP 1. Denote by $\boxed{(e_{A_i}, e_{B_i})}$ the basis (e_1, e_2) of the weight lattice N_{A_i, B_i}. The segment $[e_{A_i}, e_{B_i}]$ is the trunk $\theta(\mathcal{F}_{A_i, B_i}(C))$. The following definition uses Notations 1.4.25:

Definition 1.4.33 The **fan tree** $\boxed{(\theta_\pi(C), \mathbf{S}_\pi)}$ **of the toroidal pseudo-resolution** $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ **of** C is a pair formed by a rooted tree $\theta_\pi(C)$ and a **slope function** $\mathbf{S}_\pi : \theta_\pi(C) \to [0, \infty]$ obtained by gluing the disjoint union of the trunks $(\theta(\mathcal{F}_{A_i, B_i}(C)), \mathbf{S}_{\mathcal{F}_{A_i, B_i}(C)})_{i \in I}$ in the following way:

1. Label each marked point with the corresponding irreducible component E_k, L_j or C_l of the boundary $\partial\Sigma$ of the toroidal surface $(\Sigma, \partial\Sigma)$.
2. Identify all the points of $\bigsqcup_{i \in I} \theta(\mathcal{F}_{A_i, B_i}(C))$ which have the same label. The result of this identification is the fan tree $\theta_\pi(C)$ and the images inside it of the marked points of $\bigsqcup_{i \in I} \theta(\mathcal{F}_{A_i, B_i}(C))$ are its **marked points**. We keep for each one of them the same label as in the initial trunks.
3. The **root** of $\theta_\pi(C)$ is the point labeled by the initial smooth branch L.
4. For every $i \in I$, the restriction of the function $\mathbf{S}_\pi$ to every half-open trunk $\theta(\mathcal{F}_{A_i, B_i}(C)) \setminus \{e_{A_i}\} = (e_{A_i}, e_{B_i}] \hookrightarrow \theta_\pi(C)$ is equal to $\mathbf{S}_{\mathcal{F}_{A_i, B_i}(C)}$.
5. At the root, $\mathbf{S}_\pi(L) = \mathbf{S}_{\mathcal{F}_{L, L_1}(C)}(L) = 0$.

As in any rooted tree, the root L defines a partial order $\boxed{\preceq_L}$ on the set of vertices of the fan tree $\theta_\pi(C)$ (that is, on its set of marked points), by declaring that $P \preceq_L Q$ if and only if the unique segment $[L, P]$ joining L and P inside the tree is included in the analogous segment $[L, Q]$.

Note that the slope function $\mathbf{S}_\pi$ is discontinuous at all the marked points of $\theta_\pi(C)$ resulting from the identification of points of two different trunks, its directional limits jumping from a positive value to 0 when one passes from one trunk to another one in increasing way relative to the partial order $\preceq_L$. It follows that the fan tree of a toroidal pseudo-resolution determines the trunks $(\theta(\mathcal{F}_{A_i,B_i}(C)), \mathbf{S}_{\mathcal{F}_{A_i,B_i}(C)})_{i\in I}$.

Example 1.4.34 Consider again the toroidal pseudo-resolution process of Example 1.4.28. The construction of the trunks associated to its Newton fans is represented in Fig. 1.19 for all the crosses at which one applies STEP 2 of the algorithm, that is, for the crosses (A_i, B_i) with $i \in J$. The remaining crosses are those at which the algorithm stops while executing STEP 1. The corresponding trunks are represented on the bottom line of Fig. 1.19. Figure 1.20 shows the construction of the fan tree from the previous collection of trunks. In order to make clear the process of gluing of points with the same label, the upper part of the figure shows again the whole collection of trunks, as well as the labels of its marked points.

The following proposition is an easy consequence of Definition 1.4.33 and of Proposition 1.3.24 (recall that the notion of *dual graph* of an abstract simple normal crossings curve was explained in Definition 1.3.22):

Proposition 1.4.35 *The fan tree $\theta_\pi(C)$ is isomorphic to the dual graph of the boundary $\partial\Sigma$ of the source of the toroidal pseudo-resolution $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ of the curve singularity C, by an isomorphism which respects the labels.*

Example 1.4.36 Proposition 1.4.35 is illustrated in Fig. 1.21 with the fan tree of the bottom of Fig. 1.20 and the boundary $\partial\Sigma$ of the bottom right of Fig. 1.18. Both of them correspond to the toroidal pseudo-resolution process of Example 1.4.28. The singular points of Σ may be found out from the knowledge of the slope function on the trunks composing the fan tree. Indeed, consider the slopes β/α and δ/γ of two consecutive vertices of the trunk of one of the Newton fans of the pseudo-resolution process. Then the matrix $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ is of determinant ± 1 if and only if the intersection point o_i of the irreducible components of $\partial\Sigma$ which corresponds to this edge is non-singular on Σ. Moreover, the surface singularity (Σ, o_i) is analytically isomorphic to the germ at its orbit of dimension 0 of the affine toric surface generated by the cone $\mathbb{R}_+\langle \alpha\, e_1 + \beta\, e_2, \gamma\, e_1 + \delta\, e_2\rangle$ and the lattice $N = \mathbb{Z}\langle e_1, e_2\rangle$. As in Fig. 1.18, the singular points on $\partial\Sigma$ are indicated by small blue discs. The corresponding edges of the fan trees are represented also in blue. Note that in the previous explanation it was important to say that one has to work with the slope function on the individual trunks, instead of the slope function of the fan tree. For instance, if one looks at the intersection point of the components E_1 and E_4, the corresponding slopes are to be read on the trunk $\theta(\mathcal{F}_{E_1,L_2}(C))$ (they are therefore $0/1$ and $2/3$, and the associated

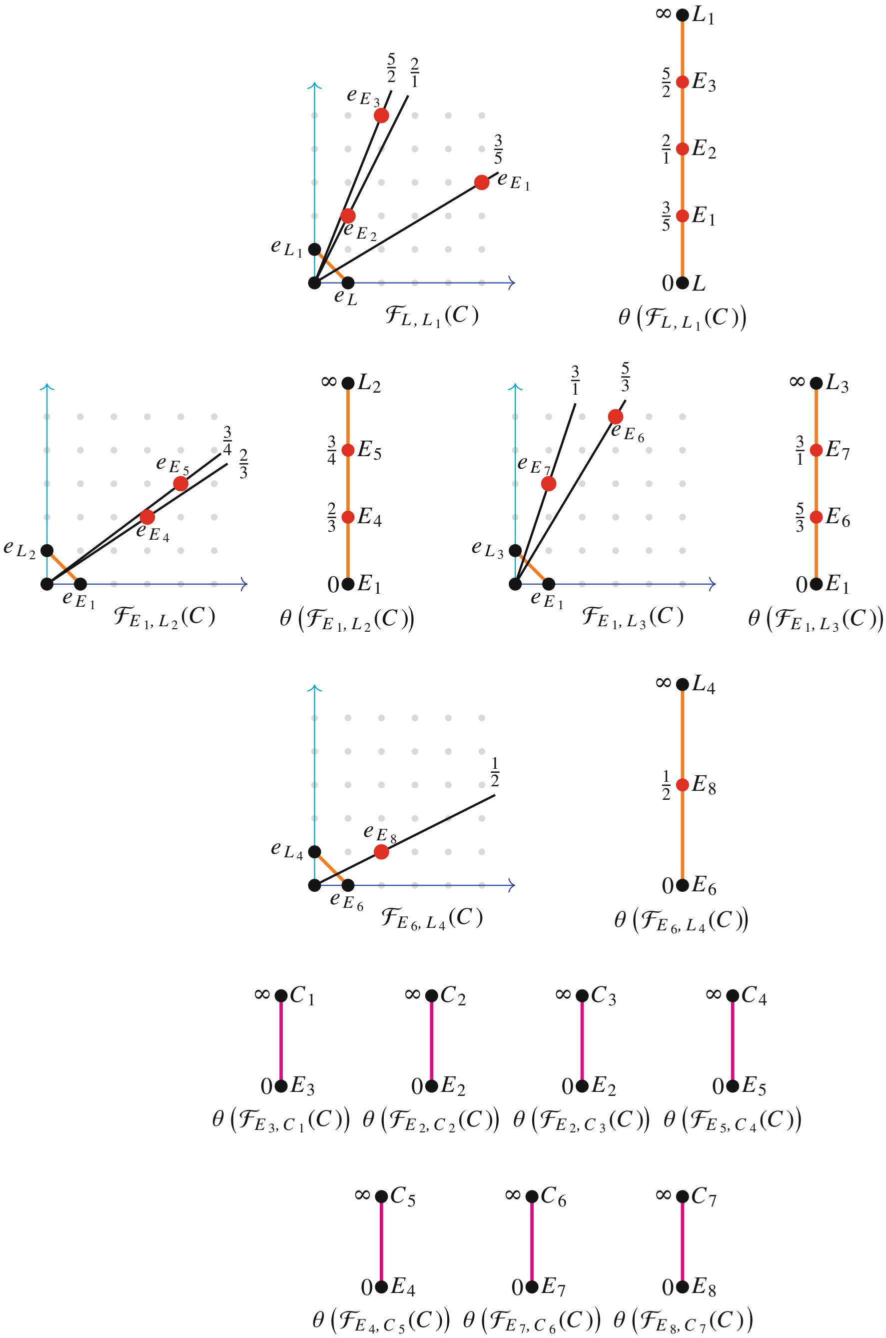


Fig. 1.19 The trunks associated to the toroidal pseudo-resolution of Example 1.4.28

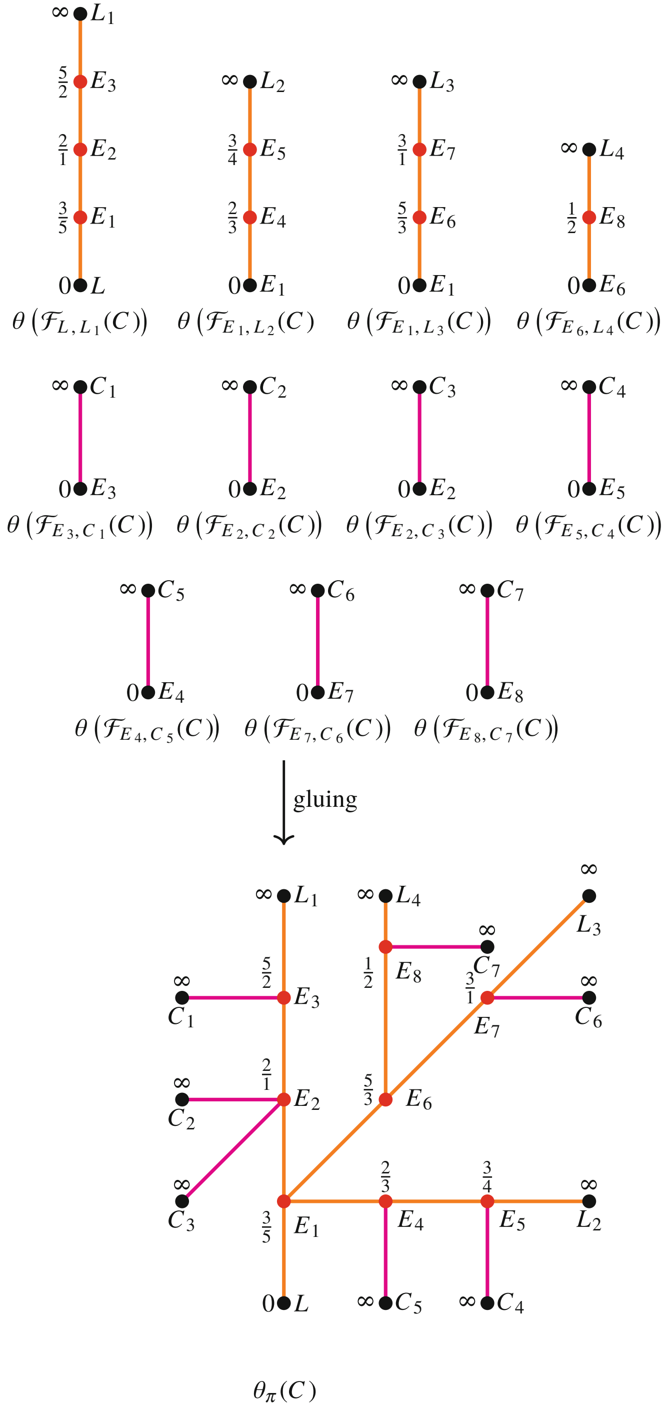


Fig. 1.20 Construction of the fan tree of the toroidal pseudo-resolution of Example 1.4.28

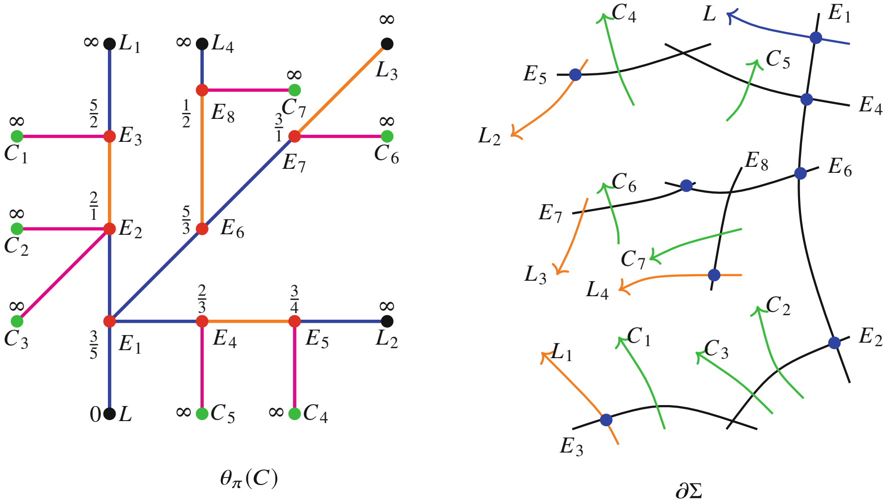


Fig. 1.21 The fan tree $\theta_\pi(C)$ is isomorphic to the dual graph of the toroidal boundary $\partial\Sigma$

matrix $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ is not unimodular), not on the fan tree $\theta_\pi(C)$ (which would give the slopes $3/5$ and $2/3$, whose associated matrix $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ is unimodular).

1.4.5 Historical Comments

The oldest method to study a plane curve singularity C, imagined by Newton around 1665, but published only in 1736 as [88], is to express it first in local coordinates (x, y) as the vanishing locus of a power series $f(x, y)$ satisfying $f(0, 0) = 0$ and $f(0, y) \neq 0$, then to compute iteratively a formal power series $\eta(x)$ with *rational* positive exponents such that $f(x, \eta(x)) = 0$. Whenever $\dfrac{\partial f}{\partial y}(0, 0) \neq 0$, there is only one such series $\eta(x)$ which has moreover only integral exponents. This series is simply the Taylor expansion at the origin of the explicit function $y(x)$ whose existence is ensured by the implicit function theorem applied to the function $f(x, y)$ in the neighborhood of $(0, 0)$. But, if $\dfrac{\partial f}{\partial y}(0, 0) = 0$, then there are at least two such series, their number being equal to the order in y of the series $f(0, y)$.

As explained on the example studied in Sect. 1.2.6, the first step of Newton's iterative method consists in finding the possible leading terms $c\, x^\alpha$ of the series $\eta(x)$. His main insight was that if one substitutes $y := c\, x^\alpha$ in the series $f(x, y)$, getting a formal power series with rational exponents in the variable x, then *there are* *at least two terms of this series with minimal exponent, and the sum of all such terms vanishes*. This fact has two consequences. First, there is a finite number of possible exponents α, which are the slopes of the rays orthogonal to the compact edges of the *Newton polygon* of $f(x, y)$. Secondly, for a fixed exponent α_K corresponding to the compact edge K, there is a finite number of values of the leading coefficient c, given by the roots of the algebraic equation $f_K(x, c\, x^{\alpha_K}) = 0$, where f_K is the restriction of f to K in the sense of Definition 1.4.2.

Newton's explanations were much developed in Cramer's 1750 book [27, Chapter VII], which seems also interesting to us in this context for its interpretation of the weights of the variables x and y as orders of magnitude for infinitely small quantities.

Figures 1.22 and 1.23 are extracted from [88, I, Section XXX] and [27, Section 103] respectively. The first one represents the only drawing of Newton polygon in

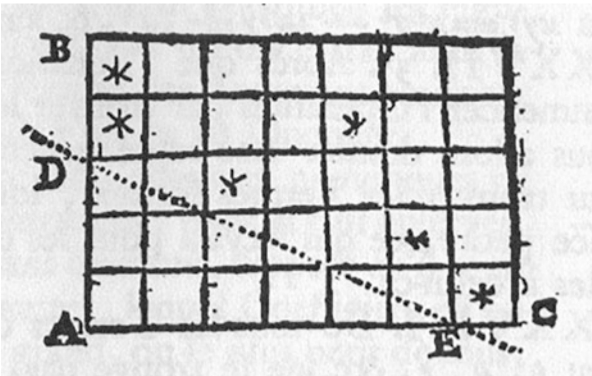


Fig. 1.22 The first Newton polygon

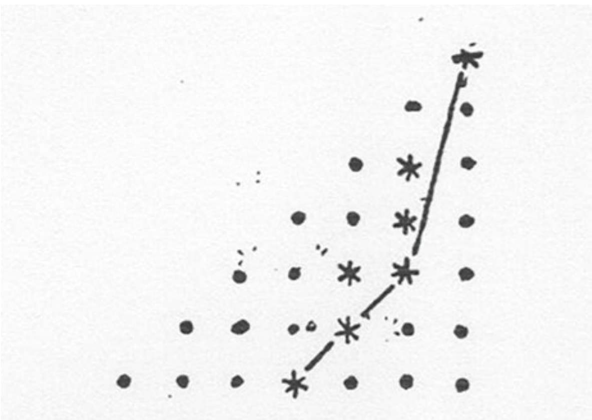
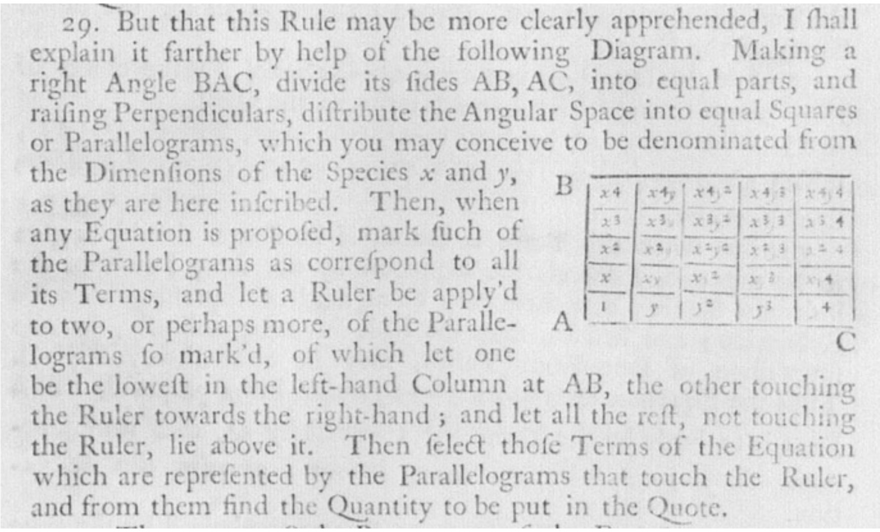


Fig. 1.23 The compact sides of a Newton polygon, as represented by Cramer



29. But that this Rule may be more clearly apprehended, I ſhall explain it farther by help of the following Diagram. Making a right Angle BAC, divide its ſides AB, AC, into equal parts, and raiſing Perpendiculars, diſtribute the Angular Space into equal Squares or Parallelograms, which you may conceive to be denominated from the Dimenſions of the Species x and y, as they are here inſcribed. Then, when any Equation is propoſed, mark ſuch of the Parallelograms as correſpond to all its Terms, and let a Ruler be apply'd to two, or perhaps more, of the Parallelograms ſo mark'd, of which let one be the loweſt in the left-hand Column at AB, the other touching the Ruler towards the right-hand; and let all the reſt, not touching the Ruler, lie above it. Then ſelect thoſe Terms of the Equation which are repreſented by the Parallelograms that touch the Ruler, and from them find the Quantity to be put in the Quote.

x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy	xy^2	xy^3	xy^4
1	y	y^2	y^3	y^4

Fig. 1.24 Newton's ruler

Newton's book. Strictly speaking, what we call the *Newton polygon* of a series in two variables was not formally introduced in the book. Newton explained only how to move a ruler in order to get a first bounded edge of the polygon (see Fig. 1.24). More details about Newton's and Cramer's ideas on this subject may be found in Ghys' 2017 book [50, Pages 43–68].

Newton wrote that his procedure may be performed iteratively in order to compute as many terms of the series $\eta(x)$ as desired. He also sketched in [88, Ch. I.LII] an explanation of the fact that, whenever $f(x, y)$ converges, the formal series with rational exponents $\eta(x)$ obtained by continuing forever the procedure also converge and satisfy indeed, all of them, the relation $f(x, \eta(x)) = 0$. But it was Puiseux, in his 1850 paper [106], who proved rigorously that one gets indeed as many series as the order in y of $f(0, y)$, that all of them are obtained by substituting some root $x^{1/n}$ of the variable x into formal power series with integral exponents, and that those formal power series are in fact convergent in a neighborhood of the origin. In order to honor his work, the formal or convergent power series in a variable x of the form $\xi(x^{1/n})$, where $\xi(x)$ is a usual power series and $n \in \mathbb{N}^*$ are called nowadays *Puiseux series* or *Newton-Puiseux series*.

Puiseux's approach to the proofs of the existence and the convergence of these series avoided the use of roots $x^{1/n}$, by performing changes of variables of the form $x = x_1^q,\ y = c_1 x_1^p + y_1$ or of the form $x = x_1^q,\ y = x_1^p(c_1 + y_1)$, where c_1 is a non-zero constant and p/q is the irreducible expression of one of the exponents α_K given by the Newton polygon of f. Both changes of variables are compositions of a birational change of variables and of the monomial change of variables $x = u^q,\ y = u^p\, v$. This monomial change of variables is birational only when $q = 1$, that is, when $\alpha_K \in \mathbb{N}^*$. Therefore Puiseux's changes of variables are in general not birational. Nevertheless, by Lemma 1.6.24 below, such a map can be seen as the local analytical expression of a birational map, with respect to a particular choice of local coordinates.

Zariski saw this non-birationality as a drawback, and in his 1939 paper [136] he introduced alternative changes of variables of the form $x = x_1^q(c_1 + y_1)^{q_1}$, $y = x_1^p(c_1 + y_1)^{p_1}$, where $(p_1, q_1) \in \mathbb{N}^* \times \mathbb{N}^*$ and $p_1 q - q_1 p = 1$. This last condition means that Zariski's changes of variables are birational.

Let us discuss now the toric approach to the study of plane curve singularities. Note that the changes of variables used by Puiseux and by Zariski are compositions of affine morphisms and of toric ones. This fact became clear after the development of toric geometry (see Sect. 1.3.5).

The systematic study of plane curve singularities using sequences of toric modifications began with Mutsuo Oka's 1995–96 papers [8, 83, 93], the first one written in collaboration with Lê and the second one with A'Campo (see also Oka's 1997 book [94, Ch. III, Sect. 4]). Oka gave an introduction to this approach in his 2010 paper [95], through the detailed examination of the case of one branch. The second author generalized this approach to *quasi-ordinary* hypersurface singularities of arbitrary dimension in his 2003 paper [52] and applied it to the study of deformations of real plane curve singularities in the 2010 papers [53] and [54], the second one written in collaboration with Risler.

Also during the 1990s, Pierrette Cassou-Noguès started studying plane curve singularities using Puiseux's non-birational toric morphisms, called *Newton maps*. References to her early works on the subject, done partly in collaboration, may be found in her 2011 paper [20] with Płoski, her 2014–15 papers [22, 23] with Veys, her 2014 paper with Libgober [21] and her 2018 paper with Raibaut [24].

In his 1997 paper [129], Veys considered the *log-canonical model* of a plane curve singularity, obtained by contracting certain exceptional divisors on its minimal embedded resolution, in order to study associated zeta functions. The modification from the log-canonical model to the ambient germ of the plane curve singularity may be seen as a morphism associated with a toroidal pseudo-resolution of this singularity. A toroidal pseudo-resolution algorithm for plane curve singularities was described by the second author in [52, Section 3.4]. A more general algorithm was given by Cassou-Noguès and Libgober in [21, Section 3]. Our Algorithm 1.4.22 of toroidal pseudo-resolution generalizes them, since it does not depend on the choice of special kinds of coordinates.

There are several approaches for the search of *optimal choices* of smooth branches in STEP 2 of Algorithm 1.4.22. *Assume first that C is a branch*, that $f \in \mathbb{C}[[x]][y]$ is the monic polynomial of degree n defining C in the local coordinate system (x, y) and that the line $L = Z(x)$ is transversal to C. Let a be a divisor of n. The *a-th approximate root* $h \in \mathbb{C}[[x]][y]$ of f is the unique monic polynomial of degree a such that the degree in y of $f - h^{n/a}$ is smaller than $n - a$. The importance of approximate roots for the study of plane curve singularities and of the algebraic embeddings of $\mathbb{C}$ in $\mathbb{C}^2$ was emphasized by Abhyankar and Moh in their 1973–75 papers [2] and [3]. Certain approximate roots of f, called *characteristic approximate roots*, have the property that their strict transforms can be chosen at

STEP 2 of Algorithm 1.4.22, providing in this way a toroidal pseudo-resolution of C with the minimal number of Newton modifications. This number is precisely the number of *characteristic exponents* of C with respect to x (see Sect. 1.6). This approach was explained by A'Campo and Oka in their 1996 paper [8].

Some properties of the approximate roots may fail when working with a base field of positive characteristic. By contrast, the more general combinatorial notion of *semiroot/maximal contact curve* can be defined over fields of arbitrary characteristic and plays a similar role (see the papers [77] of Lejeune-Jalabert and [49] of the first author and Płoski). For details on applications of approximate roots and semiroots to the study of plane curve singularities, see the paper [60] of Gwoździewicz and Płoski and [99] of the third author. Proposition 1.4.35 above implies that if $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ is a toroidal embedded resolution of C which defines its minimal resolution, then the irreducible components of the associated completion $\hat{C}_\pi = \pi(\partial\Sigma)$ may be thought as generalizations of the notion of semiroot to plane curve singularities with an arbitrary number of branches (see also the final comments in Example 1.6.33 below).

Assume now that C is an arbitrary plane curve singularity. The minimal number of Newton modifications involved in the construction of a toroidal pseudo-resolution C was characterized by Lê and Oka in [83] in terms of properties of the dual graph of its minimal embedded resolution.

Another toric approach to the study of plane curve singularities was initiated in Goldin and Teissier's 2000 paper [51], in the case of branches. They first reembedded in a special way the initial germ of surface in a higher dimensional space, then they resolved the branch by just one toric modification of that space. Their approach was done in the spirit of the philosophy of Teissier's 1973 paper [119], in which he saw all equisingular plane branches as deformations of a single branch of higher embedding dimension, the germ at the origin of their common *monomial curve*. A generalization of some of the results in [51] to the case of quasi-ordinary hypersurface singularities was obtained by the second author in [52]. The theoretical possibility of studying analogously singularities of any dimension was established by Tevelev in his 2014 paper [125]. See Teissier's comments in [124, Section 11] for more details about his toric approach to the study of singularities.

The notions of *Newton non-degenerate polynomials* and *series* were introduced by Kouchnirenko in his 1976 paper [74], using the last characterization of Proposition 1.4.20. A version of the first characterization was essential in Varchenko's theorem in [128] about the monodromy of Newton non-degenerate holomorphic series. Then Khovanskii introduced in [73] *Newton non-degenerate complete intersection singularities*, a notion which was much studied by Mutsuo Oka in a series of papers, which were the basis of his 1997 monograph [94]. Characterizations of Newton non-degenerate singularities, analogous to those of Proposition 1.4.20, are in fact true for complete intersection singularities (see Oka's book [94] or Teissier's paper [122, Section 5]). This last paper contains interesting comments about the evolution of the notion of Newton non-degeneracy, and an extension of it to arbitrary singularities, which are not necessarily complete intersections. This extension was further studied in Fuensanta Aroca, Gómez-Morales and Shabbir's paper [9].

Let us discuss now the notion of *tropicalization* trop^f introduced in Definition 1.4.4. The union of the rays of the Newton fan $\mathcal{F}(f)$ which intersect the interior of the regular cone σ_0 is the *tropical zero-locus* of the function trop^f, as defined in tropical geometry, that is, the locus of non-differentiability of the continuous piecewise linear function trop^f. It is also part of the *local tropicalization* of the zero locus $Z(f) \hookrightarrow (\mathbb{C}^2, 0)$ of f, as defined by Stepanov and the third author in [105] for complex analytic singularities of arbitrary dimension embedded in germs of affine toric varieties. The local tropicalization contains also portions at infinity, in a partial compactification of the cone defining the ambient toric variety, in order to keep track of the intersections of the singularity with all the toric orbits.

A precursor of the notion of local tropicalisation was introduced under the name of "*tropism of an ideal*" by Maurer in his 1980 article [85], which was unknown to the authors of [105] when they wrote that paper. In our case, the tropism of the ideal $(f) \subseteq \mathbb{C}[[x, y]]$ is the set of lattice points lying on the rays of $\mathcal{F}(f)$ which are different from the edges of the cone σ_0. The term "*tropism*" had been used before by Lejeune-Jalabert and Teissier in their 1973 paper [79], in the expression "*tropisme critique*". They saw this notion as a measure of anisotropy, as explained by Teissier in [65, Footnote to Sect. 1]:

> As far as I know the term did not exist before. We tried to convey the idea that giving different weights to some variables made the space "anisotropic", and we were intrigued by the structure, for example, of anisotropic projective spaces (which are nowadays called weighted projective spaces). From there to "tropismes critiques" was a quite natural linguistic movement. Of course there was no "tropical" idea around, but as you say, it is an amusing coincidence. The Greek "Tropos" usually designates change, so that "tropisme critique" is well adapted to denote the values where the change of weights becomes critical for the computation of the initial ideal. The term "Isotropic", apparently due to Cauchy, refers to the property of presenting the same (physical) characters in all directions. Anisotropic is, of course, its negation. The name of Tropical geometry originates, as you probably know, from tropical algebra which honours the Brazilian computer scientist Imre Simon living close to the tropics, where the course of the sun changes back to the equator. In a way the tropics of Capricorn and Cancer represent, for the sun, critical tropisms.

1.5 Lotuses

Throughout this section, we will assume that C is *reduced*. We explain the notion of *Newton lotus* (see Definition 1.5.4), its relation with continued fractions (see Sect. 1.5.2) and how to construct a more general *lotus* from the fan tree of a toroidal pseudo-resolution process (see Definition 1.5.26). It is a special type of simplicial complex of dimension 2, built from the Newton lotuses associated with the Newton fans generated by the process, by gluing them in the same way one glued the corresponding trunks into the fan tree. It allows to visualize the combinatorics of the decomposition of the embedded resolution morphism into point blow ups, as well as the associated Enriques diagram and the final dual graphs (see Theorem 1.5.29). We show by two examples that its structure depends on the choice of auxiliary curves introduced each time one executes STEP 2 of Algorithm 1.4.22, that is, on the choice of completion $\hat{C}_\pi$ of C (see Sect. 1.5.4). In Sect. 1.5.5 we define an operation of *truncation* of the lotus of a toroidal pseudo-resolution and we explain some of its uses. In the final Sect. 1.5.6 we give historical information about other works in which appeared objects similar to the notion of lotus.

1.5.1 The Lotus of a Newton Fan

In this subsection, whose content is very similar to that of [102, Section 5], we give a first level of explanation of the subtitle of this article, a second level being described in Sect. 1.5.3. Namely, we introduce the notion of *lotus* $\Lambda(\mathcal{F})$ of a Newton fan $\mathcal{F}$ (see Definition 1.5.4). If the fan originates from a Newton polygon $\mathcal{N}(f)$, that is, if $\mathcal{F} = \mathcal{F}(f)$ (see Definition 1.4.9), we imagine $\Lambda(\mathcal{F})$ as a *blossoming* of $\mathcal{N}(f)$. The lotus of a Newton fan $\mathcal{F}$ allows to understand visually the decomposition into blow ups of the toric modification defined by the regularized fan $\mathcal{F}^{reg}$. For instance, the dual graph of the final exceptional divisor, the Enriques diagram and the graph of the proximity relation of the associated constellation embed naturally in it, as subcomplexes of its 1-skeleton (see Propositions 1.5.11, 1.5.14 and 1.5.16).

Lotuses are built from *petals*, which are triangles with supplementary structure (see Fig. 1.25):

Definition 1.5.1 Let N be a 2-dimensional lattice and let (e_1, e_2) be a basis of it. Denote by $\boxed{\delta(e_1, e_2)}$ the convex and compact triangle with vertices $e_1, e_2, e_1 + e_2$, contained in the real plane $N_{\mathbb{R}}$. It is the **petal associated with the basis** (e_1, e_2). Its **base** is the segment $[e_1, e_2]$, oriented from e_1 to e_2. The points e_1 and e_2 are called the **basic vertices** of the petal. Its **lateral edges** are the segments $[e_i, e_1 + e_2]$, for each $i \in \{1, 2\}$.

Once the petal $\delta(e_1, e_1 + e_2)$ is constructed, the construction may be repeated starting from each one of the bases $(e_1, e_1 + e_2)$ and $(e_1 + e_2, e_2)$ of N, getting two

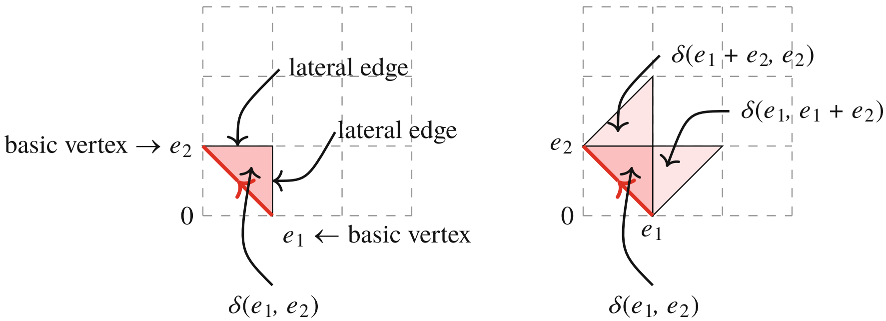


Fig. 1.25 Vocabulary and notations about petals

new petals $\delta(e_1, e_1 + e_2)$ and $\delta(e_1 + e_2, e_2)$, and so on. Note that the bases produced by this process are ordered such as to define always the same orientation of the real plane $N_{\mathbb{R}}$—we say that they are **positive bases**. In this way, one progressively constructs an infinite simplicial complex embedded in the cone σ_0: at the n-th step, one adds 2^n petals to those already constructed. Each petal, with the exception of the first one $\delta(e_1, e_2)$, has a common edge—its base—with exactly one of the petals constructed at the previous step, called its **parent**.

The pairs of vectors $(f_1, f_2) \in N^2$ which appear as bases of petals $\delta(f_1, f_2)$ during the previous process may be characterized in the following way (see [102, Remarque 5.1]):

Lemma 1.5.2 *A segment* $[f_1, f_2]$*, oriented from* f_1 *to* f_2*, is the base of a petal* $\delta(f_1, f_2)$ *constructed during the previous process if and only if* (f_1, f_2) *is a positive basis of the lattice* N *contained in the cone* σ_0*. Said differently, if a positive basis* (f_1, f_2) *of* N *is contained in the cone* σ_0 *and is different from* (e_1, e_2)*, then there exists a unique permutation* (i, j) *of* $(1, 2)$ *such that* $f_j - f_i \in \sigma_0 \cap N$*.*

We are ready to define the simplest kinds of lotuses:

Definition 1.5.3 The simplicial complex obtained as the union of all the petals constructed by the previous process starting from the basis (e_1, e_2) of N, is called **the universal lotus** $\boxed{\Lambda(e_1, e_2)}$ **relative to** (e_1, e_2) (see Fig. 1.26). A **lotus** Λ **relative to** (e_1, e_2) is either the segment $[e_1, e_2]$ or the union of a non-empty set of petals of the universal lotus $\Lambda(e_1, e_2)$, stable under the operation of taking the parent of a petal. The segment $[e_1, e_2]$ is called the **base** of Λ. If Λ is of dimension 2, then the petal $\delta(e_1, e_2)$ is called its **base petal**. The point e_1 is called the **first basic vertex** and e_2 the **second basic vertex** of the lotus. The lotus is oriented by restricting to it the orientation of $N_{\mathbb{R}}$ induced by the basis (e_1, e_2).

A lotus may be associated with any set $\mathcal{E} \subseteq [0, \infty]$ or with any Newton fan:

Definition 1.5.4 Let N be a lattice of rank 2, endowed with a basis (e_1, e_2).

- If $\lambda \in (0, \infty)$, then its **lotus**, denoted $\boxed{\Lambda(\lambda)}$, is the union of petals of the universal lotus $\Lambda(e_1, e_2)$ whose interiors intersect the ray of slope λ. If $\lambda \in \{0, \infty\}$, then its lotus $\Lambda(\lambda)$ is just $[e_1, e_2]$.
- If $\mathcal{E} \subseteq [0, \infty]$, then its **lotus** $\boxed{\Lambda(\mathcal{E})}$ is the union $\bigcup_{\lambda \in \mathcal{E}} \Lambda(\lambda)$ of the lotuses of its elements.
- If $\mathcal{F}$ is a Newton fan and $\mathcal{F} = \mathcal{F}(\mathcal{E})$ in the sense of Definition 1.3.4, we say that $\boxed{\Lambda(\mathcal{F})} := \Lambda(\mathcal{E})$ is the **lotus of the fan** $\mathcal{F}$.
- A **Newton lotus** is the lotus of a Newton fan. That is, it is a lotus relative to (e_1, e_2) with a finite number of petals.

We could have called the lotuses relative to (e_1, e_2) *finite lotuses* instead of *Newton lotuses*. We chose the second terminology because in Definition 1.5.26 below we will introduce a more general kind of lotuses with a finite number of

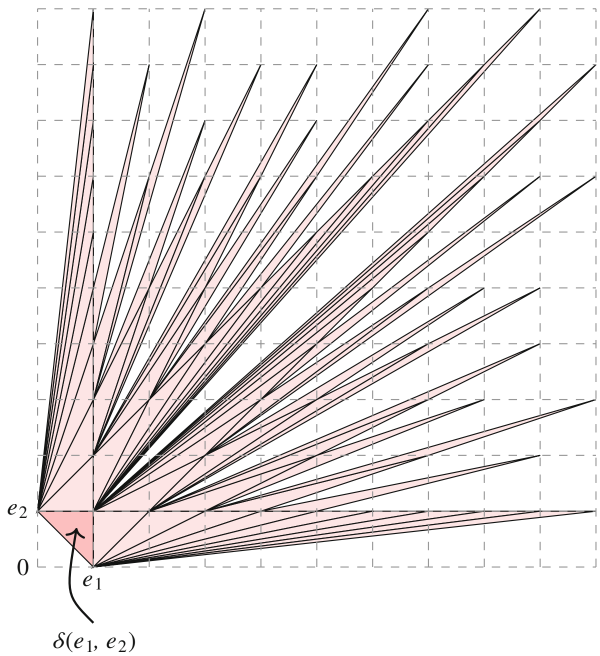


Fig. 1.26 Partial view of the universal lotus $\Lambda(e_1, e_2)$ relative to (e_1, e_2)

petals, and we want to distinguish the class of lotuses of Newton fans inside that more general class of lotuses.

A lotus $\Lambda(\mathcal{E})$, for $\mathcal{E} \subseteq [0, \infty]$, is a Newton lotus if and only $\mathcal{E}$ is a finite set of non-negative rational numbers. Note that, as illustrated for instance by Example 1.5.9 below, the structure of the lotus $\Lambda(\mathcal{E})$ does not allow to reconstruct the initial set $\mathcal{E}$. For this reason, we enrich $\Lambda(\mathcal{E})$ with several *marked* points, whose knowledge allows to reconstruct $\mathcal{E}$ unambiguously:

Definition 1.5.5 Fix a Newton lotus Λ.

- If $\Lambda \neq [e_1, e_2]$, we denote by $\boxed{\partial_+\Lambda}$ the compact and connected polygonal line defined as the complement of the open segment (e_1, e_2) in the boundary of the lotus Λ. If $\Lambda = [e_1, e_2]$, we set $\partial_+\Lambda := [e_1, e_2]$. The polygonal line $\partial_+\Lambda \subseteq \Lambda$ is called the **lateral boundary** of the lotus Λ.
- We denote by $\boxed{p_\Lambda}$ the homeomorphism $p_\Lambda : [0, \infty] \to \partial_+\Lambda$ which associates with any $\lambda \in [0, \infty]$ the unique point $p_\Lambda(\lambda) \in \partial_+\Lambda$ of slope λ. If $\Lambda = \Lambda(\mathcal{E})$ where $\mathcal{E} \subseteq \mathbb{Q}_+ \cup \{\infty\}$ is finite and $\lambda \in \mathcal{E}$, then we call $p_{\Lambda(\mathcal{E})}(\lambda)$ the **marked point of** λ (or of the ray of slope λ) in the lotus $\Lambda(\mathcal{E})$. We consider $\Lambda(\mathcal{E})$ as a **marked lotus** using those marked points.

Remark 1.5.6 Notice that if $\lambda \in \mathcal{E}$, then $p_{\Lambda(\mathcal{E})}(\lambda)$ is by construction the unique primitive element $p(\lambda)$ of the lattice N, which has slope λ relative to the basis (e_1, e_2). Therefore, it is independent of the remaining elements of the set $\mathcal{E}$.

We distinguish also by geometric properties several vertices of a Newton lotus:

Definition 1.5.7 Assume that Λ is a Newton lotus. A vertex of Λ different from e_1 and e_2 is called a **pinching point** of the lotus Λ if it belongs to a unique petal of it. If the lotus Λ is two-dimensional, then the lattice point which is connected to e_2 (resp. to e_1) inside the lateral boundary $\partial_+\Lambda$ of Λ is called the **last interior point** (resp. **first interior point**) of the lateral boundary.

Remark 1.5.8 The pinching points of a Newton lotus $\Lambda(\mathcal{E})$ are part of its marked points. Two Newton lotuses $\Lambda(\mathcal{E}_1)$ and $\Lambda(\mathcal{E}_2)$ coincide as unmarked simplicial complexes if and only if their sets of pinching points coincide.

Example 1.5.9 In Fig. 1.27 are represented the lotuses $\Lambda\,(3/5)$ and $\Lambda(\mathcal{E})$, where $\mathcal{E} = \{3/5, 2/1, 5/2\}$ is the set whose fan $\mathcal{F}(\mathcal{E})$ was drawn in Fig. 1.8. The lotus $\Lambda\,(3/5)$ has only one pinching point, which is $p\,(3/5)$. The pinching points of $\Lambda(\mathcal{E})$ are $p\,(3/5)$ and $p\,(5/2)$. Its marked points are $p\,(3/5)$, $p\,(2/1)$ and $p\,(5/2)$. This differentiates it from the lotus $\Lambda(3/5, 5/2) := \Lambda(\{3/5, 5/2\})$, which is the same simplicial complex if one forgets their respective marked points. The first interior point of $\Lambda(\mathcal{E})$ is $p\,(1/2)$ and its last interior point is $p\,(3/1)$.

By comparing Figs. 1.27 and 1.9, which we combined in Fig. 1.28, one sees that the lateral boundary of the lotus $\Lambda(3/5, 2/1, 5/2)$ is exactly the polygonal line constructed when one performed the regularization of the fan $\mathcal{F}(3/5, 2/1, 5/2)$ (see Proposition 1.3.9). This is a general phenomenon, as shown by the following proposition.

Proposition 1.5.10 *Let $\mathcal{F}$ be a fan subdividing the cone σ_0. Then the regularization $\mathcal{F}^{reg}$ of $\mathcal{F}$ is obtained by subdividing σ_0 using the rays generated by all the lattice points lying along the lateral boundary $\partial_+\Lambda(\mathcal{F})$ of the lotus $\Lambda(\mathcal{F})$.*

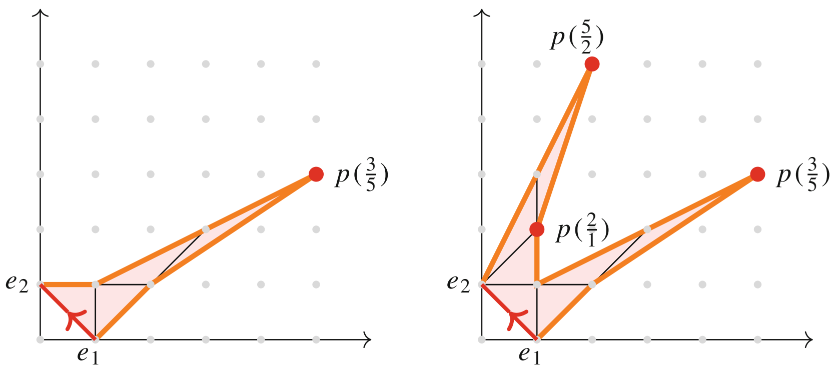


Fig. 1.27 The Newton lotuses $\Lambda\,(3/5)$, $\Lambda\,(3/5, 2/1, 5/2)$ and their marked points

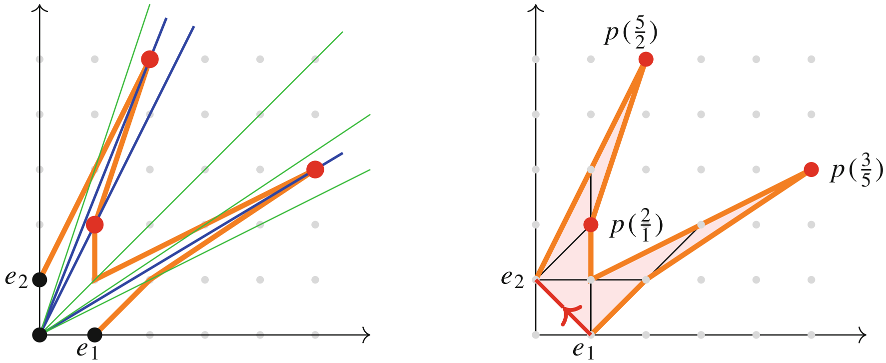


Fig. 1.28 The regularized fan $\mathcal{F}^{reg}$ $(3/5, 2/1, 5/2)$ and the Newton lotus Λ $(3/5, 2/1, 5/2)$

Proof Consider two successive marked points $p(\lambda)$ and $p(\mu)$ of the lateral boundary $\partial_+\Lambda(\mathcal{F})$. They are primitive elements of the ambient lattice N. Denote by $p(\lambda)+\mathbb{R}_+p(\lambda)$ the closed half line originating from the point $p(\lambda)$ and generated by the vector $p(\lambda)$. Consider analogously the half-line $p(\mu)+\mathbb{R}_+p(\mu)$. Let $P(\lambda, \mu)$ be the polygonal line joining the points $p(\lambda)$ and $p(\mu)$ inside $\partial_+\Lambda(\mathcal{F})$. Consider the union of the three previous polygonal lines: $Q(\lambda, \mu) := (p(\lambda)+\mathbb{R}_+p(\lambda)) \cup P(\lambda, \mu) \cup (p(\mu)+\mathbb{R}_+p(\mu))$.

As the pinching points of $\Lambda(\mathcal{F})$ belong to the marked points, this shows that there are no pinching points in the interior of the polygonal line $P(\lambda, \mu)$. Therefore, $Q(\lambda, \mu)$ is the boundary of a closed convex set $\hat{Q}(\lambda, \mu)$ contained in the cone $\mathbb{R}_+\langle p(\lambda), p(\mu)\rangle$. The complement $\mathbb{R}_+\langle p(\lambda), p(\mu)\rangle \setminus \hat{Q}(\lambda, \mu)$ is contained in the union of the complement $\Lambda(\mathcal{F}) \setminus \partial_+\Lambda(\mathcal{F})$ and the convex hull of the points $0, e_1, e_2$ deprived of the segment $[e_1, e_2]$. Therefore, the origin 0 is the only point of N contained in $\mathbb{R}_+\langle p(\lambda), p(\mu)\rangle \setminus \hat{Q}(\lambda, \mu)$. As all the vertices of $Q(\lambda, \mu)$ belong to N, this shows that $\hat{Q}(\lambda, \mu)$ is the convex hull of the set $\mathbb{R}_+\langle p(\lambda), p(\mu)\rangle \cap (N \setminus \{0\})$. One concludes using Proposition 1.3.9. $\square$

Consider again Fig. 1.28. As shown by Proposition 1.3.24, the polygonal line on the left side gives a concrete embedding of the dual graph of the boundary $\partial X_{\mathcal{F}^{reg}}$. But it does not show the order in which were performed the blow ups into which the associated modification $\psi_{\sigma_0}^{\mathcal{F}} : X_{\mathcal{F}} \to X_{\sigma_0}$ decomposes (see Theorem 1.4.30). It turns out that this order is indicated by the lotus on the right side of Fig. 1.28. To understand this fact, recall first the combinatorial description of the blow up of the orbit of dimension 0 of the smooth affine toric surface X_{σ_0}, explained in Example 1.3.27: one gets it by subdividing the cone σ_0 using the ray generated by $e_1 + e_2$. In terms of the associated bases of N, one replaces the basis (e_1, e_2) by the pair of bases $(e_1, e_1 + e_2)$ and $(e_1 + e_2, e_2)$. Graphically, this may be understood as the passage from the base $[e_1, e_2]$ of the petal $\delta(e_1, e_2)$ seen as the simplest 2-dimensional lotus (see Definition 1.5.1) to its lateral boundary $[e_1, e_1 + e_2] \cup [e_1 + e_2, e_2]$. Again by Proposition 1.3.24, we may see this passage as the replacement of the dual graph of ∂X_{σ_0} by the dual graph of the boundary of the blown up toric surface. Now, each new petal in the lotus $\Lambda(\mathcal{F})$ corresponds to the blow up of an orbit of dimension 0 of the previous toric surface. Its base may be seen as the dual graph of the irreducible components of the boundary meeting at that point. One gets:

Proposition 1.5.11 *Let $\mathcal{F}$ be a Newton fan. Then:*

- *The lateral boundary $\partial_+\Lambda(\mathcal{F})$ of the lotus $\Lambda(\mathcal{F})$ is the dual graph of the boundary $\partial X_{\mathcal{F}^{reg}}$ of the smooth toric surface $X_{\mathcal{F}^{reg}}$. Two vertices of it are joined by an edge of the lotus $\Lambda(\mathcal{F})$ if and only if the corresponding orbits have intersecting closures at some moment of the process of creation of $\partial X_{\mathcal{F}^{reg}}$ by blow ups of orbits of dimension* 0*, which are particular infinitely near points of $O_{\sigma_0} \in X_{\sigma_0}$.*
- *If one associates with each orbit of dimension* 0 *the corresponding petal of $\Lambda(\mathcal{F})$, then the parent map on the set of petals induces on the previous set of* 0*-dimensional orbits the restriction of the parent relation defined on the set of infinitely near points of O_{σ_0} (see Definition 1.4.31).*

Let us set a notation for the constellation created during a toric blow up process (see Definition 1.4.31):

Definition 1.5.12 Let $\mathcal{F}$ be a Newton fan. Denote by $\boxed{C_{\mathcal{F}}}$ the finite constellation above O_{σ_0} consisting of the 0-dimensional orbits O_σ, where σ varies among the regular 2-dimensional cones of the blow up process leading to the smooth toric surface $X_{\mathcal{F}^{reg}}$. It is **the constellation of the fan** $\mathcal{F}$.

Let σ be one of the cones mentioned in Definition 1.5.12. It is of the form $\mathbb{R}_+\langle f_1, f_2\rangle$, where (f_1, f_2) is a positive basis of the lattice N. Proposition 1.5.11 shows that one may represent the 0-dimensional orbit O_σ either by the edge $[f_1, f_2]$ of the lotus $\Lambda(\mathcal{F})$ or by the petal $\delta(f_1, f_2)$. How to understand the Enriques diagram of the constellation $C_{\mathcal{F}}$ using the lotus $\Lambda(\mathcal{F})$? It turns out that this may be done easily using the representing edges $[f_1, f_2]$. In order to explain it, let us introduce first the following definition (see Figs. 1.29 and 1.30):

Definition 1.5.13 Let $\delta(f_1, f_2)$ be a petal of the universal lotus $\Lambda(e_1, e_2)$. Assume that it is different from $\delta(e_1, e_2)$, which means that there exists a unique permutation (i, j) of $(1, 2)$ such that $f_j - f_i \in \sigma_0 \cap N$ (see Lemma 1.5.2). Then its **Enriques edge** is its lateral edge $[f_j, f_1 + f_2]$, that is, its unique lateral edge which extends an edge of its parent petal. The **Enriques tree** of a lotus Λ is:

- the union of the Enriques edges of all its petals different from $\delta(e_1, e_2)$, rooted at its vertex $e_1 + e_2$, whenever Λ is of dimension 2;
- the vertex $e_1 + e_2$ of $\delta(e_1, e_2)$, if $\Lambda = [e_1, e_2]$.

The **extended Enriques tree** of a lotus Λ is:

- the union of the Enriques subtree and of the lateral edge $[e_1, e_1 + e_2]$ of the base petal $\delta(e_1, e_2)$ of Λ, whenever Λ is of dimension 2;
- the lateral edge $[e_1, e_1 + e_2]$ of $\delta(e_1, e_2)$, if $\Lambda = [e_1, e_2]$.

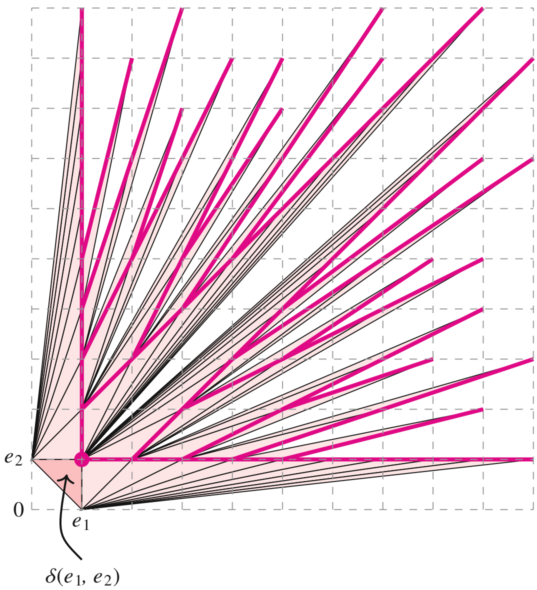


Fig. 1.29 Partial view of the Enriques subtree of the universal lotus $\Lambda(e_1, e_2)$

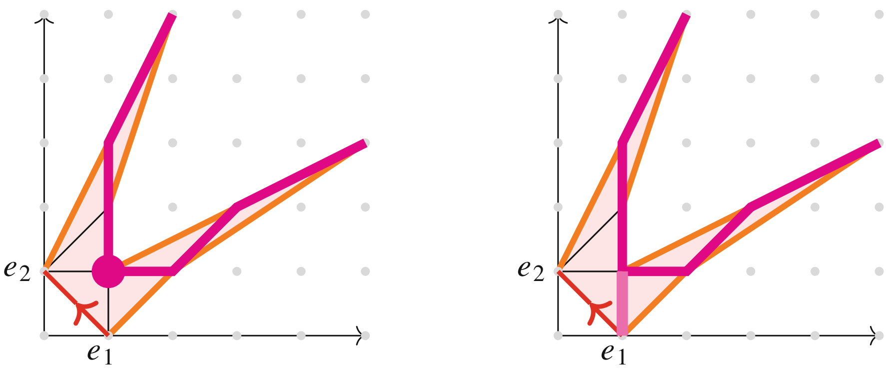


Fig. 1.30 The Enriques tree and the extended Enriques tree of the lotus $\Lambda\,(3/5, 2/1, 5/2)$

One has the following interpretation of the Enriques diagram of the constellation of the fan $\mathcal{F}$ using the lotus $\Lambda(\mathcal{F})$. It allows to understand for which reason we defined the Enriques tree of a lotus reduced to the base $[e_1, e_2]$ in the previous way:

Proposition 1.5.14 *Let $\mathcal{F}$ be a Newton fan. Then the Enriques diagram $\Gamma(C_{\mathcal{F}})$ of the constellation $C_{\mathcal{F}}$ of $\mathcal{F}$ (see Definition 1.5.12) is isomorphic to the Enriques subtree of the lotus $\Lambda(\mathcal{F})$. This isomorphism sends each orbit O_σ belonging to $C_{\mathcal{F}}$ onto the point $f_1 + f_2$, if $\sigma = \mathbb{R}_+\langle f_1, f_2\rangle$.*

Proof The basic idea is that we have a bijection between the set of infinitely near points of O_{σ_0} and the set of prime exceptional divisors created by blowing them up. Therefore, the parent binary relation may be thought as a binary relation on the set of those prime exceptional divisors. In this proposition, we restrict to the divisors which are the orbit closures $\overline{O_\rho}$, where ρ varies among the rays of the regularization $\mathcal{F}^{reg}$ of $\mathcal{F}$ which are distinct from the edges of σ_0. Each such a ray is generated by a lateral vertex of $\Lambda(\mathcal{F})$, therefore the parent binary relation among those orbit closures may be also seen as a binary relation among those lateral vertices. One may prove by induction on this number of rays, that is, on the number of petals of the associated lotus $\Lambda(\mathcal{F})$, that the pairs of related vertices are precisely those which are connected by an edge in the Enriques tree of $\Lambda(\mathcal{F})$.

The case $\mathcal{F} = \sigma_0$ corresponds to a constellation formed by O_{σ_0} alone. In this case one looks at the prime divisor created by blowing it up, that is, at $\overline{O}_{\mathbb{R}_+\langle e_1+e_2\rangle}$. This explains why we defined $\Gamma(C_{\sigma_0})$ as the vertex $e_1 + e_2$ of the petal $\delta(e_1, e_2)$. □

Remark 1.5.15 The reason why we introduced also the notion of *extended Enriques tree* in Definition 1.5.13, in addition to that of *Enriques tree*, will become clear after understanding point (8) of Theorem 1.5.29. Briefly speaking, the constellations associated to the toroidal pseudo-resolution processes have associated lotuses which are glued from lotuses of Newton fans. An analog of Proposition 1.5.14 is also true for them. The corresponding Enriques tree contains the Enriques trees of the Newton fans created by the toroidal process, but also other edges. Those supplementary edges are precisely the edges which have to be added to the Enriques tree of a Newton fan in order to get the corresponding extended Enriques tree (see Definition 1.5.26 below).

The lotus $\Lambda(\mathcal{F})$ contains also the *graph of the proximity binary relation* on the constellation $C_{\mathcal{F}}$, whose set of vertices is the given constellation, two points being joined by an edge if and only if one of them is proximate to the other one (see Definition 1.4.31):

Proposition 1.5.16 *Let $\mathcal{F}$ be a fan refining the regular cone σ_0. Then the graph of the proximity binary relation on the finite constellation $C_{\mathcal{F}}$ is isomorphic to the union of the edges of the lotus $\Lambda(\mathcal{F})$ which do not contain the vertices e_1 and e_2.*

The proof of this proposition is based on the same principles as the proof of Proposition 1.5.14 and is left to the reader.

1.5.2 Lotuses and Continued Fractions

In this subsection we explain a way to build, up to isomorphism, the lotus of a finite set of positive rational numbers in the sense of Definition 1.5.4, starting from the continued fraction expansions of its elements. Namely, given a positive rational number λ, we show how to construct an *abstract lotus* $\Delta(\lambda)$ starting from the continued fraction expansion of λ (see Definition 1.5.18) and we explain that $\Delta(\lambda)$ is isomorphic to the lotus $\Lambda(\lambda)$. Then we show how to glue two abstract lotuses $\Delta(\lambda)$ and $\Delta(\mu)$ in order to get a simplicial complex isomorphic to the lotus $\Lambda(\lambda, \mu)$ (see Proposition 1.5.23). This extends readily to arbitrary finite sets of positive rationals.

Recall first the following classical notion:

Definition 1.5.17 Let $k \in \mathbb{N}^*$ and let $a_1, \dots, a_k$ be natural numbers such that $a_1 \geq 0$ and $a_j > 0$ if $j \in \{2, \dots, k\}$. The **continued fraction** with **terms** $a_1, \dots, a_k$ is the non-negative rational number:

$$\boxed{[a_1, a_2, \dots, a_k]} := a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_k}}}.$$

Any $\lambda \in \mathbb{Q}_+^*$ may be written uniquely as a continued fraction $[a_1, a_2, \dots, a_k]$ if one imposes the constraint that $a_k > 1$ whenever $\lambda \neq 1$. One speaks then of the **continued fraction expansion** of λ. Note that its first term a_1 vanishes if and only if $\lambda \in (0, 1)$.

Definition 1.5.18 Let $\lambda \in \mathbb{Q}_+^*$. Consider its continued fraction expansion $\lambda = [a_1, a_2, \dots, a_k]$. Its **abstract lotus** $\boxed{\Delta(\lambda)}$ is the simplicial complex constructed as follows:

- Start from an affine triangle $[A_1, A_2, V]$, with vertices A_1, A_2, V.
- Draw a polygonal line $P_0 P_1 P_2 \dots P_{k-1}$ whose vertices belong alternatively to the sides $[A_1, V]$, $[A_2, V]$, and such that $P_0 := A_2$ and

$$\begin{cases} P_1 \in [A_1, V), \text{ with } P_1 = A_1 \text{ if and only if } a_1 = 0, \\ P_i \in (P_{i-2}, V) \text{ for any } i \in \{2, \dots, k-1\}. \end{cases}$$

  By convention, we set also $P_{-1} := A_1$, $P_k := V$. The resulting subdivision of the triangle $[A_1, A_2, V]$ into k triangles is the **zigzag decomposition associated with** λ.
- Decompose then each segment $[P_{i-1}, P_{i+1}]$ (for $i \in \{0, \dots, k-1\}$) into a_{i+1} segments, and join the interior points of $[P_{i-1}, P_{i+1}]$ created in this way to P_i. One obtains then a new triangulation of the initial triangle $[A_1, A_2, V]$, which is by definition the abstract lotus $\Delta(\lambda)$.

The **base** of the abstract lotus $\Delta(\lambda)$ is the segment $[A_1, A_2]$, oriented from A_1 to A_2. One orients also the other edges of $\Delta(\lambda)$ in the following way:

- $[P_{i-1}, P_i]$ is oriented from P_i to P_{i-1}, for each $i \in \{1, \dots, k-1\}$.
- An edge joining P_i to a point of the open segment (P_{i-1}, P_{i+1}) is oriented towards P_i.
- An edge contained in a segment $[V, A_j]$ is oriented towards A_j, for each $j \in \{1, 2\}$.

The abstract lotus $\Delta(\lambda)$ of $\lambda \in \mathbb{Q}_+^*$ is a simplicial complex of pure dimension 2, isomorphic to a convex polygon triangulated by diagonals intersecting only at vertices and with a distinguished oriented base. It is well-defined, up to combinatorial isomorphism of polygons triangulated by diagonals intersecting only at vertices, respecting the bases and their orientations. The orientations of its other edges are in fact determined by the orientation of the base. Those orientations will not be important in the sequel, excepted in Proposition 1.5.21 below. For this reason we do not draw them in our examples of abstract lotuses.

Example 1.5.19 Figures 1.31 and 1.32 represent the previous constructions applied to the numbers $\lambda = [4, 2, 5]$ and $\mu = [3, 2, 1, 4]$. On the left are shown the initial zigzag decompositions and on the right the final abstract lotuses $\Delta(\lambda)$ and $\Delta(\mu)$.

The abstract lotus of a positive rational number is isomorphic with its lotus:

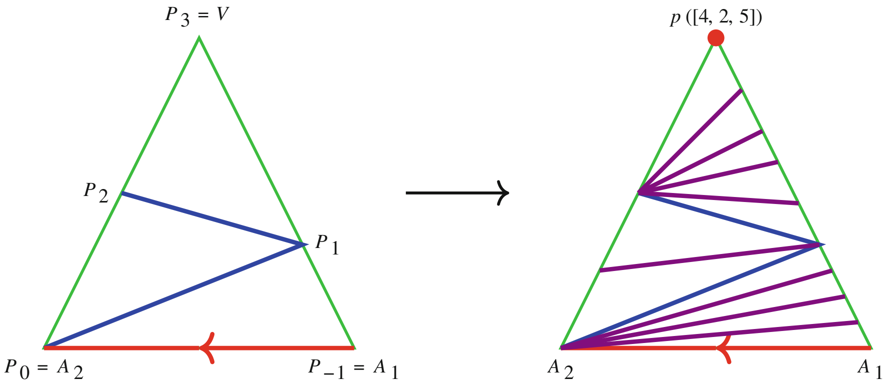


Fig. 1.31 The construction of the abstract lotus $\Delta([4, 2, 5])$

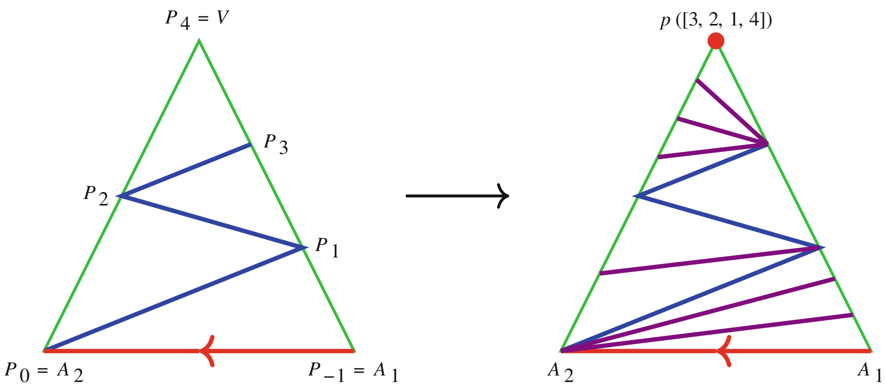


Fig. 1.32 The construction of the abstract lotus $\Delta([3, 2, 1, 4])$

Proposition 1.5.20 *There is a unique isomorphism between the lotus $\Lambda(\lambda)$ and the abstract lotus $\Delta(\lambda)$, seen as simplicial complexes with a marked point and an oriented base.*

Proof The isomorphism sends A_i to e_i for $i = 1, 2$. The proof may be done by induction on k, the number of terms in the continued fraction expansion of λ. We leave the details to the reader. □

The previous isomorphism does not always send the orientations of the edges of $\Lambda(\lambda)$ as chosen after Definition 1.5.1 onto the orientations of the edges of $\Delta(\lambda)$ as fixed in Definition 1.5.18. The possibility of defining various canonical orientations on the edges of a lotus of the form $\Lambda(\lambda)$ may be useful in applications.

The rational number $\lambda > 0$ may be recovered in the following way from the structure of the corresponding abstract lotus:

Proposition 1.5.21 *Assume that $\lambda = p_2/p_1$ with $p_1, p_2 \in \mathbb{N}^*$ coprime. Then, for each $j \in \{1, 2\}$, the positive integer p_j is equal to the number of oriented paths not containing the base $[A_1, A_2]$ and going from V to A_j inside the 1-skeleton of $\Delta(\lambda)$, oriented as in Definition 1.5.18.*

This proposition may be easily proved by induction on the number of petals of $\Delta(\lambda)$. It shows a way in which the numbers leading to the construction of a Newton lotus may be interpreted as combinatorial invariants of the lotus, seen purely as a marked simplicial complex with oriented base.

Example 1.5.22 In Fig. 1.33 is represented the case $(p_1, p_2) = (2, 3)$ of Proposition 1.5.21. We have drawn twice the lotus $\Delta(3/2) = \Delta([1, 2])$. On the right are drawn the 2 oriented paths starting from V and arriving at A_1. On the left are drawn the 3 oriented paths starting from V and arriving at A_2. We see that the constraint not to contain the base is necessary, otherwise one would obtain 2 more paths from V to A_2 by adding the base to the paths from V to A_1.

Suppose now that one has two numbers $\lambda, \mu \in \mathbb{Q}_+^*$. If $\lambda = [a_1, \ldots, a_k]$ and $\mu = [b_1, \ldots, b_l]$, let $j \in \{0, \ldots, \min\{k, l\}\}$ be maximal such that $a_i = b_i$ for all $i \in \{1, \ldots, j\}$. We may assume, up to permutation of λ and μ, that $k = j$ or $a_{j+1} < b_{j+1}$. Define then:

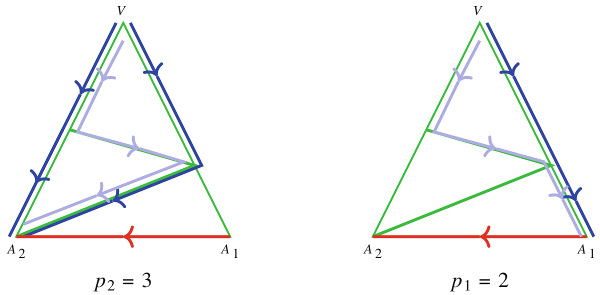


Fig. 1.33 An illustration of Proposition 1.5.21 for $p_2/p_1 = 3/2$

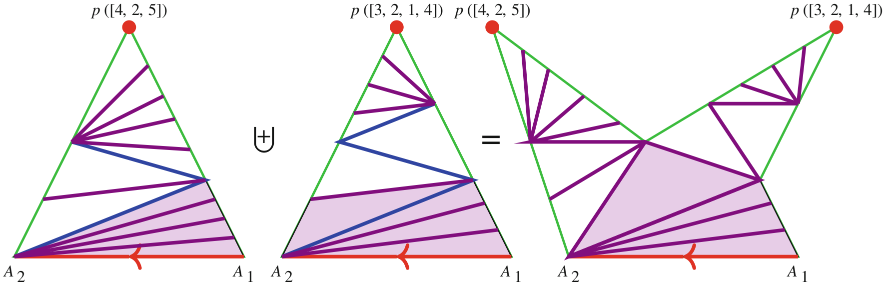


Fig. 1.34 The abstract lotus $\Delta([4, 2, 5], [3, 2, 1, 4])$

$$\boxed{\lambda \wedge \mu} = \boxed{\mu \wedge \lambda} := \begin{cases} [a_1, \ldots, a_j], \text{ if } k = j, \\ \left[a_1, \ldots, a_j, a_{j+1}\right], \text{ if } k = j + 1, \\ \left[a_1, \ldots, a_j, a_{j+1} + 1\right], \text{ if } k > j + 1. \end{cases} \tag{1.48}$$

Next proposition explains that the symmetric binary operation $\wedge$ on $\mathbb{Q}_+^*$ allows to describe the intersection of two lotuses of the form $\Lambda(\lambda)$:

Proposition 1.5.23 *For any $\lambda, \mu \in \mathbb{Q}_+^*$, one has:*

$$\Lambda(\lambda) \cap \Lambda(\mu) = \Lambda(\lambda \wedge \mu).$$

Therefore, the lotus $\Lambda(\lambda, \mu)$ is isomorphic as a simplicial complex with an oriented base to the triangulated polygon obtained by gluing $\Delta(\lambda)$ and $\Delta(\mu)$ along $\Delta(\lambda \wedge \mu)$.

Proof Assume that $\lambda = [a_1, \ldots, a_k]$. Proposition 1.5.20 shows in particular that the lotus $\Lambda(\lambda)$ has $n := a_1 + \cdots + a_k$ petals. Denote by $(\lambda_i)_{1 \le i \le n}$ the sequence of positive rationals such that the successive non-basic vertices of the petals of $\Lambda([a_1, \ldots, a_k])$ are the primitive vectors $p(\lambda_1), \ldots, p(\lambda_n)$. The sequence of continued fraction expansions of $(\lambda_i)_{1 \le i \le n}$ is:

$$[1], [2], \ldots, [a_1], [a_1, 1], [a_1, 2], \ldots, [a_1, a_2], [a_1, a_2, 1], \ldots, [a_1, \ldots, a_k]. \tag{1.49}$$

One may prove this fact at the same time as Proposition 1.5.20, by making now an induction on the number n of petals of $\Lambda([a_1, \ldots, a_k])$, instead of the number k of terms of the continued fraction.

The proposition results then by combining the previous fact with formula (1.48). □

Example 1.5.24 Let us consider the two rational numbers $\lambda = [4, 2, 5]$ and $\mu = [3, 2, 1, 4]$ of Example 1.5.19. Then $j = 0, k = 3, l = 4$, therefore $j + 1 < \min\{k, l\}$ and $\lambda \wedge \mu = [3 + 1] = 4$. The lotus $\Lambda(\lambda, \mu)$ is therefore isomorphic to the triangulated polygon with an oriented base of the right side of Fig. 1.34.

Iterating the gluing operation, one may construct an **abstract lotus** $\boxed{\Delta(\lambda_1, \ldots, \lambda_k)}$ combinatorially equivalent to any given Newton lotus $\Lambda(\lambda_1, \ldots, \lambda_k)$, seen as a triangulated polygon with marked points and oriented base. One gets an abelian monoid of (abstract) lotuses, the monoid operation $\boxed{\uplus}$ generalizing the gluing operation of Fig. 1.34. Namely, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are finite subsets of $\mathbb{Q}_+ \cup \{\infty\}$, then:

$$\boxed{\Delta(\mathcal{E}_1) \uplus \Delta(\mathcal{E}_2)} := \Delta(\mathcal{E}_1 \cup \mathcal{E}_2). \tag{1.50}$$

The neutral element of this monoid is the segment $[A_1, A_2] = \Delta(\emptyset) = \Delta(0) = \Delta(\infty) = \Delta(\{0, \infty\})$.

1.5.3 The Lotus of a Toroidal Pseudo-Resolution

In this subsection we reach a second level of explanation of the subtitle of this article, the first level having been reached in Sect. 1.5.1 above. Namely, we define a new kind of lotus by gluing the lotuses associated to the Newton fans produced by Algorithm 1.4.22 (see Definition 1.5.26). We illustrate this definition by our recurrent example (see Example 1.5.28) and by the case of an arbitrary branch (see Example 1.5.30). Finally, we show how this lotus allows to visualize many objects associated to the regularized algorithm and with the decomposition into blow ups of points of the embedded resolution produced by it (see Theorem 1.5.29).

Consider again a reduced curve singularity C on the smooth germ of surface (S, o). Fix a smooth branch L on (S, o), and run Algorithm 1.4.22. Denote as before by $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ a resulting toroidal pseudo-resolution of C. We associated to it a fan tree $(\theta_\pi(C), \mathbf{S}_\pi)$, as explained in Definition 1.4.33. One may associate an analogous fan tree $(\theta_{\pi^{reg}}(C), \mathbf{S}_{\pi^{reg}})$ to the toroidal resolution $\pi^{reg} : (\Sigma^{reg}, \partial\Sigma^{reg}) \to (S, L + L')$ defined in Sect. 1.4.3 (see Proposition 1.4.29). One sees that the trunks used in the two constructions are the same, as well as the gluing rules. What changes is that $\theta_{\pi^{reg}}(C)$ has more vertices than $\theta_\pi(C)$, those labeled by the irreducible components of the exceptional divisor of the modification $\eta : \Sigma^{reg} \to \Sigma$ which resolves the singularities of the surface Σ. Therefore:

Proposition 1.5.25 *Seen as rooted trees endowed with* $[0, \infty]$*-valued functions, the fan trees* $(\theta_\pi(C), \mathbf{S}_\pi)$ *and* $(\theta_{\pi^{reg}}(C), \mathbf{S}_{\pi^{reg}})$ *coincide. The second one contains more vertices than the first one, labeled by the irreducible components of the exceptional divisor of the minimal resolution* $\eta : \Sigma^{reg} \to \Sigma$*. The fan tree* $\theta_{\pi^{reg}}(C)$ *of the toroidal resolution* π^{reg} *is isomorphic to the dual graph of the boundary* $\partial\Sigma^{reg}$ *by an isomorphism which respects the labels of the irreducible components.*

The disadvantage of the fan tree $(\theta_{\pi^{reg}}(C), \mathbf{S}_{\pi^{reg}})$ is that one cannot see on it at a glance the partial order of the blow ups leading to the resolution $\pi^{reg} : \Sigma \to S$ of C. We explained in Sect. 1.5.1 that this order may be visualized by using the notion of lotus, for each Newton modification of the regularized algorithm obtained by replacing STEP 3 with STEP 3^{reg}. In order to visualize the blow up structure of the resolution process leading to the modification $\pi^{reg} : (\Sigma^{reg}, \partial\Sigma^{reg}) \to (S, L + L')$, we glue those lotuses using the same rules as those allowing to construct the fan tree from its trunks (see Definition 1.4.33):

Definition 1.5.26 Let C be a reduced curve singularity and (L, L') be a cross on the smooth germ (S, o). The **lotus** $\boxed{\Lambda_\pi(C)}$ **of the toroidal pseudo-resolution** π : $(\Sigma, \partial\Sigma) \to (S, L + L')$ **of** C is a simplicial complex of dimension 2 endowed with a marked oriented edge called its **base**. It is obtained by gluing the disjoint union of the lotuses $(\Lambda(\mathcal{F}_{A_i,B_i}(C)))_{i\in I}$ in the following way:

1. Label each vertex of those lotuses with the corresponding irreducible component E_k, L_j or C_l of the boundary $\partial\Sigma^{reg}$ of the smooth toroidal surface $(\Sigma^{reg}, \partial\Sigma^{reg})$.
2. Identify all the vertices of $\bigsqcup_{i\in I} \Lambda(\mathcal{F}_{A_i,B_i}(C))$ which have the same label. The result of this identification is $\Lambda_\pi(C)$ and the images inside it of the labeled points of $\bigsqcup_{i\in I} \Lambda(\mathcal{F}_{A_i,B_i}(C))$ are its vertices. We keep for each one of them the same label as in the initial lotuses.

Introduce the following terminology for the anatomy of $\Lambda_\pi(C)$:

- The **petals** of $\Lambda_\pi(C)$ are the images by the gluing morphism of the petals of the initial lotuses $(\Lambda(\mathcal{F}_{A_i,B_i}(C)))_{i\in I}$.
- Its **base** is the edge labeled by the initial cross (L, L_1) and its **basic petal** is the petal having it as base.
- Its **basic vertices** are the images inside it of the basic vertices of the 2-dimensional lotuses $(\Lambda(\mathcal{F}_{A_j,B_j}(C)))_{j\in J}$ which were not identified with other vertices.
- Its **lateral boundary** $\boxed{\partial_+\Lambda_\pi(C)}$ is the image by the gluing morphism of the union of the lateral boundaries $(\partial_+\Lambda(\mathcal{F}_{A_i,B_i}(C)))_{i\in I}$ in the sense of Definition 1.5.5.
- Its **lateral vertices** are the vertices of $\Lambda_\pi(C)$ which are not basic.
- Its **membranes** are the images inside it of the lotuses $\Lambda(\mathcal{F}_{A_i,B_i}(C))$ used to construct it.
- Its **Enriques tree** is the union of the Enriques tree of $\Lambda(\mathcal{F}_{A_1,B_1}(C))$ (remember that $(A_1, B_1) = (L, L')$) and of the extended Enriques trees of the other Newton fans $\Lambda(\mathcal{F}_{A_i,B_i}(C))$ (see Definition 1.5.13).

We introduce the notion of *Enriques tree* of a lotus in order to be able to state point (8) of Theorem 1.5.29 below. See also Remark 1.5.15.

Remark 1.5.27 The lateral boundary $\partial_+\Lambda_\pi(C)$ is a *covering subtree* of the 1-skeleton of the lotus $\Lambda_\pi(C)$, that is, a subtree containing all of its vertices. The membranes of $\Lambda_\pi(C)$ may be obtained by removing all the vertices of $\Lambda_\pi(C)$ and by taking the closures inside $\Lambda_\pi(C)$ of the connected components of the resulting topological space. The lotus $\Lambda_\pi(C)$ is a *flag complex*, that is, it may be reconstructed

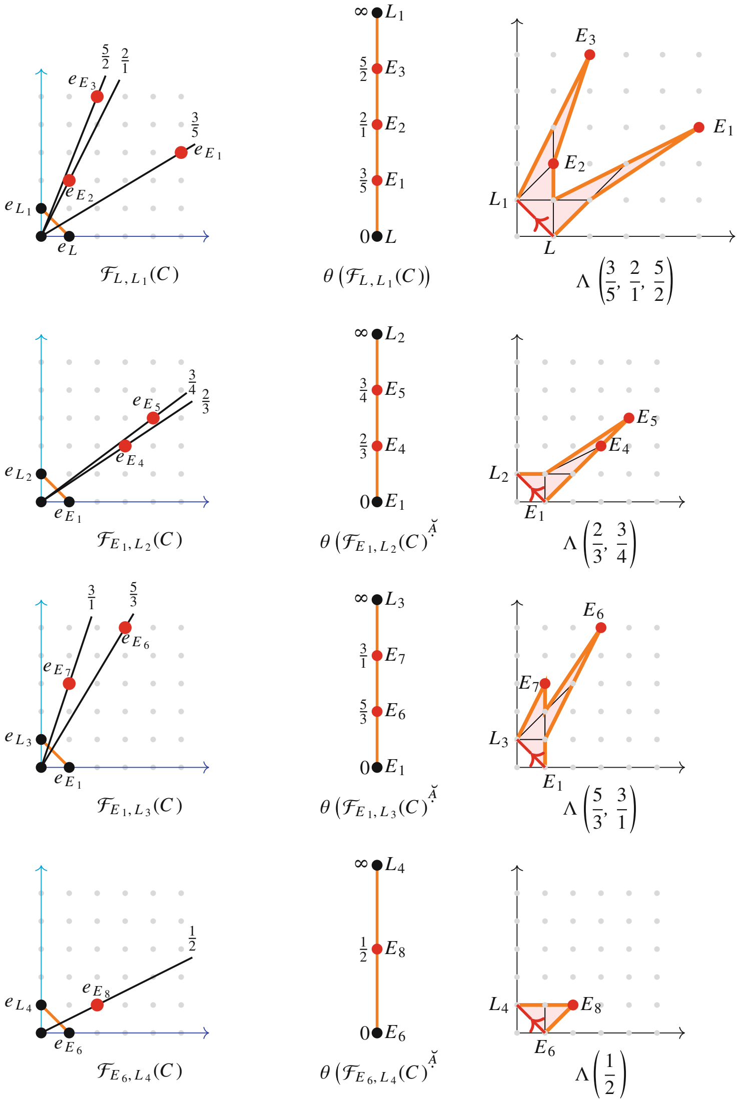


Fig. 1.35 The 2-dimensional Newton lotuses of Example 1.5.28

from its 1-skeleton by filling each complete subgraph with k vertices by a $(k-1)$-dimensional simplex. It turns out that there are such complete subgraphs only for $k \in \{1, 2\}$, for which values of k the filling process adds nothing new, and for $k = 3$, for which one gets all the petals of the lotus.

Example 1.5.28 Consider the toroidal pseudo-resolution process of Example 1.4.28. The construction of the corresponding fan tree was explained in

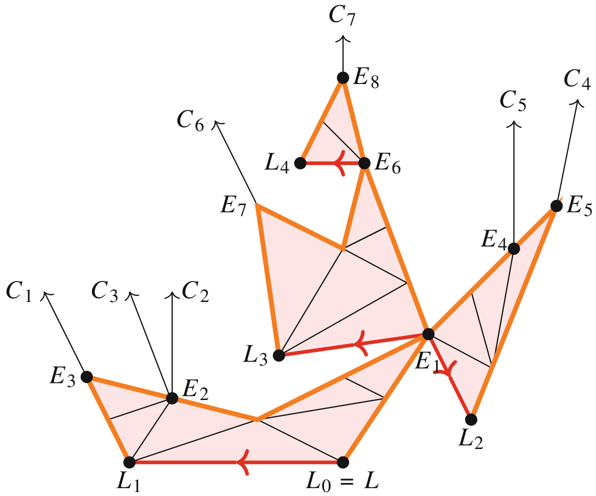


Fig. 1.36 The lotus of the toroidal pseudo-resolution of Example 1.5.28

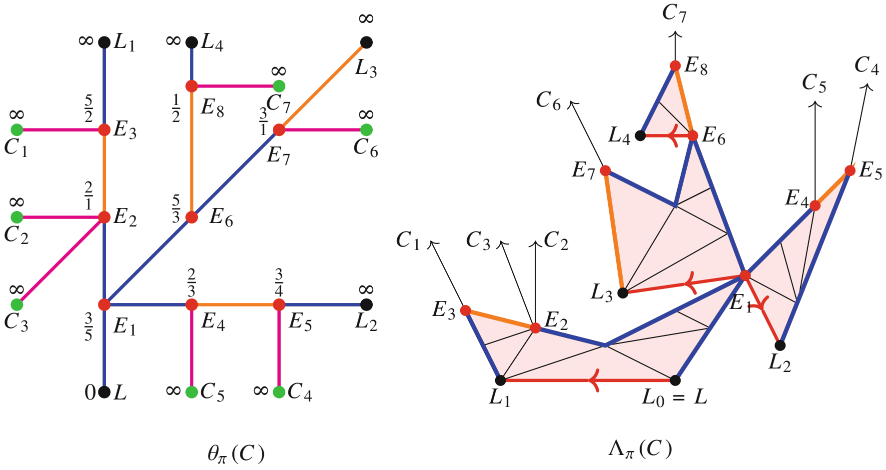


Fig. 1.37 Comparison of the fan tree and the lotus of Example 1.5.28

Example 1.4.36 and illustrated in Fig. 1.20. The left column of Fig. 1.35 represents the Newton fans produced each time one runs STEP 2 of Algorithm 1.4.22. The middle column shows the associated trunks and the right column the corresponding lotuses.

The associated lotus $\Lambda_\pi(C)$ is represented in Fig. 1.36. It has 4 membranes of dimension 2 and 7 membranes of dimension 1. The oriented base of each lotus $\Lambda(\mathcal{F}_{A_i,B_i}(C))$ used to construct it is indicated in red. The base of $\Lambda_\pi(C)$ is the oriented edge whose vertices are labeled by L and L_1. The basic vertices of $\Lambda_\pi(C)$

are those labeled by L, L_1, L_2, L_3, L_4. The part of the lateral boundary $\partial_+ \Lambda_\pi(C)$ contained in the 2-dimensional lotuses $(\Lambda(\mathcal{F}_{A_j,B_j}(C)))_{j\in J}$ is represented in orange. In order to get the whole lateral boundary, one has to add the 1-dimensional lotuses of the fans associated to the crosses at which one stops at STEP 1, that is, the segments $[E_3, C_1]$, $[E_2, C_2]$, $[E_2, C_3]$, $[E_5, C_4]$, $[E_4, C_5]$, $[E_7, C_6]$ and $[E_8, C_7]$.

In Fig. 1.37 are represented side by side the fan tree $\theta_\pi(C)$ and the lotus $\Lambda_\pi(C)$. Note that the fan tree is homeomorphic (forgetting the values of the slope function at its vertices) with the lateral boundary $\partial_+ \Lambda_\pi(C)$, by a homeomorphism which preserves the labels. This is a general fact, as formulated in point (4) of Theorem 1.5.29 below. This homeomorphism is not an isomorphism of trees because some of the edges of the fan tree—the blue ones—get subdivided in the lateral boundary of the lotus. Those are precisely the edges which correspond to the singular points of the surface Σ. One may see on the lateral boundary the structure of the exceptional divisor of the minimal resolution of each such point.

For instance, the intersection point of the curves E_1 and E_6 on Σ gets resolved by replacing that point with an exceptional divisor with two components. Their self-intersection numbers in the smooth surface Σ^{reg} are -4 and -3, as results from point (5) of Theorem 1.5.29.

Here comes the announced visualization of the structure of the decomposition of the modification $\pi^{reg} : \Sigma^{reg} \to S$ into blow ups of points in terms of the anatomy of the lotus $\Lambda_\pi(C)$ (see Definition 1.5.26):

Theorem 1.5.29 *Let C be a reduced curve singularity on the smooth germ of surface (S, o). Consider a toroidal pseudo-resolution $\pi : (\Sigma, \partial\Sigma) \to (S, L+L')$ of C produced by Algorithm 1.4.22. Its lotus $\Lambda_\pi(C)$ represents the following aspects of the associated embedded resolution $\pi^{reg} : (\Sigma^{reg}, \partial\Sigma^{reg}) \to (S, L + L')$:*

1. *Its basic edges represent the crosses with respect to which STEP 2 of Algorithm 1.4.22 was applied.*
2. *Its basic vertices represent the branches $(L_j)_{j\in J}$ of the crosses used during the process, which were introduced each time one executed STEP 2.*
3. *Its lateral vertices represent the irreducible components E_k of the exceptional divisor $(\pi^{reg})^{-1}(o)$ of the smooth modification $\pi^{reg} : \Sigma^{reg} \to S$.*
4. *Its lateral boundary $\partial_+ \Lambda_\pi(C)$ is the dual graph of the boundary divisor $\partial\Sigma^{reg}$ and is homeomorphic with the fan tree $\theta_{\pi^{reg}}(C)$, by a homeomorphism which respects the labels.*
5. *The opposite of the number of petals of $\Lambda_\pi(C)$ containing a given lateral vertex is the self-intersection number of the irreducible component of $(\pi^{reg})^{-1}(o)$ represented by that lateral vertex.*
6. *The edges of $\Lambda_\pi(C)$ represent the affine charts used in the decomposition of π into a composition of blow ups of points, and the pairs of irreducible components of $(\pi^{reg})^{-1}(\sum_{j\in J} L_j)$ which are strict transforms of crosses used at some stage of the composition of blow ups.*

7. *The graph of the proximity binary relation on the constellation which is blown up is the full subgraph of the* 1*-skeleton of the lotus* $\Lambda_\pi(C)$ *on its set of non-basic vertices.*
8. *The Enriques tree of* $\Lambda_\pi(C)$ *is the Enriques diagram of the constellation of infinitely near points at which are based the crosses introduced during the blow up process leading to the boundary* $\partial\Sigma^{reg}$.

Proof Points (1) and (2) result from Proposition 1.4.18. Points (3) and (4) result from Propositions 1.4.35, 1.5.10 and 1.5.25. Point (5) results from Corollary 1.2.28 and Proposition 1.2.37. A prototype of this result had been stated in [102, Thm. 6.2]. Points (6) and (7) result from Proposition 1.5.16. Point (8) results from Proposition 1.5.14. □

Example 1.5.30 Assume that C is a branch. Its fan tree $\theta_\pi(C)$ is a segment $[L, C]$. Denote its interior vertices by $P_1 \prec_L \cdots \prec_L P_k = P$, with $k \geq 1$. Here $\preceq_L$ denotes the total order on $\theta_\pi(C)$ induced by the root L. Consider the continued fraction expansions of their slopes $\mathbf{S}_\pi(P_j) = [p_j, q_j, \ldots]$, for all $j \in \{1, \ldots, k\}$. Then the lotus $\Lambda_\pi(C)$ is represented in Fig. 1.38. We explain in Examples 1.6.32 and 1.6.33 below how to give examples of branches which admit a pseudo-resolution process with such a lotus.

Example 1.5.31 Let us consider again our recurrent example of toroidal pseudo-resolution. Its associated lotus was represented in Fig. 1.36. In Fig. 1.39 are represented the Enriques trees and extended Enriques trees of its membranes of

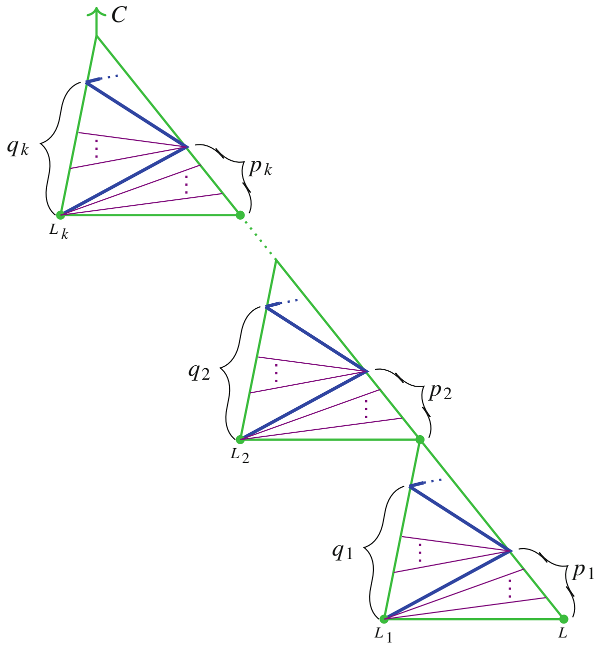


Fig. 1.38 The lotus of toroidal pseudo-resolution for one branch from Example 1.5.30

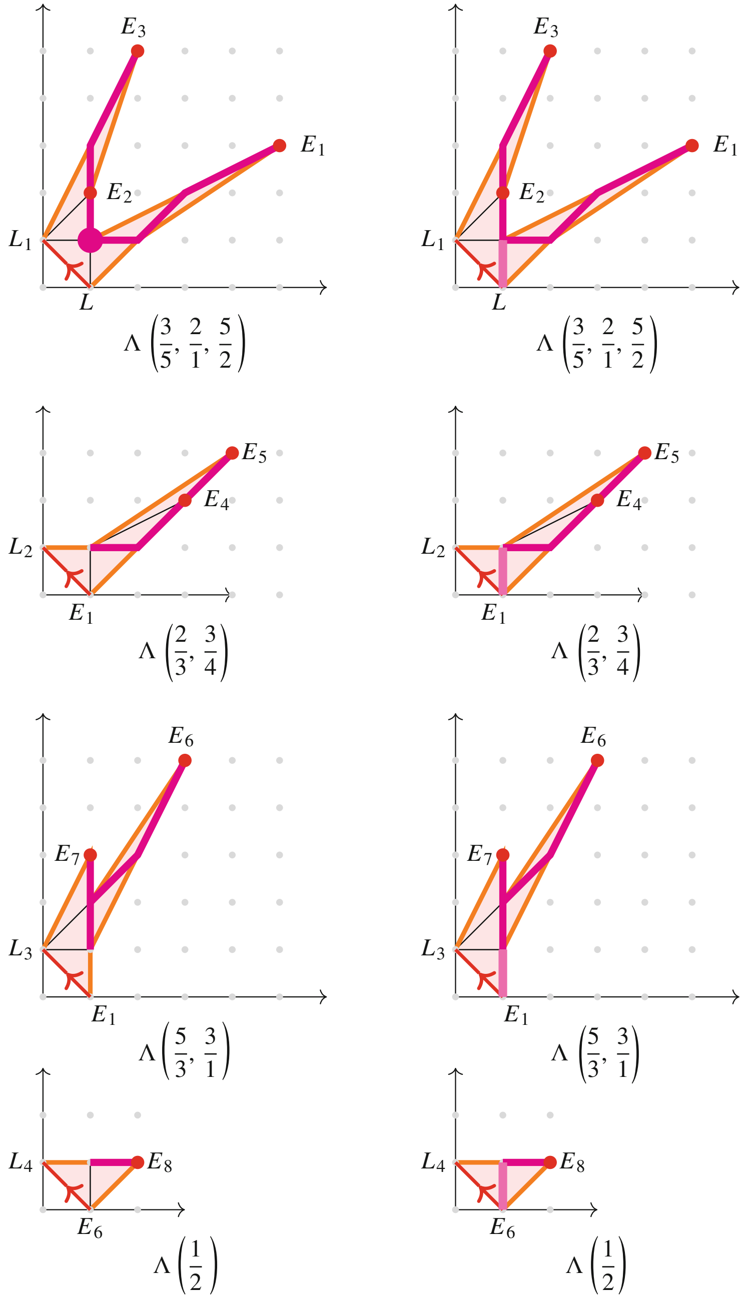


Fig. 1.39 The Enriques trees and the extended Enriques trees in Example 1.5.31

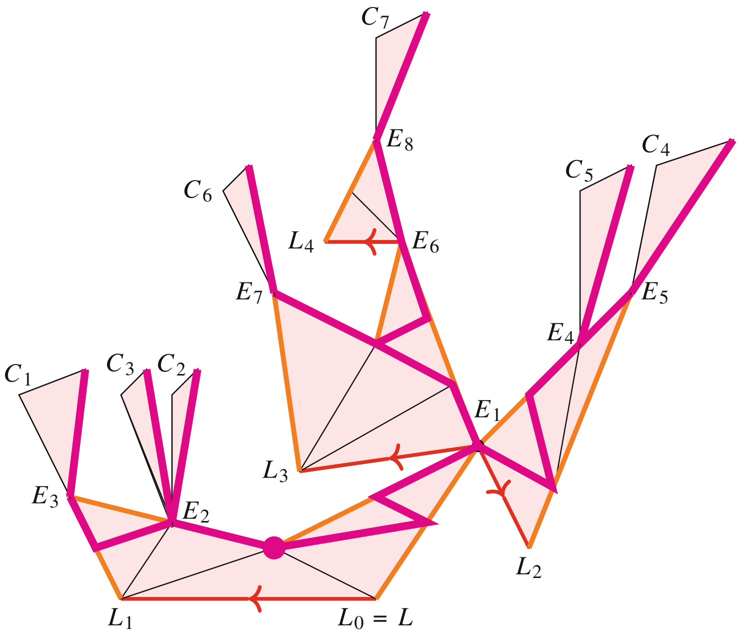


Fig. 1.40 The Enriques tree of the toroidal pseudo-resolution of Example 1.5.31

dimension 2. Finally, in Fig. 1.40 is represented its full Enriques tree. In this figure we have also represented the petals associated to the pairs (E_i, C_j), in order to draw the end edges of the Enriques tree.

1.5.4 The Dependence of the Lotus on the Choice of Completion

In this subsection we show using two examples that the lotus $\Lambda_\pi(C)$ of a toroidal pseudo-resolution process π of a plane curve singularity $C \hookrightarrow S$ depends on the choice of auxiliary curves added each time one executes STEP 2 of Algorithm 1.4.22, that is, on the choice of completion $\hat{C}_\pi$ of C (see Definition 1.4.15).

In the following two Examples 1.5.32 and 1.5.33, we build the lotuses $\Lambda_\pi(C)$ associated with two distinct embedded resolutions $\pi : (\Sigma, \partial\Sigma) \to (S, \partial S)$ of the curve singularity $C = Z(f)$, defined by the power series $f := y^2 - 2xy + x^2 - x^3 \in \mathbb{C}[[x, y]]$, relative to local coordinates (x, y) on the germ (S, o). These examples illustrate the fact that the associated lotus $\Lambda_\pi(C)$ (see Definition 1.5.26), which is based on the toroidal structure of Σ, depends on the choices of auxiliary curves done at STEP 2 of the Algorithm 1.4.22, that is, on the choice of *completion* $\hat{C}_\pi$ of C (see Definition 1.4.15). In both examples we run Algorithm 1.4.22 with $L = Z(x)$,

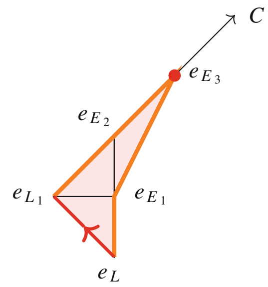


Fig. 1.41 The lotus $\Lambda_\pi(C)$ of Example 1.5.32

replacing STEP 3 by STEP 3^{reg} as we explained in Sect. 1.4.3, and taking different choices of auxiliary curves. The output, which determines the toroidal boundary on Σ, provides two different lotuses. On both of them we recognize the same weighted dual graph of the final total transform of C, thanks to point (4) of Theorem 1.5.29.

Example 1.5.32 We start the algorithm by choosing $L_1 := Z(y - x)$. The cross (L, L_1) at o is defined by the local coordinate system $(x, y_1 := y - x)$. Relative to these coordinates, C has local equation $y_1^2 - x^3 = 0$. The Newton polygon $\mathcal{N}_{L,L_1}(C)$ has only one edge and its orthogonal ray has slope 3/2, hence $\mathcal{F}_{L,L'}(C) \simeq \mathcal{F}(3/2)$. The first trunk is just the segment $[e_L, e_{L_1}]$ with its point of slope 3/2 marked.

The Newton modification $\pi := \psi^{C,reg}_{L,L_1} : (\Sigma, \partial\Sigma) \to (S, \partial S)$ has three exceptional divisors E_1, E_2 and E_3 which correspond to the rays of the regularization $\mathcal{F}^{reg}(3/2) = \mathcal{F}(1, 2, 3/2)$ of the fan $\mathcal{F}(3/2)$ of slopes 1 and 2 and 3/2 respectively. In this case, the strict transform C_{L,L_1} of C is smooth and intersects transversally the component E_3 of the exceptional divisor, that is, the Newton modification π is an embedded resolution of C. Note that when running the Algorithm 1.4.22, we include the cross (E_3, C_{L,L_1}) in the toroidal structure of the boundary of Σ.

The lotus $\Lambda_\pi(C)$ is built by gluing the lotus $\Lambda_{L,L_1}(C) = \Lambda(3/2)$ with the lotus $[e_{E_3}, C]$ associated to the cross (E_3, C_{L,L_1}), identifying the points labeled by E_3 (see Fig. 1.41).

Example 1.5.33 We start the algorithm by choosing $L_1 := Z(y)$ and the cross (L, L_1) on (S, o). The Newton polygon $\mathcal{N}_{L,L_1}(C)$ has only one edge and its orthogonal ray has slope 1, hence $\mathcal{F}_{L,L_1}(C) \simeq \mathcal{F}(1)$. The first trunk is the segment $[e_L, e_{L_1}]$ with its midpoint marked. The first lotus is just the petal $\Lambda_1 := \Lambda(1) = \delta(e_L, e_{L_1})$ with base $[e_L, e_{L_1}]$.

The Newton modification ψ^C_{L,L_1} is the usual blow up of the point o. We restrict it to the chart $\mathbb{C}^2_{v_1,v_2}$, where $x = v_1$, $y = v_1 v_2$. The strict transform $C_1 := C_{L,L_1}$ is defined in this chart by the equation $v_2^2 - 2v_2 + 1 - v_1 = 0$. The exceptional divisor $E_1 := Z(v_1)$ intersects the strict transform C_1 at the point o_1 defined by $v_2 = 1$. When running the algorithm, we have to choose a smooth branch B_2 such that (E_1, B_2) defines a cross at o_1. We set $B_2 := Z(v_2 - 1)$ and $u_1 := v_2 - 1$.

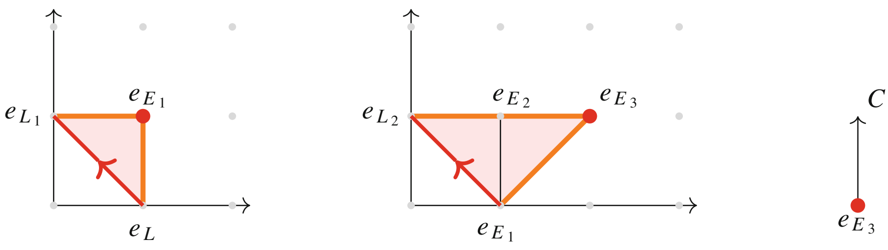
Then, the local coordinates (v_1, u_1) define the cross (E_1, B_2). We denote by L_2 the projection to S of the line $B_2 = Z(u_1)$, which is parametrized by $v_1 = t$ and $v_2 = 1$. One gets that L_2, which is parametrized by $x = t,\ y = t$, has local equation $y - x = 0$.

The strict transform C_1 has local equation $u_1^2 - v_1 = 0$. The Newton polygon $\mathcal{N}_{E_1,B_2}(C_1)$ has only one edge and its orthogonal ray has slope $1/2$, hence its associated fan is $\mathcal{F}_{E_1,B_2}(C_1) \simeq \mathcal{F}(1/2)$. The second trunk is just the segment $[e_{E_1}, e_{L_2}]$ with a marked point of slope $1/2$. The modification $\psi_{E_1,B_2}^{C_1,reg}$ defined by the regularization of this fan has two exceptional divisors E_2 and E_3 corresponding to the rays of the regularization of the fan $\mathcal{F}(1/2)$ of slopes 1 and $1/2$ respectively. When we consider the regularization of the fan $\mathcal{F}_{E_1,B_2}(C_1)$, we have to mark an additional point of slope 1 in the second trunk $[e_{E_1}, e_{L_2}]$. The associated lotus is $\Lambda_2 := \Lambda(1/2)$, with base $[e_{E_1}, e_{L_2}]$.

In this example, the composition $\pi := \psi_{E_1,B_2}^{C_1,reg} \circ \psi_{L,L_1}^{C} : (\Sigma, \partial\Sigma) \to (S, \partial S)$ is an embedded resolution of C, since the strict transform C_2 of C is smooth and intersects transversally the exceptional divisor of π at a point $o_2 \in E_3$. Notice that when running the algorithm, we have to consider also the cross (E_3, C_2) at o_2. Its trunk coincides with its associated lotus. It is just the segment $\Lambda_3 := [e_{E_3}, C]$, with no marked points.

The lotus $\Lambda_\pi(C)$ is represented in Fig. 1.43. It is obtained from $\Lambda_1,\ \Lambda_2$ and Λ_3 (see Fig. 1.42) by identifying the points with the same label.

Remark 1.5.34 The lotus $\Lambda_\pi(C)$ may be embedded canonically into the set of **semivaluations** of the local $\mathbb{C}$-algebra $\hat{\mathcal{O}}_{S,o}$ (semi-valuations are defined similarly to valuations, but dropping the last condition from Definition 1.2.19). Indeed, its base membrane $\Lambda(\mathcal{F}_{L,L_1}(C))$ embeds into the regular cone σ_0^{L,L_1} of Definition 1.3.32, which may be interpreted valuatively by associating to each $w \in \sigma_0^{L,L_1}$ the valuation ν_w defined by Eq. (1.32). Each other membrane may be similarly interpreted valuatively, and one may show that one gets in this way an embedding. Details may be found in [102, Section 7].



Λ_2 Λ_2 Λ_3

Fig. 1.42 The Newton lotuses $\Lambda_1,\ \Lambda_2$ and Λ_3 of Example 1.5.33

1.5.5 Truncated Lotuses

In this subsection we introduce an operation of *truncation* of the lotus of a toroidal pseudo-resolution of a plane curve singularity C, and we explain how to use it in order to visualize the dual graph of the total transform of C on the associated embedded resolution, as well as the Enriques diagram of the constellation of infinitely near points blown up for creating this resolution, in a way different from that formulated in point (8) of Theorem 1.5.29.

Recall first from Definition 1.5.26 the construction of the lotus $\Lambda_{\pi}(C)$ of a toroidal pseudo-resolution $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$ of the curve singularity $C \hookrightarrow S$. As stated in point (4) of Theorem 1.5.29, its lateral boundary $\partial_{+}\Lambda_{\pi}(C)$ is isomorphic to the dual graph of the boundary divisor $\partial\Sigma^{reg}$. Here Σ^{reg} denotes the minimal resolution of Σ, and $\partial\Sigma^{reg}$ is the total transform on it of the boundary divisor $\partial\Sigma$ of the toroidal surface $(\Sigma, \partial\Sigma)$. The divisor $\partial\Sigma^{reg}$ is also the total transform of the completion $\hat{C}_{\pi}$ of C relative to π, that is, the sum of the total transform of C by the smooth modification $\pi^{reg} : \Sigma^{reg} \to S$ and of the strict transforms of the branches L_j introduced while running Algorithm 1.4.22.

How to get the dual graph of the total transform of C on Σ^{reg} from the lateral boundary $\partial_{+}\Lambda_{\pi}(C)$? One has simply to remove the ends of $\partial_{+}\Lambda_{\pi}(C)$ which are labeled by the branches L_j, as well as the edges which connect them to other vertices of $\partial_{+}\Lambda_{\pi}(C)$. This *truncation operation* performed on the tree $\partial_{+}\Lambda_{\pi}(C)$ may be seen as the restriction of a similar operation performed on the whole lotus $\Lambda_{\pi}(C)$. Let us explain this truncation operation on $\Lambda_{\pi}(C)$, as well as some of its uses.

Consider first a petal $\delta(e_1, e_2)$ associated to a base (e_1, e_2) of a lattice N (see Definition 1.5.1). Its **axis** is the median $[(e_1 + e_2)/2, e_1 + e_2]$ of the petal, joining the vertex $e_1 + e_2$ to the midpoint of the opposite edge. This axis decomposes the petal into two **semipetals**.

The **semipetals** of a lotus are the semipetals of all its petals. Using this vocabulary, as well as that introduced in Definition 1.5.26 about the anatomy of lotuses of toroidal pseudo-resolutions, we may define now the operation of truncation of such a lotus:

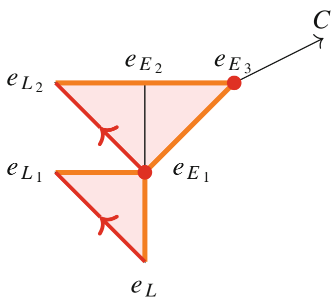


Fig. 1.43 The lotus $\Lambda_{\pi}(C)$ of Example 1.5.33

Definition 1.5.35 Let $\Lambda_\pi(C)$ be the lotus of a toroidal pseudo-resolution π of the plane curve singularity $C \hookrightarrow (S, o)$. Its **truncation** $\boxed{\Lambda_\pi^{tr}(C)}$ is the union of the axis of its basic petal, of all the semipetals which do not contain basic vertices and of all the membranes which are segments, that is, of the edges of $\Lambda_\pi(C)$ which have an extremity labeled by a branch of C. The **lateral boundary** $\boxed{\partial_+ \Lambda_\pi^{tr}(C)}$ of $\Lambda_\pi^{tr}(C)$ is the part of the lateral boundary of $\Lambda_\pi(C)$ which remains in $\Lambda_\pi^{tr}(C)$.

Truncating the lotus $\Lambda_\pi(C)$ corresponds to forgetting its points whose corresponding semivaluations depend on the choice of the branches L_j. One keeps only those semivaluations determined by the given curve singularity C and by the infinitely near points through which pass its strict transforms during the blow up process (see Remark 1.5.34). In fact, the third author had introduced truncated lotuses in [102]—under the name of *sails*—as objects which represent the combinatorial type of a blow up process of a finite constellation, without considering any supplementary branches passing through the points of the constellation.

By construction, the lateral boundary $\partial_+ \Lambda_\pi^{tr}(C)$ is isomorphic to the dual graph of the total transform $(\pi^{reg})^*(C)$. One may read again the self-intersection number of each irreducible component of the exceptional divisor of π^{reg} as the opposite of the number of petals, semi-petals and axis containing the vertex representing it.

Note that both lotuses of Figs. 1.41 and 1.43 have the same truncations. The reason is that their associated toroidal pseudo-resolutions lead to the same embedded resolution of C by regularization and that the truncated lotus is a combinatorial object encoding the decomposition of this resolution into blow ups of points (Fig. 1.44).

Example 1.5.36 For instance, in Fig. 1.45 is shown the truncation of the lotus of Fig. 1.36. Its lateral boundary is emphasized using thick orange segments. The component of the exceptional divisor represented by the unique vertex of the lotus contained in the axis has self-intersection number -4, as this vertex is contained in the axis, in two semi-petals and in one petal of $\Lambda_\pi^{tr}(C)$.

Consider now the Enriques tree of the toroidal pseudo-resolution π. Its edges are certain lateral edges of the 2-dimensional petals of $\Lambda_\pi(C)$ and of the 2-

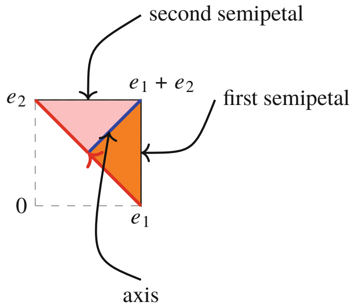


Fig. 1.44 The two semipetals and the axis of the petal $\delta(e_1, e_2)$

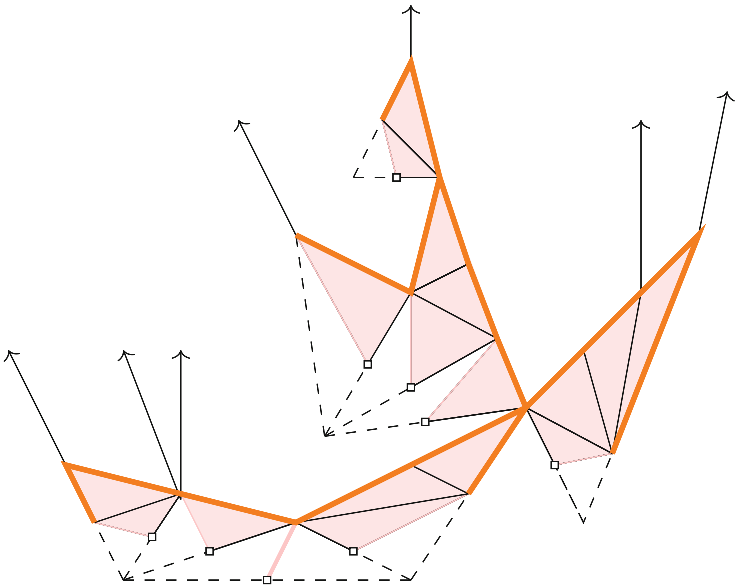


Fig. 1.45 The truncation of the lotus of Fig. 1.36 (see Example 1.5.36)

dimensional petals constructed from the 1-dimensional petals of $\Lambda_\pi(C)$ as bases (see Definition 1.5.26). For each edge $[A, B]$ of the Enriques tree, one may consider instead the homothetic segment $(1/2)[A, B]$, joining the points $(1/2)A$ and $(1/2)B$. This homothety is well-defined if one interprets the elements of the segment $[A, B]$ as valuations (see Remark 1.5.34). If both A and B are vertices of the same petal, then the segment $(1/2)[A, B]$ joins two edge midpoints of this petal. Otherwise, the interior points of the segment $(1/2)[A, B]$ are disjoint from the lotus $\Lambda_\pi(C)$.

The union of such segments $(1/2)[A, B]$—which were called *ropes* by the third author in [102]—is isomorphic to the Enriques tree of π. Therefore it is another representation of the Enriques diagram of the constellation whose blow up creates the resolution π^{reg}.

It is convenient to draw in a same picture both the truncation $\Lambda_\pi^{tr}(C)$ and the union of the ropes. For instance, for the lotus of Fig. 1.36 this union is represented on the right side of Fig. 1.46. For comparison, the Enriques tree is represented on the left side. An advantage of the right-side drawing is that the ropes whose interiors lie outside the truncation are exactly the ropes which were represented by Enriques as curved arcs. One may similarly determine from this drawing which edges go straight in Enriques' convention. For details, one may consult [102, Thm. 6.2]. Note that the *kites* of the title of [102] (in French *cerf-volants*) were the unions of truncated lotuses and of their ropes, as represented on the right side of Fig. 1.46.

Assume now that the combinatorial type of a plane curve singularity is given either using the dual graph of its total transform by an embedded resolution, weighted by the self-intersection numbers of the components of its exceptional

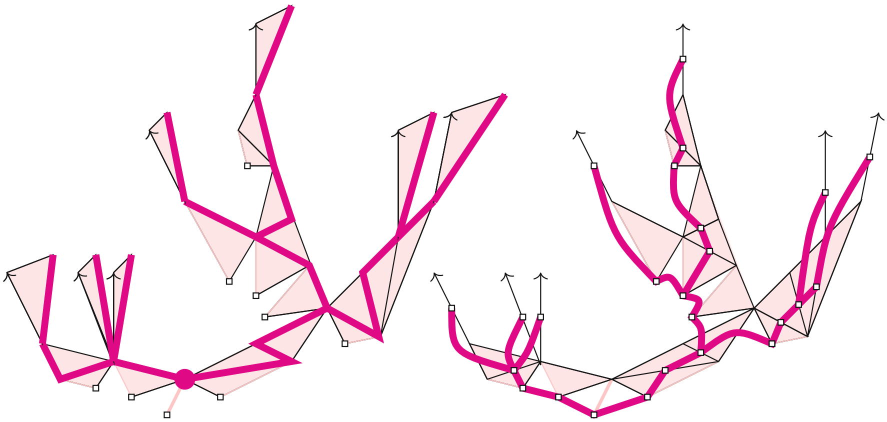


Fig. 1.46 Two ways of visualizing the Enriques tree on a truncated lotus

divisor, or using the Enriques diagram of the decomposition of the resolution morphism into blow ups of infinitely near points of o. How to get a series $f \in \mathbb{C}[[x, y]]$ defining a curve singularity with the given combinatorial type?

One may apply the following steps:

- Pass from the given tree to the associated truncated lotus. If the given tree is an Enriques diagram, it may be more convenient for drawing purposes to think about it as the union of ropes of the truncated lotus which is searched for.
- Complete the truncated lotus into a lotus having it as truncation. This step is not canonical, as shown by the comparison of Figs. 1.41 and 1.43 above.
- Proceed as in Example 1.6.29 below, by constructing the fan tree of the lotus, then the associated Eggers-Wall tree and writing finally a finite set of Newton-Puiseux series whose associated Eggers-Wall tree is isomorphic with this one.

1.5.6 Historical Comments

The study of plane curve singularities by using sequences of blow ups of points was initiated by Max Noether in his 1875 paper [89], and became common in the meantime, as shown by the works [90] of Noether, [35] of Enriques and Chisini, [126] of Du Val and [134, Sections I.2, II.2], [135] of Zariski.

Nowadays, a modification of $\mathbb{C}^2$ obtained as a sequence of blow ups of points is studied most of the time through the structure of its exceptional divisor. One encodes the incidences between its components, as well as their self-intersection numbers in a *weighted dual graph*, which is a tree (see [104] for a description of the development of this idea). When one looks at an *embedded resolution* of the plane curve singularity C, one adds new vertices to this graph, corresponding to the strict transforms of the branches of C.

The dual trees of exceptional divisors were not the first graphs associated with a process of blow ups of points. Another kind of tree, an *Enriques diagram*, encoding the proximity relation between the infinitely near points which are blown up in the process (see Definition 1.4.31), was associated with such a process in the 1917 book [35] of Enriques and Chisini. An example of an Enriques diagram, extracted from [35, Page 383], may be seen in Fig. 1.47. Details about the notion of Enriques diagram may be found in Casas' book [19] or in the third author's papers [96, 102], the second written in collaboration with Pe Pereira. The proximity relation was extended to higher dimensions by Semple in his 1938 paper [112]. Details about this generalization and about other approaches to the study of curve singularities of higher embedding dimension may be found in Campillo and Castellanos' 2005 book [18].

In order to understand the relation between the Enriques diagram of a finite constellation and the dual graph of the blow up of the constellation, the third author introduced the notion of *kite* in his 2011 paper [102]. A kite was defined by gluing *lotuses* into a *sail*, and attaching then *ropes* to this sail. The ropes were lying inside each lotus as the veins in a leaf, and they allowed to visualize the Enriques diagram. In turn, the dual graph could be visualised as the lateral boundary of the sail. A sail was composed not only of *petals*, but also of *axes* and *semi-petals*. The lotuses were also used in Castellini's thesis [25], written under the supervision of the third author. Castellini was able to do everything with petals, eliminating the use of axes,

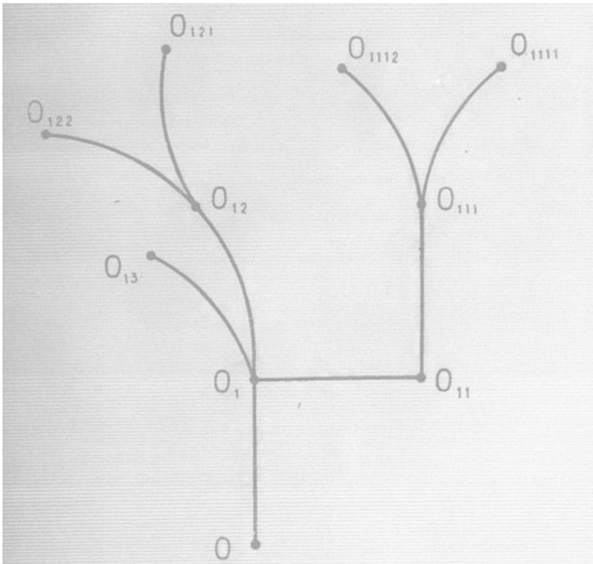


Fig. 1.47 An Enriques diagram

semi-petals and ropes, as what we call here the *Enriques tree* of a lotus proved to be more convenient to visualize the Enriques diagram. Also, the terminology was simplified, the gluing of lotuses resulting again in lotuses, instead of sails, as we do in the present paper.

It turns out that lotuses already appeared in disguise before the paper [102]. Their oldest ancestor is probably the *proximity relation*, defined in Enriques and Chisini's book [35, Page 381]. Indeed (see Theorem 1.5.29 (7)), the graph of the proximity relation among all the points whose blow up composes the embedded resolution produced by the second algorithm described in our paper may be identified with the full subgraph of the 1-skeleton of the associated lotus on the set of vertices which are not basic. The oldest drawings of such proximity graphs seem to be those of Du Val's 1944 paper [127] (see Figs. 1.48 and 1.49, in which one may also recognize what we call the "Enriques tree" of a lotus, drawn with continuous segments). Before, the proximity binary relation was related to the exceptional divisor of the associated blow up process in Barber and Zariski's 1935 paper [12] and Du Val's 1936 paper [126]. Du Val introduced the notion of *proximity matrix*, equivalent to that of proximity binary relation. In his 1939 paper [136], Zariski began a new ideal-theoretical and valuation-theoretical trend in the study of infinitely near points. A geometrical presentation of the previous approaches of study of infinitely near points was given by Lejeune-Jalabert in her 1995 paper [78].

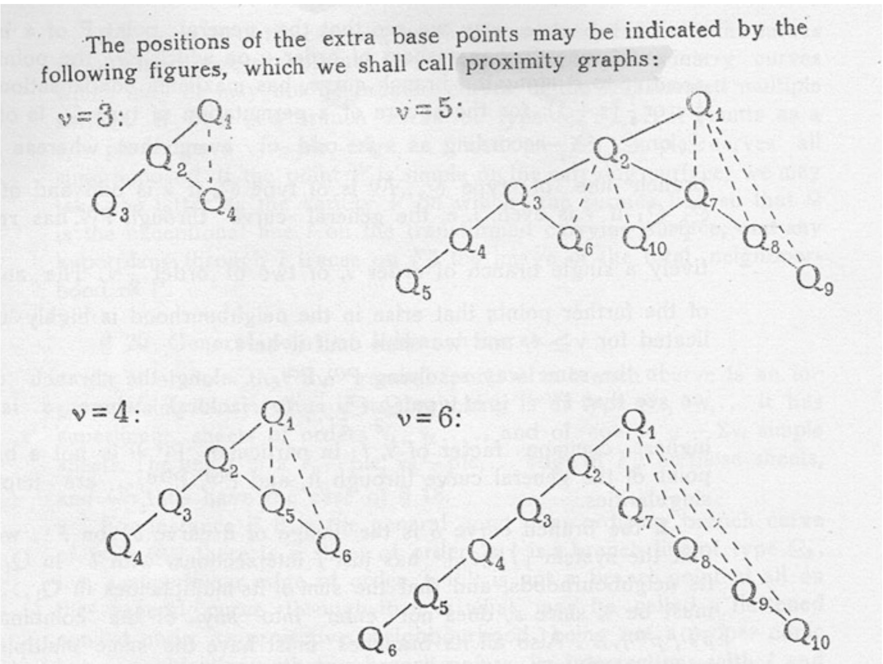


Fig. 1.48 Du Val's "proximity graphs"

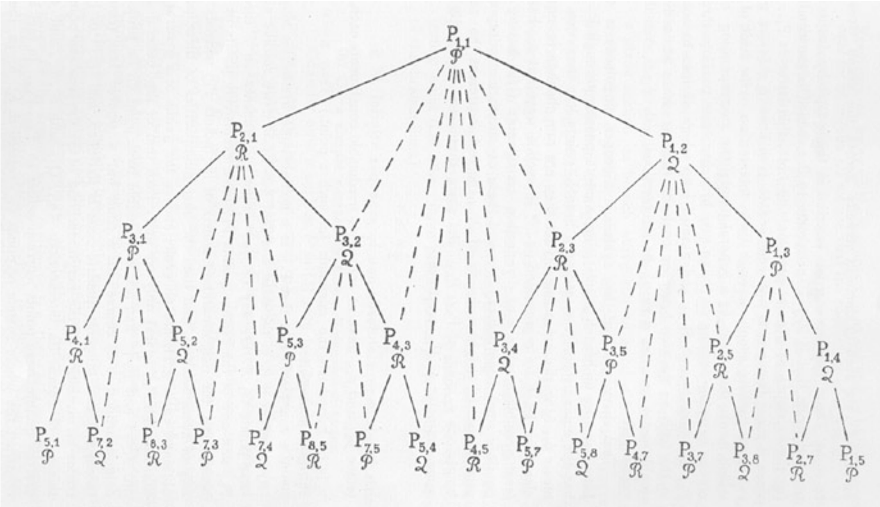


Fig. 1.49 Du Val's version of universal lotus

The graph of the proximity relation was mentioned again by Deligne in his 1973 paper [29], by Morihiko Saito in his 2000 paper [108] and by Wall in his 2004 book [131, Sections 3.5, 3.6]. One may find drawings of simple such graphs only in the first and the third reference.

Another occurrence of lotuses in disguise may be found in Schulze-Röbbecke's 1977 Diplomarbeit [111] written under the supervision of Brieskorn. In that paper are described particular divides (generic immersions of segments in a disc) obtained by applying to *branches* A'Campo's method of constructing δ-constant deformations explained in the 1974–75 papers [6] and [7]. The diagram of Fig. 1.50, extracted from page 57 of [111], indicates the general shape of the divides constructed in that paper. One may recognize inside it part of the lotus associated to a toroidal resolution process of a branch. In his already mentioned 2015 PhD thesis [25], Castellini could extend Schulze-Röbbecke's description to arbitrary plane curve singularities, using in a crucial way the notion of lotus of a blow up process.

Let us discuss now the relation of the universal lotus introduced in Definition 1.5.3 with other objects and constructions. The Enriques tree of the universal lotus $\Lambda(e_1, e_2)$ is an embedding into the cone σ_0 of almost all the *Stern-Brocot tree* defined by Graham, Knuth and Patashnik in [57, Page 116], in reference to the 1858 paper [117] of Stern and the 1860 paper [16] of Brocot. This tree represents the successive generation of the positive rational numbers starting from the sequence $(0/1, 1/0)$. At each step of the generating process, one performs the *Farey addition* $(a/b, c/d) \to (a+c)/(b+d)$ on the pairs of successive terms of the increasing sequence of rationals obtained at the previous steps. The vertices of the Stern-Brocot tree correspond bijectively with the positive rationals. For each Farey

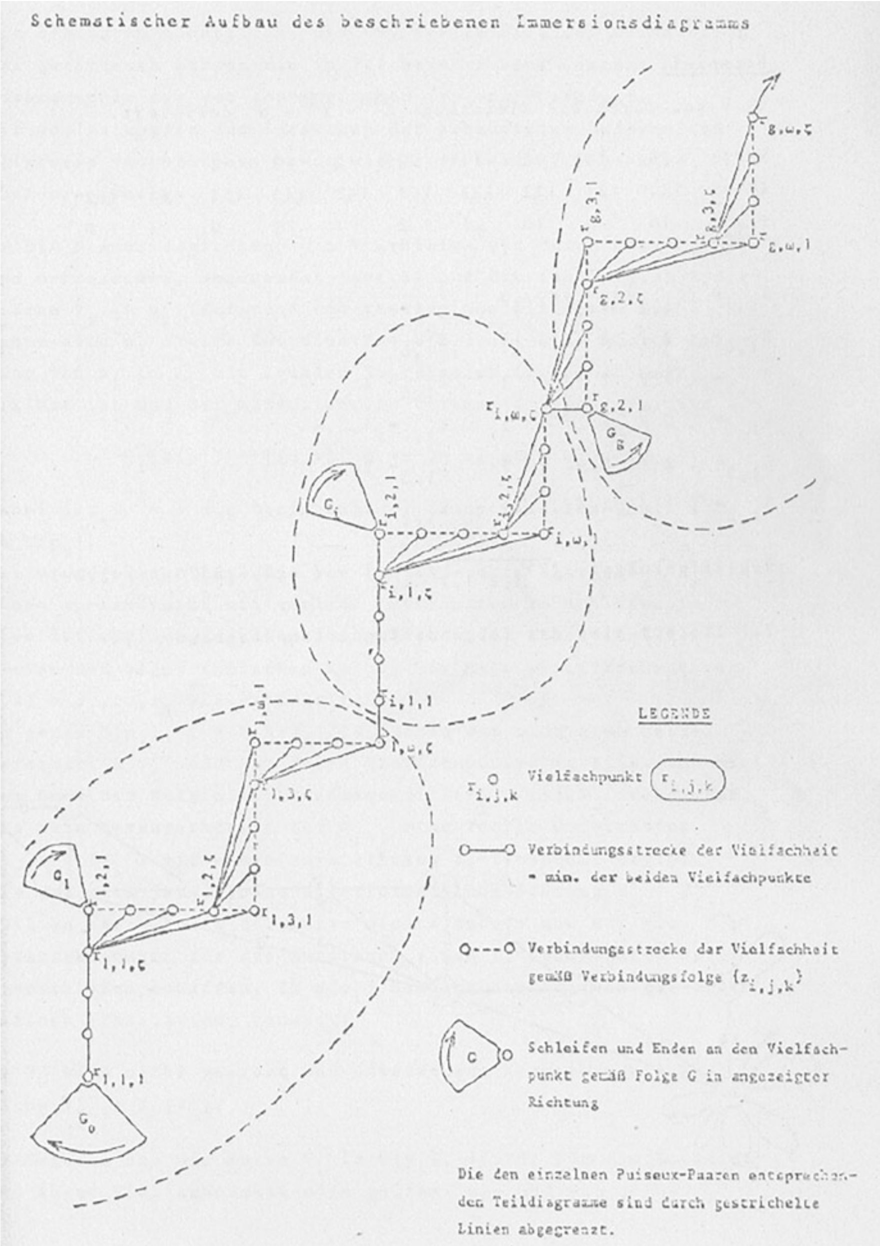


Fig. 1.50 The general shape of Schulze-Röbbecke's divides

addition $(a/b, c/d) \to (a+c)/(b+d)$ in which c/d was created after a/b, one joins the vertices corresponding to c/d and to $(a+c)/(b+d)$. The embedding of the Stern-Brocot tree represented in Fig. 1.29 is obtained by sending each vertex corresponding to $\lambda \in \mathbb{Q} \cap (0, \infty)$ to the primitive vector $p(\lambda) \in N \cap \sigma_0$ (see Notations 1.3.2) and each edge to a Euclidean segment. Another embedding in the cone σ_0 of the same part of the Stern-Brocot tree as above was described in [102, Rem. 5.7]. That embedding may be obtained from the embedding of Fig. 1.29 by applying a homothety of factor $1/2$.

The sequence of continued fractions (1.49) appearing in the proof of Proposition 1.5.23 was called the *slow approximation* ("*approximation lente*") of $[a_1, \dots, a_k]$ in Lê, Michel and Weber's paper [82, Appendice]. They used such sequences in order to describe the construction of the dual graph of the minimal embedded resolution of a plane curve singularity starting from the generic characteristic exponents of its branches and the orders of coincidence between such branches.

The *zigzag decompositions* introduced in Definition 1.5.18 are a variant of the *zigzag diagrams* of the third's author 2007 paper [101, Section 5.2]. Those diagrams allow to relate geometrically the usual continued fractions to the so-called *Hirzebruch-Jung continued fractions*. Those Hirzebruch-Jung continued fractions are the traditional tool, going back to Jung's 1908 paper [68] and Hirzebruch's 1953 paper [62], to describe the regularization of a 2-dimensional strictly convex cone. They are also crucial for the understanding of *lens spaces*, which becomes obvious once one sees that those 3-manifolds are exactly the links of toric surface singularities. See Weber's survey [133] for more details and historical explanations about the relations between lens spaces and complex surface singularities.

In [87, Section 9.1], Neumann and Wahl described a method for reconstructing the dual graph of the minimal resolution of a complex normal surface singularity whose link is an integral homology sphere from the so-called *splice diagram* of the link. This method is based on the construction of a finite sequence of rationals interpolating between two given positive rational numbers λ and μ. It may be described in the following way using lotuses of sequences of positive rational numbers:

- Construct by successive additions of petals the lotus $\Lambda(\lambda, \mu)$ as the union of $\Lambda(\lambda)$ and $\Lambda(\mu)$.
- Consider the increasing sequence of slopes of vertices of $\Lambda(\lambda, \mu)$ lying between λ and μ, that is, of vertices of the lateral boundary $\partial_+\Lambda(\lambda, \mu)$ (see Definition 1.5.5) lying on the arc joining the primitive vectors $p(\lambda)$ and $p(\mu)$ of N.

In [40, Section 2.2], Fock and Goncharov described the *tropical boundary hemisphere of the Teichmüller space of the punctured torus* as an infinite simplicial complex with integral vertices embedded in the real affine space associated to a two-dimensional affine lattice. This simplicial complex is a union of universal lotuses (see [40, Fig. 1]).

1.6 Relations of Fan Trees and Lotuses with Eggers-Wall Trees

In Sect. 1.6.1 we explain how to associate an *Eggers-Wall* tree $\Theta_L(C)$ to a plane curve singularity $C \hookrightarrow (S, o)$, relative to a smooth branch L. It is a rooted tree endowed with three structure functions, the *index* $\mathbf{i}_L$, the *exponent* $\mathbf{e}_L$ and the *contact complexity* $\mathbf{c}_L$. In Sect. 1.6.2 we express the Newton polygon of C relative to a cross (L, L') in terms of the Eggers-Wall tree $\Theta_L(C + L')$ of $C + L'$ relative to L (see Corollary 1.6.17). In Sect. 1.6.5 we prove that the fan tree $\theta_\pi(C)$ associated with a toroidal pseudo-resolution process of C is canonically isomorphic with the Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$ of the completion of C relative to this process (see Theorem 1.6.27), and we explain how to compute the triple $(\mathbf{i}_L, \mathbf{e}_L, \mathbf{c}_L)$ of functions starting from the slope function of the fan tree (see Proposition 1.6.28). As a prerequisite, in Sects. 1.6.3 and 1.6.4 we prove *renormalization formulae*, which compare the Eggers-Wall tree of C relative to L and those of its strict transform relative to the exceptional divisor of a Newton modification.

1.6.1 Finite Eggers-Wall Trees and the Universal Eggers-Wall Tree

In this subsection we define the *Eggers-Wall tree* $\Theta_L(C)$ of a *reduced* plane curve singularity $C \hookrightarrow (S, o)$ relative to a smooth branch L (see Notations 1.6.7). It is constructed from the Newton-Puiseux series of C relative to a local coordinate system (x, y) such that $L = Z(x)$ (see Definition 1.6.3), but it is independent of this choice (see Proposition 1.6.6). It is a rooted tree whose root is labeled by L and whose leaves are labeled by the branches of C. It is endowed with three functions, the *index* $\mathbf{i}_L$, the *exponent* $\mathbf{e}_L$ and the *contact complexity* $\mathbf{c}_L$, which allow to compute the characteristic exponents of the Newton-Puiseux series mentioned above and the intersection numbers of the branches of C (see Proposition 1.6.11). Finally, we introduce the *universal Eggers-Wall tree* of (S, o) relative to L (see Definition 1.6.12), as the projective limit of the Eggers-Wall trees of the plane curve singularities contained in S. For more details and proofs one may consult our papers [45, Subsection 4.3] and [46, Section 3].

Let L be a smooth branch on (S, o). Assume in the whole subsection that C is *reduced*. Let (x, y) be a local coordinate system on (S, o), such that $L = Z(x)$, and let $f \in \mathbb{C}[[x, y]]$ be a defining function of C in this coordinate system. As a consequence of the Newton-Puiseux Theorem 1.2.20, one has:

Theorem 1.6.1 *Assume that C does not contain L, that is, that x does not divide $f(x, y)$. Then there exists a finite set* $\boxed{\mathcal{Z}_x(f)}$ *of Newton-Puiseux series of* $\mathbb{C}[[x^{1/\mathbb{N}}]]$ *and a unit $u(x, y)$ of the local ring $\mathbb{C}[[x, y]]$, such that:*

$$f(x, y) = u(x, y) \prod_{\eta(x) \in \mathcal{Z}_x(f)} (y - \eta(x)). \tag{1.51}$$

The set $\mathcal{Z}_x(f)$ is obviously independent of the defining function f of C. For this reason, we will denote it instead $\mathcal{Z}_x(C)$. It is the disjoint union of the sets $\mathcal{Z}_x(C_l)$, when C_l varies among the branches of C. It allows to associate to f the following objects:

Definition 1.6.2 Let (x, y) be a local coordinate system of (S, o) such that $L = Z(x)$ and let C be a reduced curve singularity on (S, o) not containing L.

- The finite subset $\boxed{\mathcal{Z}_x(C)} := \mathcal{Z}_x(f)$ from the statement of Theorem 1.6.1 is called the set of **Newton-Puiseux roots** of C relative to x.
- The **order of coincidence** $\boxed{k_x(\xi, \xi')}$ of two Newton-Puiseux series ξ, ξ' is equal to $\nu_x(\xi - \xi')$.
- The **order of coincidence** $\boxed{k_x(C_l, C_m)}$ of two distinct branches C_l and C_m of C is the maximal order of coincidence of Newton-Puiseux roots of the two branches: $\max\{k_x(\xi, \xi'),\ \xi \in \mathcal{Z}_x(C_l),\ \xi' \in \mathcal{Z}_x(C_m)\}$.
- The **set of characteristic exponents** $\boxed{\mathrm{Ch}_x(C_l)}$ of a branch C_l of C relative to the variable x is the set of orders of coincidence of pairs of distinct Newton-Puiseux roots of it: $\{k_x(\xi, \xi'),\ \xi, \xi' \in \mathcal{Z}_x(C_l),\ \xi \neq \xi'\}$.

This shows that for each $\xi \in \mathcal{Z}_x(C_l)$, there exists some $\xi' \in \mathcal{Z}_x(C_m)$ such that $\nu_x(\xi - \xi') = k_x(C_l, C_m)$. Therefore, knowing a Newton-Puiseux root of C_l determines some Newton-Puiseux root of C_m until their order of coincidence $k_x(C_l, C_m)$. This fact motivates the following construction of a rooted tree endowed with two functions:

Definition 1.6.3 Let (x, y) be a local coordinate system such that $L = Z(x)$ and C be a reduced curve singularity on (S, o).

- The **Eggers-Wall tree** $\boxed{\Theta_x(C_l)}$ of a branch $C_l \neq L$ of C relative to x is a compact segment endowed with a homeomorphism $\boxed{\mathbf{e}_x} : \Theta_x(C_l) \to [0, \infty]$ called the **exponent function**, and with **marked points**, which are the preimages by the exponent function of the characteristic exponents of C_l relative to x. The point $(\mathbf{e}_x)^{-1}(0)$ is labeled by L and $(\mathbf{e}_x)^{-1}(\infty)$ is labeled by C_l. The **index function** $\boxed{\mathbf{i}_x} : \Theta_x(C_l) \to \mathbb{N}^*$ whose value $\mathbf{i}_x(P)$ on a point $P \in \Theta_x(C_l)$ is equal to the lowest common multiple of the denominators of the exponents of the marked points belonging to the half-open segment $[L, P)$.
- The **Eggers-Wall tree** $\boxed{\Theta_x(L)}$ is reduced to a point labeled by L, at which $\mathbf{e}_x(L) = 0$ and $\mathbf{i}_x(L) = 1$.
- The **Eggers-Wall tree** $\boxed{\Theta_x(C)}$ of C relative to x is obtained from the disjoint union of the Eggers-Wall trees $\Theta_x(C_l)$ of its branches by identifying, for each pair of distinct branches C_l and C_m of C, their points with equal exponents not greater than the order of coincidence $k_x(C_l, C_m)$. Its **marked points** are

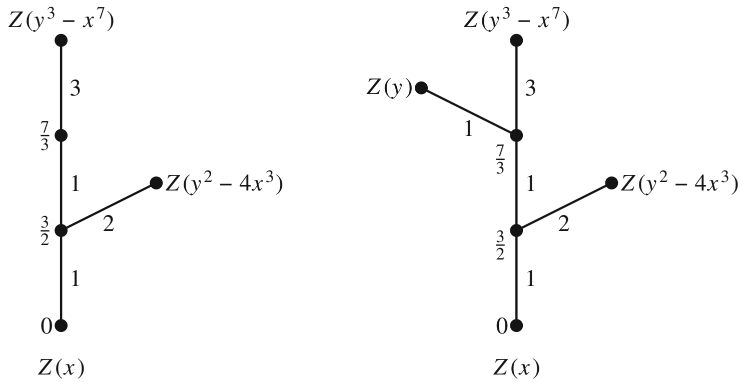


Fig. 1.51 The Eggers-Wall trees of $Z(f(x, y))$ and $Z(xyf(x, y))$ from Example 1.6.4

its ramification points and the images of the marked points of the trees $\Theta_x(C_l)$ by the identification map. Its **labeled points** are analogously the images of the labeled points of the trees $\Theta_x(C_l)$, the identification map being label-preserving. The tree is rooted at the point labeled by L. It is endowed with an **exponent function** $\boxed{\mathbf{e}_x} : \Theta_x(C) \to [0, \infty]$ and an index function $\boxed{\mathbf{i}_x} : \Theta_x(C) \to \mathbb{N}^*$ obtained by gluing the exponent functions and index functions on the trees $\Theta_x(C_l)$ respectively.

Note that, by construction, the exponent function is surjective in restriction to every segment $[L, C_l] = \Theta_x(C_l)$ of $\Theta_x(C)$ such that $C_l \neq L$ and that the ends of $\Theta_x(C)$ are labeled by the branches of C and by the smooth reference branch L. The marked points of $\Theta_x(C)$ which are images of marked points of the subtrees $\Theta_x(C_l)$ may be recovered from the knowledge of the index function, as its set of points of discontinuity. Therefore, the index function is constant on each open edge between two consecutive marked points of $\Theta_x(C)$. Moreover, it is continuous from below relative to the partial order $\preceq_L$ defined by the root L of $\Theta_x(C)$.

The Eggers-Wall tree allows to determine visually the characteristic exponents of each branch C_l. One has simply to follow the segment going from the root to the leaf representing the branch and to read all the vertex weights of the discontinuity points of the index function. In particular, if an internal vertex of such a segment is not a ramification vertex of the tree, then its exponent is necessarily a characteristic exponent of C_l.

Example 1.6.4 Consider again the plane curve singularity $C = Z(f(x, y))$ of Sect. 1.2.6. That is, $f(x, y) = (y^2 - 4x^3)(y^3 - x^7)$. Its Eggers-Wall tree is drawn on the left side of Fig. 1.51. On the right side is drawn the Eggers-Wall tree of the singularity $Z(xy(y^2 - 4x^3)(y^3 - x^7))$, which is the sum of C and of the coordinate axes.

Look at the segment joining the root to the branch $Z(y^3 - x^7)$, on the left side of Fig. 1.51. It contains two internal vertices, with exponents $3/2$ and $7/3$. The vertex of exponent $7/3$ is not a ramification vertex of the tree, therefore $7/3$ is a characteristic exponent of this branch. In turn, $3/2$ is not a characteristic exponent of this branch, as the value of the index function does not increase when crossing the corresponding vertex. Note that, by contrast, it increases when crossing the same vertex on the segment joining the root to the leaf corresponding to the branch $Z(y^2 - 4x^3)$, which shows that $3/2$ is a characteristic exponent of that branch.

We have represented both the Eggers-Wall tree of C and of its union with the coordinate axes in order to show that *the second one is homeomorphic to the dual graph of the total transform of the union by its minimal embedded resolution*, while our example shows that this is not true if one looks at the total transform of C alone (see Fig. 1.7). The previous homeomorphism is a general phenomenon, valid for any plane curve singularity, as seen by combining Proposition 1.4.35 and Theorem 1.6.27 below. Note that in full generality one needs to add to C more branches than simply the coordinate axes, considering a *completion* in the sense of Definition 1.4.15.

Example 1.6.5 Consider a plane curve singularity C whose branches C_i, $1 \leq i \leq 3$, are defined by the Newton-Puiseux series ξ_i, where:

$$\xi_1 = x^{7/2} - x^4 + 2x^{17/4} + x^{14/3}, \quad \xi_2 = x^{5/2} + x^{8/3}, \quad \xi_3 = x^2.$$

The sets of characteristic exponents of the branches are $\mathrm{Ch}_x(C_1) = \{7/2,\ 17/4,\ 14/3\}$, $\mathrm{Ch}_x(C_2) = \{5/2, 8/3\}$, $\mathrm{Ch}_x(C_3) = \emptyset$. One has $k_x(C_1, C_2) = 5/2$, $k_x(C_1, C_3) = k_x(C_2, C_3) = 2$. The Eggers-Wall trees $\Theta_x(C_1)$ and $\Theta_x(C)$ relative to x are drawn in Fig. 1.52. We represented the value of the corresponding exponent near each marked or labeled point, and the value of the corresponding index function near each edge.

In fact, the objects introduced in Definition 1.6.3 depend only on C and L, not on the coordinate system (x, y) such that $L = Z(x)$ (see [45, Proposition 103]):

Proposition 1.6.6 *Let (x, y) be a local coordinate system such that $L = Z(x)$ and C be a reduced curve singularity on (S, o). Then the tree $\Theta_x(C)$ endowed with the pair of functions $(\mathbf{i}_x, \mathbf{e}_x)$ is independent of the choice of local coordinate system such that $L = Z(x)$.*

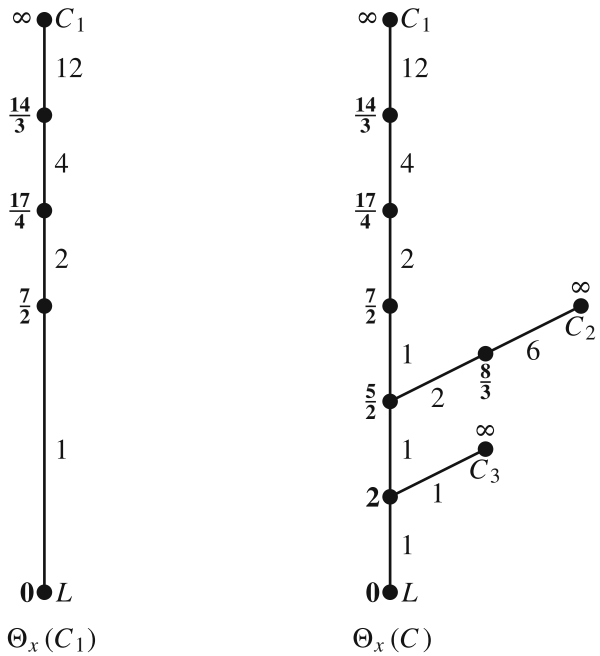


Fig. 1.52 The Eggers-Wall tree of the curve singularities C_1 and C of Example 1.6.5

Proposition 1.6.6 motivates us to introduce the following notations:

Notations 1.6.7 Let L be a smooth branch and C be a reduced curve singularity on (S, o). We denote $\boxed{(\Theta_L(C), \mathbf{i}_L, \mathbf{e}_L)}$:= $(\Theta_x(C), \mathbf{i}_x, \mathbf{e}_x)$, for any local coordinate system (x, y) on (S, o) such that $L = Z(x)$. We say that this rooted tree endowed with two structure functions is the **Eggers-Wall tree of** C **relative to** L.

Remark 1.6.8 Let L be a smooth branch and C be a reduced curve singularity on (S, o). Then for any point $Q \in \Theta_L(C)$, we have:

$$\mathbf{i}_L(Q) = \min\{\mathbf{i}_L(A)\,,\ A \text{ is a branch on } S \text{ such that } Q \preceq_L A\}, \tag{1.52}$$

where $Q \preceq_L A$ has a meaning in the Eggers-Wall-tree $\Theta_L(C + A) \supseteq \Theta_L(C)$. Indeed, if $Q \preceq_L C_l$ for a branch C_l of C, and if B is a branch on S parametrized by the truncation of a Newton-Puiseux series $\xi \in Z_x(C_l)$, obtained by keeping only the terms of ξ of exponent $< \mathbf{e}_L(Q)$, then $Q \preceq_L B$ and $\mathbf{i}_L(Q) = \mathbf{i}_L(B)$.

The exponent function and the index function determine a third function on the tree $\Theta_L(C)$, the *contact complexity* function (see [46, Def. 3.19]):

Definition 1.6.9 Let C be a reduced curve singularity on (S, o). The **contact complexity function** $\boxed{\mathbf{c}_L}$: $\Theta_L(C) \to [0, \infty]$ is defined by the formula:

$$\mathbf{c}_L(P) := \int_L^P \frac{d\mathbf{e}_L}{\mathbf{i}_L}.$$

Note that in restriction to a segment $[L, C_l] = \Theta_L(C_l)$ of $\Theta_L(C)$, the contact complexity function is a bijection $[L, C_l] \to [0, \infty]$.

Remark 1.6.10 It follows immediately from Definition 1.6.9 that the contact complexity function together with the index function determine the exponent function by the following formula:

$$\mathbf{e}_L(P) = \int_L^P \mathbf{i}_L d\mathbf{c}_L. \tag{1.53}$$

The importance of the contact complexity function stems from the following property, which in different formulation goes back at least to Smith [115, Section 8], Stolz [118, Section 9] and Max Noether [90]:

Proposition 1.6.11 *Let L be a smooth branch and C be a reduced curve singularity on (S, o), not containing L. Let A and B be two distinct branches of C. Denote by $\boxed{A \wedge_L B}$ the infimum of the points of $\Theta_L(C)$ labeled by A and B, relative to the partial order $\preceq_L$ defined by the root L. Then:*

$$\mathbf{c}_L(A \wedge_L B) = \frac{A \cdot B}{(L \cdot A) \cdot (L \cdot B)}. \tag{1.54}$$

Proof One may find a proof of Proposition 1.6.11 in [131, Thm. 4.1.6]. Let us just sketch the main idea. Fix a local coordinate system (x, y) on (S, o), such that $L = Z(x)$. Start from a normalization of the branch A of the form $u \to (u^n, \zeta(u))$ (see the explanations leading to formula (1.2)). Therefore, $\zeta(x^{1/n})$ is a Newton-Puiseux root of A. By Theorem 1.6.1, one has a defining function of the branch B of the form $\prod_{\eta(x) \in \mathcal{Z}_x(B)} (y - \eta(x))$. Proposition 1.2.8 implies that:

$$A \cdot B = \nu_u \left(\prod_{\eta(x) \in \mathcal{Z}_x(B)} (\zeta(u) - \eta(u^n)) \right) = \sum_{\eta(x) \in \mathcal{Z}_x(B)} \nu_u \left(\zeta(u) - \eta(u^n) \right).$$

The finite multi-set of rational numbers whose elements are summed may be expressed in terms of the characteristic exponents of A and B which are not greater than the order of coincidence of A and B. A little computation finishes the proof.

□

If C and D are two reduced plane curve singularities on (S, o), with $C \subseteq D$, then by construction one has a natural embedding of rooted trees $\Theta_L(C) \subseteq \Theta_L(D)$. The uniqueness of the segment joining two points of a tree allows to define a canonical retraction $\Theta_L(D) \to \Theta_L(C)$. One may consider then either the *direct limit of the previous embeddings*, or the *projective limit of the previous retractions*, for varying C and D. Both limits have natural topologies. The direct limit, which may be thought simply as the union of all Eggers-Wall trees $(\Theta_L(C))_C$, is not compact, but the projective limit is compact. It is in fact a compactification of the direct limit.

For this reason, the projective limit is more suitable in many applications. Let us introduce a special notation for this notion, which will be used in Sect. 1.6.3 below.

Definition 1.6.12 Let L be a smooth branch on (S, o). The **universal Eggers-Wall tree** $\boxed{\Theta_L}$ **of** (S, o) **relative to** L is the projective limit of the Eggers-Wall trees $\Theta_L(C)$ of the various reduced curve singularities C on (S, o), relative to the natural retraction maps $\Theta_L(D) \to \Theta_L(C)$ associated to the inclusions $C \subseteq D$.

1.6.2 From Eggers-Wall Trees to Newton Polygons

In this subsection we explain how the Newton polygon $\mathcal{N}_{L,L'}(C)$ of a plane curve singularity C relative to the cross (L, L') (see Definition 1.4.14) may be determined from the Eggers-Wall tree $\Theta_L(C + L')$ (see Corollary 1.6.17).

The **Minkowski sum** $\boxed{K_1 + K_2}$ of two subsets of a real vector space is the set of sums $v_1 + v_2$, where each v_i varies independently among the elements of K_i. It is a commutative and associative operation. When both subsets are convex, their Minkowski sum is again convex.

The following property is classical and goes back at least to Dumas' 1906 paper [30, Section 3] (where it was formulated in a slightly different, p-adic, context):

Proposition 1.6.13 *If C and D are germs of effective divisors on (S, o), then:*

$$\mathcal{N}_{L,L'}(C + D) = \mathcal{N}_{L,L'}(C) + \mathcal{N}_{L,L'}(D).$$

Proof This is a direct consequence of formula (1.35) and Proposition 1.4.12. $\square$

One may extend the notion of Newton polygon to series in two variables with non-negative rational exponents whose denominators are bounded. They have again only a finite number of edges. The simplest Newton polygons are those with at most one compact edge:

Definition 1.6.14 Assume that $a, b \in \mathbb{Q}_+^*$. One associates them the following **elementary Newton polygons** (see Fig. 1.53):

$$\boxed{\left\{\frac{a}{b}\right\}} := \mathcal{N}(x^a + y^b), \quad \boxed{\left\{\frac{a}{\infty}\right\}} := \mathcal{N}(x^a), \quad \boxed{\left\{\frac{\infty}{b}\right\}} := \mathcal{N}(y^b).$$

The quotient a/b is the **inclination** of the elementary Newton polygon $\left\{\frac{a}{b}\right\}$.

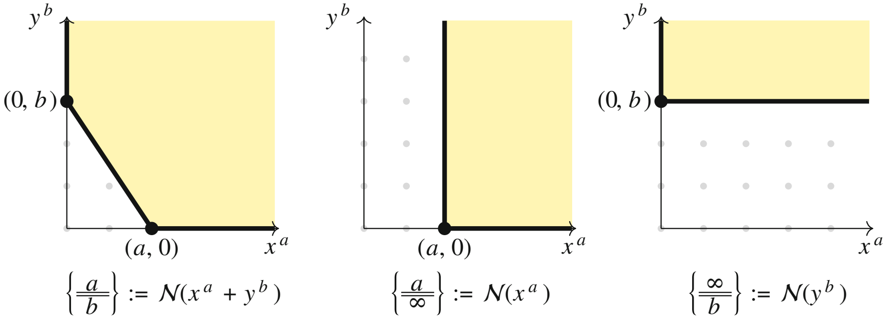


Fig. 1.53 The elementary Newton polygons $\{\frac{a}{b}\}$, $\{\frac{a}{\infty}\}$, $\{\frac{\infty}{b}\}$

Note that for any $a \in \mathbb{Q}_+^* \cup \{\infty\}$, $b \in \mathbb{Q}_+^*$ and any $d \in \mathbb{N}^*$, one has: $d\left\{\frac{a}{b}\right\} = \left\{\frac{da}{db}\right\}$, where the left-hand side is the Minkowski sum of $\left\{\frac{a}{b}\right\}$ with itself d times. This allows to write:

$$\left\{\frac{a}{b}\right\} = b\left\{\frac{a/b}{1}\right\} \tag{1.55}$$

whenever $b \in \mathbb{N}^*$. The elementary Newton polygons are generators of the semigroup of Newton polygons, with respect to Minkowski sum. In fact one has more:

Proposition 1.6.15 *Each Newton polygon $\mathcal{N}$ may be written in a unique way, up to permutations of the terms, as a Minkowski sum of elementary Newton polygons with pairwise distinct inclinations. Their compact edges are translations of the compact edges of $\mathcal{N}$.*

Proof This is a consequence of the following property, which in turn may be proved by induction on $p \in \mathbb{N}^*$: *If $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_p$ are elementary Newton polygons with finite non-zero strictly increasing inclinations, then their Minkowski sum $\mathcal{N}$ has exactly p compact edges which are translations of the compact edges of $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_p$. Moreover, they are met in this order when one lists them starting from the unique vertex of $\mathcal{N}$ lying on the vertical axis.* □

The next proposition explains how to compute the Newton polygon of a branch C relative to a cross (L, L'), starting from the Eggers-Wall tree of $C + L'$ relative to L:

Lemma 1.6.16 *Let (L, L') be a cross and let $C \neq L$ be a branch on (S, o). Then the Newton polygon $\mathcal{N}_{L,L'}(C)$ may be expressed as follows in terms of the Eggers-Wall tree $(\Theta_L(C + L'), \mathbf{e}_L, \mathbf{i}_L)$:*

$$\mathcal{N}_{L,L'}(C) = \mathbf{i}_L(C) \left\{ \frac{\mathbf{e}_L(C \wedge_L L')}{1} \right\}.$$

The fan $\mathcal{F}_{L,L'}(C)$ has a unique ray in the interior of the cone σ_0, and its slope is equal to $\mathbf{e}_L(C \wedge_L L')$. That is:

$$\mathcal{F}_{L,L'}(C) = \mathcal{F}\big(\mathbf{e}_L(C \wedge_L L')\big).$$

Proof This is a consequence of Theorem 1.6.1. Indeed, let $f \in \mathbb{C}[[x]][y]$ be a defining function for C relative to a local coordinate system (x, y) defining the cross (L, L'). We know that its set of Newton-Puiseux roots $\mathcal{Z}_x(f)$ has $C \cdot L = \mathbf{i}_L(C)$ elements. All of them have the same support, since C is a branch, which implies that they form a single orbit under the Galois action of multiplication of $x^{1/\mathbf{i}_L(C)}$ by the group of $\mathbf{i}_L(C)$-th roots of 1. The order of any such series is equal to $k_x(L', C) = \mathbf{e}_L(C \wedge_L L')$. We deduce from Proposition 1.6.13 that the Newton polygon $\mathcal{N}_{L,L'}(C)$ is equal to the Minkowski sum of the factors of f in formula (1.51). The first assertion follows since the Newton polygon of $y - \eta(x)$ is equal to $\left\{ \frac{\mathbf{e}_L(C \wedge_L L')}{1} \right\}$, for any series $\eta(x) \in \mathcal{Z}_x(f)$, and then by taking into account formula (1.55). The second assertion is an immediate consequence of the first one. □

As a corollary we get the announced expression of the Newton polygon relative to (L, L') of a reduced curve singularity C in terms of the Eggers-Wall tree $\Theta_L(C+L')$ of $C + L'$ relative to L:

Corollary 1.6.17 *Let (L, L') be a cross and let C be a reduced curve singularity on (S, o) not containing the branch L. The Newton polygon $\mathcal{N}_{L,L'}(C)$ of the germ C with respect to the cross (L, L') is equal to the Minkowski sum:*

$$\sum_l \mathbf{i}_L(C_l) \left\{ \frac{\mathbf{e}_L(C_l \wedge_L L')}{1} \right\}, \tag{1.56}$$

where C_l runs through the branches of C.

Proof By Proposition 1.6.13, the Newton polygon $\mathcal{N}_{L,L'}(C)$ is the Minkowski sum of the Newton polygons of its branches. One uses then Lemma 1.6.16 for each such branch. □

Note that the previous result extends to not necessarily reduced curve singularities C if one defines their Eggers-Wall tree as the Eggers-Wall tree of their reduction, each leaf being endowed with the multiplicity of the corresponding branch in the divisor C. Then, in the right-hand side of Eq. (1.56), each branch C_l has to be counted with its multiplicity.

1.6.3 Renormalization of Eggers-Wall Trees

Let (L, L') be a cross on (S, o). In this subsection we will denote sometimes by $\boxed{\Theta_{o,L}(C)}$ the Eggers-Wall tree denoted before by $\Theta_L(C)$, in order to emphasize the point at which it is based. Indeed, we want to compare the previous tree with the Eggers-Wall tree $\Theta_{o_w,E_w}(C_w)$ of the germ (C_w, o_w) of the strict transform C_w of C at a smooth point o_w of the exceptional divisor E_w of a Newton modification relative to the cross (L, L'), with respect to the germ at o_w of the exceptional divisor E_w itself. Notice that if C is a reduced curve, then the strict transform C_w may consist of several germs of curves, one for each point of intersection of C_w with E_w. We show that the universal Eggers-Wall tree Θ_{o_w,E_w} in the sense of Definition 1.6.12 embeds naturally in the universal Eggers-Wall tree $\Theta_{o,L}$ and we explain how to relate their triples of structure functions (index, exponent and contact complexity). We conceive the passage from $\Theta_{o,L}(C)$ to $\Theta_{o_w,E_w}(C_w)$ as a *renormalization operation*, which explains the title of this subsection. We will give another proof of the renormalization Proposition 1.6.22 in Sect. 1.6.4, in terms of Newton-Puiseux series.

Let us fix a cross (L, L') on (S, o). Fix also a weight vector $\boxed{w} = \boxed{c_w} e_1 + \boxed{d_w} e_2 \in \sigma_0 \cap N_{L,L'}$. Denote by $\boxed{\pi_w : (S_w, \partial S_w) \to (S, L + L')}$ the modification obtained by subdividing σ_0 along the ray $\mathbb{R}_+ w$. If A is a branch on S, we denote by $\boxed{A_w}$ the strict transform of A by π_w. We look at the modification π_w in the toroidal category, relative to the boundaries $\partial S := L + L'$ and $\partial S_w := L_w + E_w + L'_w$, where $\boxed{E_w}$ is the exceptional divisor of the morphism π_w.

Denote by $\boxed{W}$ the point of $\Theta_L(L')$ corresponding to w, that is, the unique point of $\Theta_L(L')$ whose exponent is the slope of the ray $\mathbb{R}_+ w$ in the basis (e_1, e_2):

$$\mathbf{e}_L(W) = \frac{d_w}{c_w}. \tag{1.57}$$

Since (L, L') is a cross on (S, o) and $W \in \Theta_L(L')$, one has that $\mathbf{i}_L(W) = 1$. Therefore, by Definition 1.6.9, the contact complexity of W is:

$$\mathbf{c}_L(W) = \frac{d_w}{c_w}. \tag{1.58}$$

Recall that $A \wedge_L B$ denotes the infimum of the points A and B of the universal Eggers-Wall tree $\Theta_{o,L}$ relative to the partial order $\preceq_L$ induced by the root L. We need the following lemma:

Lemma 1.6.18 *Let A be a branch on (S, o) different from L, L'. The following properties are equivalent:*

1. *The strict transform A_w of A by π_w intersects $E_w \setminus (L_w \cup L'_w)$.*
2. *The fan $\mathcal{F}_{L,L'}(A)$ is the subdivision of σ_0 along the ray $\mathbb{R}_+ w$.*

3. $A \wedge_L L' = W$.

In addition, if these properties hold, then the order of vanishing of A along E_w is equal to $d_w \mathbf{i}_L(A)$ and the intersection number $E_w \cdot A_w$ is $\mathbf{i}_L(A)/c_w$.

Proof The equivalence of these three properties is immediate from Propositions 1.4.18 and 1.6.16. Recall that the order of vanishing $\mathrm{ord}_{E_w}(A)$ is by definition the multiplicity of E_w in the divisor $(\pi_w^* L)$, that is, the value taken by the divisorial valuation ord_{E_w} defined by E_w on a defining function f of A. Thanks to Proposition 1.4.18, this value is equal to $\mathrm{trop}^A_{L,L'}(w)$, which may be written $d_w \mathbf{i}_L(A)$ by Lemma 1.6.16. By Proposition 1.4.18, $E_w \cdot A_w$ is equal to the integral length of the compact edge of the Newton polygon $\mathcal{N}_{L,L'}(A)$. The equality $E_w \cdot A_w = \mathbf{i}_L(A)/c_w$ follows by using Lemma 1.6.16 again. □

Lemma 1.6.19 *Let A and B be two branches on (S, o). Consider the point $W \in \Theta_L(L')$ fixed above, determined by relation (1.57). Assume that $W = A \wedge_L L' = B \wedge_L L'$ inside the universal Eggers-Wall tree Θ_L. Then the following conditions are equivalent:*

1. $A \wedge_L B = W$.
2. $A \cdot B = \dfrac{d_w}{c_w}(L \cdot A)(L \cdot B)$.
3. $A_w \cap E_w \neq B_w \cap E_w$.

Proof

Proof of *1* $\Rightarrow$ *2* This implication is a consequence of Formulae (1.54) and (1.58).

Proof of *2* $\Rightarrow$ *1* Let us denote by W' the point $A \wedge_L B$. The assumption $W \preceq_L A$, $W \preceq_L B$ implies that $W \preceq_L W'$. By Formula (1.54), we get $\mathbf{c}_L(W') = d_w/c_w = \mathbf{c}_L(W)$. Since the function $\mathbf{c}_L$ is strictly increasing on $[L, A]$, the inequalities $L \preceq_L W \preceq_L W' \preceq_L A$ imply that $W = W'$.

Proof of *1* $\Leftrightarrow$ *3* Let (x, y) be a system of local coordinates defining the cross (L, L'). Denote by f_A a defining function of A with respect to this system and by K_A the compact edge of the Newton polygon $\mathcal{N}_{L,L'}(A)$. By the proof of Lemma 1.6.16, if α_A is the coefficient of x^{d_w/c_w} in a fixed Newton-Puiseux series of A, then the restriction of f_A to the compact edge K_A is equal to:

$$\left(\prod_{\gamma^{c_w}=1} (y - \alpha_A \, \gamma \, x^{d_w/c_w})\right)^{\mathbf{i}_L(A)/c_w} = (y^{c_w} - \alpha_A^{c_w} x^{d_w})^{\mathbf{i}_L(A)/c_w}.$$

We consider similar notations for the branch B. By Proposition 1.4.18, the point of intersection of the strict transform of A_w with E_w is parametrized by the coefficient $\alpha_A^{c_w}$. The desired equivalence follows since $\alpha_A^{c_w} \neq \alpha_B^{c_w}$ if and only if for every $\gamma \in \mathbb{C}$ with $\gamma^{c_w} = 1$, one has that $\alpha_A \neq \gamma \cdot \alpha_B$, which is also equivalent to the property $k_L(A, B) = d_w/c_w$ by the definition of the order of coincidence (see Definition 1.6.2). $\square$

Proposition 1.6.20 *Let A and B be two branches on S. Consider the point $W \in \Theta_L(L')$ fixed above. Assume that $W = A \wedge_L L' = B \wedge_L L'$. Then:*

1. $L \cdot A = c_w(E_w \cdot A_w)$.
2. $A \cdot B = A_w \cdot B_w + \dfrac{d_w}{c_w}(L \cdot A)(L \cdot B)$.
3. $A_w \cdot B_w > 0$ *if and only if* $W \prec_L A \wedge_L B$.
4. $\mathbf{c}_L(A \wedge_L B) = \dfrac{1}{c_w^2}\mathbf{c}_{E_w}(A_w \wedge_{E_w} B_w) + \dfrac{d_w}{c_w}$.

Proof Notice first that the hypothesis and Lemma 1.6.18 imply that the strict transforms A_w, B_w of A and B by π_w intersect $E_w \setminus (L_w \cup L'_w)$. If C is a branch on (S, o), denote by $\boxed{(\pi_w^* C)_{ex}}$ the *exceptional part* of the total transform divisor $(\pi_w^* C) = (\pi_w^* C)_{ex} + C_w$ of C on S_w.

Proof of *1* We have:

$$\begin{aligned} L \cdot A &\overset{(i)}{=\!=} (\pi_w^* L) \cdot (\pi_w^* A) = \\ &\overset{(ii)}{=\!=} (\pi_w^* L) \cdot A_w = \\ &\overset{(iii)}{=\!=} (\pi_w^* L)_{ex} \cdot A_w = \\ &\overset{(iv)}{=\!=} \mathrm{ord}_{E_w}(L)(E_w \cdot A_w) = \\ &\overset{(v)}{=\!=} \nu_w(\chi^{\epsilon_1})(E_w \cdot A_w) = \\ &\overset{(vi)}{=\!=} ((c_w e_1 + d_w e_2) \cdot \epsilon_1)(E_w \cdot A_w) = \\ &\overset{(vii)}{=\!=} c_w(E_w \cdot A_w). \end{aligned}$$

Let us explain each one of the previous equalities:

- Equality (i) results from the birational invariance of the intersection product, if one works with total transforms of divisors.
- Equality (ii) is a consequence of the equality $(\pi_w^* L) \cdot (\pi_w^* A)_{ex} = 0$, which results from the *projection formula* (see [61, Appendix A1]), applied to the divisors L on S, $(\pi_w^* A)_{ex}$ on S_w and to the proper morphism π_w.
- Equality (iii) follows from the hypothesis $L_w \cdot A_w = 0$ and the bilinearity of the intersection product.
- Equality (iv) is a consequence of the equality $(\pi_w^* L)_{ex} = \mathrm{ord}_{E_w}(L) E_w$.
- Equality (v) results from the equalities $\mathrm{ord}_{E_w} = \nu_w$ (see Eq. (1.32)) and $x = \chi^{\epsilon_1}$.
- Equality (vi) results from the fact that $w = c_w e_1 + d_w e_2$.
- Equality (vii) results from the fact that (ϵ_1, ϵ_2) is the dual basis of (e_1, e_2).

Proof of *2*. Let us choose a branch A' on (S, o) such that:

$$\mathbf{i}_L(A) = \mathbf{i}_L(A') \text{ and } W = A \wedge_L L' = A' \wedge_L L'. \tag{1.59}$$

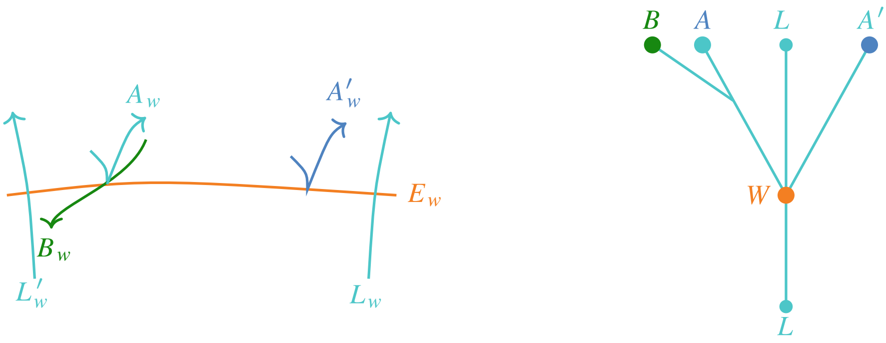


Fig. 1.54 The choice of branch A' in the proof of Proposition 1.6.20 (2)

Using Lemma 1.6.19, we can translate this hypothesis in terms of the total transform of the branches A, A' by π_w. On the left side of Fig. 1.54 is represented the total transform of $L + L' + A + A' + B$ by π_w and on its right side is represented the Eggers-Wall tree $\Theta_L(L + L' + A + A' + B)$, for some branch B. Then:

$$
\begin{aligned}
A \cdot B &\overset{(i)}{=\!=} (\pi_w^* A) \cdot (\pi_w^* B) = \\
&\overset{(ii)}{=\!=} (\pi_w^* A) \cdot B_w = \\
&\overset{(iii)}{=\!=} A_w \cdot B_w + (\pi_w^* A)_{ex} \cdot B_w = \\
&\overset{(iv)}{=\!=} A_w \cdot B_w + (\pi_w^* A')_{ex} \cdot B_w = \\
&\overset{(v)}{=\!=} A_w \cdot B_w + (\pi_w^* A') \cdot B_w = \\
&\overset{(vi)}{=\!=} A_w \cdot B_w + A' \cdot B = \\
&\overset{(vii)}{=\!=} A_w \cdot B_w + (L \cdot A')(L \cdot B)\, \mathbf{c}_L(W) = \\
&\overset{(viii)}{=\!=} A_w \cdot B_w + (L \cdot A)(L \cdot B) \frac{d_w}{c_w}.
\end{aligned}
$$

Let us explain each one of the previous equalities:

- Equalities (i) and (ii) are analogs of the equalities (i) and (ii) in the proof of point (1) above.
- Equality (iii) results from the bilinearity of the intersection product.
- Equality (iv) results from the hypothesis (1.59) and Lemma 1.6.18, which imply that $\mathrm{ord}_{E_w}(A) = \mathrm{ord}_{E_w}(A')$. Then one concludes using the equality $(\pi_w^* C)_{ex} \cdot B_w = \mathrm{ord}_{E_w}(C)(E_w \cdot B_w)$, for each $C \in \{A, A'\}$.
- Equality (v) results from the fact that, by construction, A'_w and B_w are disjoint.
- Equality (vi) results from the projection formula.
- Equality (vii) results from Lemma 1.6.19.
- Equality $(viii)$ results from Eq. (1.58) and from the equality $L \cdot A = L \cdot A'$, which is a consequence of the hypothesis (1.59) and the equality $L \cdot C = \mathbf{i}_L(C)$ for each $C \in \{A, A'\}$.

Proof of *3* By hypothesis, the strict transforms A_w and B_w intersect the set $E_w \setminus (L_w \cup L'_w)$, which is equal to the torus orbit $O_{\mathbb{R}_+ w}$. By the proof of Proposition 1.4.18, this implies that w is orthogonal to the compact edges of the Newton polygons $\mathcal{N}_{L,L'}(A)$ and $\mathcal{N}_{L,L'}(B)$. Lemma 1.6.16 implies that $\mathbf{e}_L(W) = \mathbf{e}_L(A \wedge_L L') = \mathbf{e}_L(B \wedge_L L')$. As the three points $W, A \wedge_L L', B \wedge_L L'$ belong to the segment $[L, L']$ and that $\mathbf{e}_L$ is strictly increasing on it, we get the equalities $W = A \wedge_L L' = B \wedge_L L'$. This implies that $W \preceq_L A,\ W \preceq_L B$. The claim follows from point (2) by using Lemma 1.6.19.

Proof of *4*. Dividing both sides of the formula of point (2) by the product $(L \cdot A)(L \cdot B)$, we get:

$$\frac{A \cdot B}{(L \cdot A) \cdot (L \cdot B)} = \frac{A_w \cdot B_w}{(L \cdot A) \cdot (L \cdot B)} + \frac{d_w}{c_w}.$$

Using point (1), we get:

$$\frac{A_w \cdot B_w}{(L \cdot A) \cdot (L \cdot B)} = \frac{1}{c_w^2} \frac{A_w \cdot B_w}{(E_w \cdot A_w) \cdot (L_w \cdot B_w)}.$$

By applying formula (1.54) twice we obtain the desired formula:

$$\mathbf{c}_L(A \wedge_L B) = \frac{1}{c_w^2} \mathbf{c}_{E_w}(A_w \wedge_{E_w} B_w) + \frac{d_w}{c_w}. \tag{1.60}$$

□

Let us define in combinatorial terms a natural embedding of the universal Eggers-Wall tree Θ_{o_w, E_w} into the universal Eggers-Wall tree $\Theta_{o,L}$ (see 1.6.12):

Definition 1.6.21 Let A_w be a branch on the germ of surface (S_w, o_w). Denote by A its image by the modification π_w. The **natural embedding** of the universal Eggers-Wall tree Θ_{o_w, E_w} into the universal Eggers-Wall tree $\Theta_{o,L}$ is defined by sending each point Q of the Eggers-Wall segment $\Theta_{o_w, E_w}(A_w)$ to the unique point Q' of $\Theta_{o,L}(A)$ which satisfies:

$$\mathbf{c}_L(Q') = \frac{1}{c_w^2} \mathbf{c}_{E_w}(Q) + \frac{d_w}{c_w}. \tag{1.61}$$

If (C_w, o_w) is a reduced curve on (S_w, o_w), then the embedding of the Eggers-Wall tree $\Theta_{o_w, E_w}(C_w)$ in $\Theta_{o,L}(C)$ is well-defined thanks to Formula (1.60) applied to any pair $A_w,\ B_w$ of branches of (C_w, o_w). That is, the embeddings of the Eggers-Wall segments of its branches glue into an embedding of $\Theta_{o_w, E_w}(C_w)$ in $\Theta_{o,L}(C)$. Notice that the root E_w of $\Theta_{o_w, E_w}(C_w)$ corresponds to the point $W \in \Theta_{o,L}(L')$ defined by relation (1.57) and that the leaf of Θ_{o_w, E_w} labeled by A_w corresponds to the leaf of $\Theta_{o,L}$ labeled by A.

The following proposition describes how to pass from the functions $(\mathbf{i}_{E_w}, \mathbf{e}_{E_w})$ on the tree $\Theta_{o_w,E_w}(C_w)$ to the functions $(\mathbf{i}_L, \mathbf{e}_L)$ on $\Theta_{o,L}(C)$:

Proposition 1.6.22 *Let (C_w, o_w) be a reduced curve singularity on (S_w, o_w). Identify the tree $\Theta_{o_w,E_w}(C_w)$ with the subtree of $\Theta_{o,L}(C)$ defined by the natural embedding of Definition 1.6.21. One has the following relations in restriction to this subtree:*

1. $\mathbf{i}_L = c_w \, \mathbf{i}_{E_w}$.
2. $\mathbf{e}_L = \dfrac{1}{c_w}\mathbf{e}_{E_w} + \dfrac{d_w}{c_w}$.

Proof **Proof of *1*.** We show first the assertion for an end of $\Theta_{o_w,E_w}(C_w)$ corresponding to a branch B_w of C_w. By the definition of the index function, we have the equalities $\mathbf{i}_L(B) = L \cdot B$ and $\mathbf{i}_{E_w}(B_w) = E_w \cdot B_w$. Combining these equalities with point (1) of Proposition 1.6.20, we get:

$$\mathbf{i}_L(B) = c_w \mathbf{i}_{E_w}(B_w). \tag{1.62}$$

Let $Q \neq E_w$ be any rational point of $\Theta_{o_w,E_w}(C_w)$. By the equality (1.52), there exists a branch A_w on the germ of surface (S_w, o_w) such that $\mathbf{i}_{E_w}(A_w) = \mathbf{i}_{E_w}(Q)$. We get:

$$\mathbf{i}_L(Q) \overset{(1.52)}{\leq} \mathbf{i}_L(A) \overset{(1.62)}{=} c_w \mathbf{i}_{E_w}(A_w) = c_w \mathbf{i}_{E_w}(Q).$$

This implies that $\mathbf{i}_L(Q) \leq c_w \mathbf{i}_{E_w}(Q)$. Analogously, using again equality (1.52), there exists a branch B on the germ (S, o) such that $W \prec_L Q \prec_L B$ and $\mathbf{i}_L(B) = \mathbf{i}_L(Q)$. By Definition 1.6.21 of the natural embedding of Θ_{o_w,E_w} in $\Theta_{o,L}$, this implies that $Q \prec_{E_w} B$. Therefore:

$$\mathbf{i}_L(Q) = \mathbf{i}_L(B) \overset{(1.62)}{=} c_w \mathbf{i}_{E_w}(B_w) \overset{(1.52)}{\geq} c_w \mathbf{i}_{E_w}(Q).$$

It follows that $\mathbf{i}_L(Q) = c_w \mathbf{i}_{E_w}(Q)$. We have shown that the equality in point (1) holds in restriction to the rational points of $\Theta_{o_w,E_w}(C_w)$, and by the continuity properties of the index functions, it holds for every point of $\Theta_{o_w,E_w}(C_w)$.
Proof of *2*. Let P be a point of $\Theta_{o_w,E_w}(C_w)$. This implies that $W \preceq_L P$. By the integral formula (1.53), we get:

$$\mathbf{e}_L(P) = \int_L^P \mathbf{i}_L d\mathbf{c}_L = \int_L^W \mathbf{i}_L d\mathbf{c}_L + \int_W^P \mathbf{i}_L d\mathbf{c}_L.$$

Using again Eq. (1.53), we have:

$$\int_L^W \mathbf{i}_L d\mathbf{c}_L = \mathbf{e}_L(W) = \frac{d_w}{c_w}. \tag{1.63}$$

We compute the second integral $\int_W^P \mathbf{i}_L d\mathbf{c}_L$ by making a change of variable. Differentiating formula (1.61), we get $d\mathbf{c}_L = (1/c_w^2) d\mathbf{c}_{E_w}$. Using the expression for $\mathbf{i}_L$ of point (1), we obtain:

$$\int_W^P \mathbf{i}_L d\mathbf{c}_L = \frac{1}{c_w} \int_W^P \mathbf{i}_{E_w}\, d\mathbf{c}_{E_w} = \frac{1}{c_w} \mathbf{e}_{E_w}(P), \tag{1.64}$$

where we have used again the integral formula (1.53). We end the proof by combining the equalities (1.63) and (1.64):

$$\mathbf{e}_L(P) = \frac{d_w}{c_w} + \frac{1}{c_w} \mathbf{e}_{E_w}(P).$$

□

Remark 1.6.23 Identify the tree Θ_{o_w, E_w} with the subtree of the universal Eggers-Wall tree $\Theta_{o,L}$ defined by the embedding of Definition 1.6.21. As a consequence of Proposition 1.6.22, the two formulae stated in it also hold on Θ_{o_w, E_w}.

1.6.4 Renormalization in Terms of Newton-Puiseux Series

We give a different proof of Proposition 1.6.22 by using Newton-Puiseux series. This proof relates the Newton modifications in the toroidal category of Definition 1.4.14 with the *Newton maps*, which appear sometimes in the algorithmic construction of Newton-Puiseux series (see Sect. 1.6.6).

We keep the notations introduced at the beginning of Sect. 1.6.3. Let A be a branch on (S, o) such that A_w intersects E_w at a point $o_w \in E_w \setminus (L_w \cup L'_w)$. Consider local coordinates (x, y) defining the cross (L, L'). Recall from Definition 1.6.2 that $\mathcal{Z}_x(A)$ denotes the set of Newton-Puiseux roots of A relative to x. Let us choose $\eta \in \mathcal{Z}_x(A)$. It may be expressed as:

$$\eta = \sum_{k \geq m} \alpha_k x^{k/n}, \tag{1.65}$$

where $n = A \cdot L,\ m = A \cdot L'$. Hence $\alpha_m \neq 0$. All the series in $\mathcal{Z}_x(A)$ have the same support, since they form a single orbit under the Galois action of multiplication of $x^{1/n}$ by the complex n-th roots of 1 (see Remark 1.2.21).

Let us denote $p := \gcd(n, m)$. Our hypothesis that A_w meets $E_w \setminus (L_w \cup L'_w)$ implies that:

$$n = c_w \cdot p, \quad m = d_w \cdot p. \tag{1.66}$$

The branch A is defined by $f = 0$, where:

$$f = \prod_{\gamma^n=1} (y - (\gamma \cdot \eta)(x)) = (y^{c_w} - \alpha_m^{c_w} x^{d_w})^p + \dots. \tag{1.67}$$

We have only written on the right-hand side of (1.67) the restriction of f to the unique compact edge of the Newton polygon of $f(x, y)$.

Lemma 1.6.24 *There exist local coordinates (x_1, y_1) on the germ (S_w, o_w) such that $E_w = Z(x_1)$ and the map π_w is defined by:*

$$\begin{cases} x = x_1^{c_w}, \\ y = x_1^{d_w}(\alpha_m + y_1). \end{cases} \tag{1.68}$$

Proof Consider a vector $w' = a_w e_1 + b_w e_2$ such that:

$$b_w c_w - a_w d_w = 1. \tag{1.69}$$

Therefore the cone $\sigma = \mathbb{R}_+\langle w, w'\rangle$ is regular and included in one cone of the fan $\mathcal{F}_{L,L'}(A)$. As explained in the proof of Proposition 1.4.18, we can look at the intersection of A_w with the orbit $O_{\mathbb{R}_+ w} = E_w \setminus (L_w \cup L'_w)$ in the open subset corresponding to this orbit on the toric surface $X_\sigma = \mathbb{C}^2_{u,v}$. The toric morphism $\psi^\sigma_{\sigma_0}$ is the monomial map defined by

$$\begin{cases} x = u^{c_w} v^{a_w} \\ y = u^{d_w} v^{b_w} \end{cases}$$

(see Example 1.3.26). The orbit $O_{\mathbb{R}_+ w}$, seen on the surface $\mathbb{C}^2_{u,v}$, is the pointed axis $\mathbb{C}^*_v$. The maximal monomial in (u, v) which divides $(\psi^\sigma_{\sigma_0})^* f$ is equal to $(u^{c_w d_w} v^{a_w d_w})^p$. After factoring out this monomial and setting $u = 0$ we get:

$$(v^{a_w c_w - b_w d_w} - \alpha_m^{c_w})^p \overset{(1.69)}{=} (v - \alpha_m^{c_w})^p. \tag{1.70}$$

This shows that the point o_w has coordinates $(u, v) = (0, \alpha_m^{c_w})$. The formulae

$$\begin{cases} u = x_1 (y_1 + \alpha_m)^{-a_w}, \\ v = (y_1 + \alpha_m)^{c_w}, \end{cases} \tag{1.71}$$

define local coordinates (x_1, y_1) at o_w, since the jacobian determinant of $(u, v-\alpha_m^{c_w})$ with respect to (x_1, y_1) does not vanish at $(0, 0)$. Notice also that $Z(x_1) = Z(u) = E_w$. By (1.69) we get:

$$\begin{cases} x = x_1^{c_w}(y_1 + \alpha_m)^{-a_w c_w}(y_1 + \alpha_m)^{a_w c_w} = x_1^{c_w} \\ y = x_1^{d_w}(y_1 + \alpha_m)^{b_w c_w - d_w a_w} = x_1^{d_w}(\alpha_m + y_1). \end{cases}$$

□

Proposition 1.6.25 *With respect to the coordinates (x_1, y_1) introduced in Lemma 1.6.24, the series*

$$\eta_w := \sum_{k>m} \alpha_m x_1^{(k-m)/p},$$

is a Newton-Puiseux series parametrizing the branch A_w on (S_w, o_w).

Proof By formula (1.70), we have that $(A_w \cdot E_w)_{o_w} = p$. It follows that the Newton-Puiseux series in $\mathcal{Z}_{x_1}(A_w)$ must have exponents in $(1/p)\mathbb{N}^*$. By composing (1.68) with the change of variable

$$x_1 = x_2^p, \tag{1.72}$$

we get:

$$\begin{cases} x = x_2^n, \\ y = x_2^{d_w p}(\alpha_m + y_1). \end{cases} \tag{1.73}$$

Apply the substitution (1.73) to the factor $y - (\gamma \cdot \eta)(x^{1/n})$, using that $x_2 = x^{1/n}$ by definition, and factor out the monomial $x_2^{d_w p}$. We get the series

$$(\alpha_m + y_1) - \alpha_m \gamma^m - \sum_{k>m} \alpha_k \gamma^k x_2^{k-m} \in \mathbb{C}[[x_2, y_1]]. \tag{1.74}$$

This series has vanishing constant term if and only if $\gamma^m = 1$. Since $\gamma^n = 1$ and $\gcd(n, m) = p$, one may check that this condition holds if and only if $\gamma^p = 1$, and in this case for any $k > m$ one has $\gamma^k = \gamma^{k-m}$. It follows that the series (1.74) which are non-units are precisely the conjugates of the series $y_1 - \eta_w(x_1^{1/p})$ under the Galois action, since $x_2 = x_1^{1/p}$ by definition (1.72). Therefore, the product of all the conjugates of $y_1 - \eta_w(x_1^{1/p})$ under the Galois action defines a polynomial in $\mathbb{C}[[x_1]][y_1]$ which divides the strict transform of f by the map (1.68). The remaining factor is a series with nonzero constant term, and must belong to the ring $\mathbb{C}[[x_1, y_1]]$ since it is invariant under the Galois action. □

Corollary 1.6.26 *Let A, B be two branches on (S, o) such that $o_w \in A_w \cap B_w \cap E_w$. Then:*

$$k_x(A, B) = \frac{d_w}{c_w} + c_w \cdot k_{x_1}(A_w, B_w).$$

Proof By point (3) of Proposition 1.6.20, the inequality $A_w \cdot B_w > 0$ (which results from the hypothesis that $o_w \in A_w \cap B_w$) implies that $k_x(A, B) > d_w/c_w$. It follows that if we fix a Newton-Puiseux series $\eta \in \mathcal{Z}_x(A)$, then there exists $\xi \in \mathcal{Z}_x(B)$ with the same order and the same leading coefficient. We can apply Lemma 1.6.24, using this leading coefficient, to define suitable local coordinates (x_1, y_1) at the point o_1. The formula results from Proposition 1.6.25 by taking into account the facts that $\eta_w \in \mathcal{Z}_{x_1}(A_w)$ and $\xi_w \in \mathcal{Z}_{x_1}(B_w)$. □

Corollary 1.6.26 implies readily Proposition 1.6.22.

1.6.5 From Fan Trees to Eggers-Wall Trees

In this subsection we assume that C is *reduced*. We explain that there exists a canonical isomorphism from the fan tree $\theta_\pi(C)$ of a toroidal pseudo-resolution π of C produced by running Algorithm 1.4.22, to the Eggers-Wall tree of the completion $\hat{C}_\pi$ of C (see Theorem 1.6.27). We also explain how to compute the index, exponent and contact complexity functions on the Eggers-Wall tree from the slope function on the fan tree (see Proposition 1.6.28).

Let L be a smooth branch on the germ (S, o). Assume that we run Algorithm 1.4.22, arriving at a toroidal pseudo-resolution $\pi : (\Sigma, \partial\Sigma) \to (S, L + L')$. Consider the corresponding completion $\hat{C}_\pi$, in the sense of Definition 1.4.15. There are two trees associated with this setting which have their ends labeled by the branches of $\hat{C}_\pi$, the fan tree $\theta_\pi(C)$ and the Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$. How are they related? It turns out that they are isomorphic:

Theorem 1.6.27 *There is a unique isomorphism from the fan tree $\theta_\pi(C)$ to the Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$, which preserves the labels of the ends of both trees by the branches of $\hat{C}_\pi$.*

Proof At the first step of Algorithm 1.4.22, one chooses a smooth branch L' such that (L, L') is a cross on (S, o). By definition, the branch L' is a component of the completion $\hat{C}_\pi$. Let us consider the segment $[L, L']$ of $\Theta_L(\hat{C}_\pi)$ and the first trunk $\theta_{\mathcal{F}_{L,L'}(C)} = [e_L, e_{L'}]$. We have a homeomorphism

$$\Psi_o : [e_L, e_{L'}] \to [L, L'] = \Theta_L(L')$$

sending a vector $w \in [e_L, e_{L'}]$ to the unique point $W \in [L, L']$ whose exponent $\mathbf{e}_L(W)$ is equal to the slope of w with respect to the basis $(e_L, e_{L'})$ of $N_{L,L'}$. By

Corollary 1.6.17, the map Ψ_o defines also a bijection between the set of marked points of the trunk, according to Definition 1.4.33, and the set of the marked points of the tree $\Theta_L(\hat{C}_\pi)$ which belong to the segment $[L, L']$ according to Definition 1.6.3.

Let o_i be a point of $\partial S_{\mathcal{F}_{L,L'}(C)}$, lying on the strict transform of C. The point o_i is considered at the fourth step of Algorithm 1.4.22. Let A_i denote the germ of $\partial S_{\mathcal{F}_{L,L'}(C)}$ at o_i and let (A_i, B_i) be the cross at o_i chosen when one passes again through the first and second steps of Algorithm 1.4.22. By definition, $L_i := \pi_{L,L'}(B_i)$ is a branch of $\hat{C}_\pi$. We denote by $\boxed{\hat{C}_{\pi,o_i}}$ (resp. $\boxed{C_{o_i}}$) the germ of the strict transform of $\hat{C}_\pi$ (resp. C) at the point o_i. We use the Notations 1.4.25. Let us consider the segment $[A_i, L_i]$ of the Eggers-Wall tree $\Theta_{o_i,A_i}(\hat{C}_{\pi,o_i})$ and the trunk $\theta_{\mathcal{F}_{A_i,B_i}(C_{o_i})} = [e_{A_i}, e_{B_i}]$. Arguing as before, we obtain a homeomorphism $\Psi_{o_i} : [e_{A_i}, e_{B_i}] \to [A_i, L_i]$ which sends $w \in [e_{A_i}, e_{B_i}]$ to the unique point $W \in [A_i, L_i]$ such that $\mathbf{e}_{A_i}(W)$ is equal to the slope of w with respect to the basis (e_{A_i}, e_{B_i}) of the lattice N_{A_i,B_i}. In addition, we get also that the homeomorphism Ψ_{o_i} defines a bijection between the marked points of the trunk $\theta_{\mathcal{F}_{A_i,B_i}(C_{o_i})}$ and the marked points of $\Theta_{o_i,A_i}(\hat{C}_{\pi,o_i})$ on the segment $[A_i, L_i]$. By Proposition 1.6.22, we have an embedding of the Eggers-Wall tree $\Theta_{o_i,A_i}(\hat{C}_{\pi,o_i})$ such that the root A_i of this tree is sent to the marked point $L' \wedge_L L_i$ of $\Theta_L(\hat{C}_\pi)$. By Definition 1.4.33, the point e_{A_i} of the trunk $\theta(\mathcal{F}_{A_i,B_i}(C))$ is identified with the marked point labeled by A_i on $\theta(\mathcal{F}_{L,L'}(C))$, during the construction of the fan tree $\theta_\pi(C)$.

If $\mathcal{T}$ is a tree and $P_1, \ldots, P_s \in \mathcal{T}$, we denote by $[P_1, \ldots, P_s]$ the smallest subtree of $\mathcal{T}$ containing $P_1, \ldots, P_s$. We apply this notation for the subtree $[e_L, e_{L'}, e_{B_j}]$ of $\theta_\pi(C)$ and the subtree $[L, L', L_j]$ of $\Theta_L(\hat{C}_\pi)$. The previous discussion implies that the homeomorphisms Ψ_o and Ψ_{o_i} can be glued into a homeomorphism

$$[e_L, e_{L'}, e_{B_j}] \to [L, L', L_j],$$

which sends the ramification vertex e_{A_i} of the tree $[e_L, e_{L'}, e_{B_j}]$ to the ramification vertex $L' \wedge_L L_j$ of $[L, L', L_j]$. We repeat this construction each time we pass through a cross at the first and second steps during the iterations of Algorithm 1.4.22. By induction, we get a finite number of homeomorphisms Ψ_{o_j}, which glue into a homeomorphism $\Psi : \theta_\pi(C) \to \Theta_L(\hat{C}_\pi)$ which respects the labelings of the ends of both trees by the branches of $\hat{C}_\pi$. □

Identify the two rooted trees $\theta_\pi(C)$ and $\Theta_L(\hat{C}_\pi)$ by the isomorphism of Theorem 1.6.27. For every point $P \in \theta_\pi(C)$, define the set $\boxed{\delta_P} \subset [L, P)$ as the finite subset of discontinuity points of the restriction of the slope function $\mathbf{S}_\pi$ to the segment $[L, P)$. If $\lambda \in \mathbb{Q}^*$, denote by $\boxed{\mathrm{den}(\lambda)}$ the denominator q of λ, when one writes it in the form p/q, with $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$, and $p,\ q$ coprime. The fan tree $\theta_\pi(C)$ comes endowed with only one function, the *slope function* $\mathbf{S}_\pi$, while the Eggers-Wall tree is endowed with the *index* $\mathbf{i}_L$, the *exponent* $\mathbf{e}_L$ and the *contact complexity* $\mathbf{c}_L$ functions. These functions are related by:

Proposition 1.6.28 *For every $P \in \theta_\pi(C)$, one has:*

1. $\mathbf{i}_L(P) = \prod_{Q \in \delta_P} \mathrm{den}(\mathbf{S}_\pi(Q))$.
2. $\mathbf{e}_L(P) = \int_L^P \frac{1}{\mathbf{i}_L} \, d\mathbf{S}_\pi$.
3. $\mathbf{c}_L(P) = \int_L^P \frac{1}{\mathbf{i}_L^2} \, d\mathbf{S}_\pi$.

Proof In order to follow the proof, one has to keep in mind the isomorphism of the fan tree with the Eggers-Wall tree built in Theorem 1.6.27. If the set δ_P is empty, that is, if the slope function $\mathbf{S}_\pi$ is continuous in restriction to $[L, P)$, then P belongs to the first trunk $[L, L']$. By definition, for any $Q \in [L, L']$ we have:

$$\mathbf{i}_L(Q) = 1, \quad \mathbf{e}_L(Q) = \mathbf{S}_\pi(Q). \tag{1.75}$$

Hence the equalities (1), (2) and (3) hold trivially for P.

We prove the assertions (1) and (2) by induction on the number of elements of the set δ_P of discontinuity points. Assume that $\delta_P = \{W = W_1, W_2, \dots, W_k\}$ with $k \geq 1$, and $W \prec_L W_2 \prec_L \cdots \prec_L W_k \prec_L P$. By construction, the point W belongs to the first trunk of $\theta_\pi(C)$. Then, using the notation (1.57), we have $\mathbf{e}_L(W) = d_w/c_w = \mathbf{S}_\pi(W)$, with $c_w = \mathrm{den}(\mathbf{S}_\pi(W))$. We decompose the integral of the second member of equality (2) in the form:

$$\int_L^P \frac{1}{\mathbf{i}_L} d\mathbf{S}_\pi = \int_L^W \frac{1}{\mathbf{i}_L} d\mathbf{S}_\pi + \int_W^P \frac{1}{\mathbf{i}_L} d\mathbf{S}_\pi.$$

By (1.75), one has:

$$\int_L^W \frac{1}{\mathbf{i}_L} d\mathbf{S}_\pi = \mathbf{e}_L(W) = \frac{d_w}{c_w}. \tag{1.76}$$

With the notations of Sect. 1.6.3, we consider the reduced curve C_w at (S_w, o_w), consisting of those branches A_w which are the strict transforms of branches A of C such that $W \prec_L P \prec_L A$ (see point (3) of Proposition 1.6.20). Proposition 1.6.22 implies that:

$$\mathbf{i}_L(Q) = c_w \mathbf{i}_{E_w}(Q), \text{ for } Q \in [W, P] \subset \Theta_{E_w}(C_w). \tag{1.77}$$

Hence:

$$\int_W^P \frac{1}{\mathbf{i}_L} d\mathbf{S}_\pi = \frac{1}{c_w} \int_W^P \frac{1}{\mathbf{i}_{E_w}} d\mathbf{S}_\pi = \frac{1}{c_w} \mathbf{e}_{E_w}(P). \tag{1.78}$$

To understand the last equality of (1.78), apply the induction hypothesis to the integral $\int_W^P (1/\mathbf{i}_{E_w}) d\mathbf{S}_\pi$, with respect to the set $\{W_2, \dots, W_k\}$ of discontinuity points of the restriction of the slope function $\mathbf{S}_\pi$ to $[W, P)$. The equality (2) follows from (1.76), (1.78) and point (2) of Proposition 1.6.22.

The equality (1) follows similarly by (1.77) and the induction hypothesis applied to $\mathbf{i}_{E_w}(P)$.

Let us prove the equality (3). By point (2) one has $d\mathbf{e}_L = (1/\mathbf{i}_L) d\mathbf{S}_\pi$. Therefore:

$$\mathbf{c}_L(P) = \int_L^P \frac{1}{\mathbf{i}_L} d\mathbf{e}_L = \int_L^P \frac{1}{\mathbf{i}_L^2} d\mathbf{S}_\pi.$$

□

Example 1.6.29 Consider the toroidal pseudo-resolution process of Example 1.4.28. Figure 1.55 shows the fan tree $\theta_\pi(C)$ and the corresponding Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$, for which are indicated the values of the exponent and the index functions. We computed them using Proposition 1.6.28. For instance, we have $\mathbf{i}_L(E_6) = 1 \cdot 5 = 5$, $\mathbf{e}_L(E_6) = \dfrac{3}{5} + \dfrac{1}{5} \cdot \dfrac{5}{3} = \dfrac{14}{15}$, $\mathbf{i}_L(E_8) = 1 \cdot 5 \cdot 3 = 15$ and $\mathbf{e}_L(E_8) = \dfrac{14}{15} + \dfrac{1}{15} \cdot \dfrac{1}{2} = \dfrac{29}{30}$.

Proposition 1.6.28 allows us to define a concrete reduced curve singularity C which admits the toroidal resolution process described in Example 1.4.28, whose lotus was represented in Fig. 1.36 and whose Enriques tree was represented in Fig. 1.40. Namely, we fix local coordinates (x, y) and we choose Newton-Puiseux series $\eta_1(x), \dots, \eta_7(x)$ defining branches $C_1, \dots, C_7$, then we take supplementary series $\lambda_1(x), \dots, \lambda_4(x)$ defining branches $L_1, \dots L_4$, such that the Eggers-Wall tree $\Theta_L(C_1 + \cdots + C_7 + L_1 + \cdots + L_4)$ is that on the right side of Fig. 1.55. For instance, one may choose:

$$\begin{gathered}
\eta_1(x) := x^{5/2}, \quad \eta_2(x) := x^2, \quad \eta_3(x) := -x^2, \quad \eta_4(x) := x^{3/5} + x^{3/4}, \\
\eta_5(x) := x^{3/5} + x^{11/15} \quad \eta_6(x) := 2x^{3/5} + x^{6/5}, \quad \eta_7(x) := 2x^{3/5} + x^{14/15} + x^{29/30}, \\
\lambda_1(x) := 0, \quad \lambda_2(x) := x^{3/5}, \quad \lambda_3(x) := 2x^{3/5}, \quad \lambda_4(x) := 2x^{3/5} + x^{14/15}.
\end{gathered}$$

Remark 1.6.30 The right part of Fig. 1.55 shows the Eggers-Wall tree of the completion of a plane curve singularity generated by a toroidal pseudo-resolution process. One may verify that it satisfies the following property which characterizes the Eggers-Wall trees of such completions: *each vertex which is not an end of the tree is contained in the interior of a segment in restriction to which the index function is constant* (in particular, such an Eggers-Wall tree has no vertices of valency 2). When one has such an Eggers-Wall tree, it originates from a fan tree as described in Proposition 1.6.28. But this fan tree is not unique. One has to determine first which segments of the Eggers-Wall tree are trunks of the fan tree, and there may be different choices. For instance, in Fig. 1.55 one could decide that the segment $[L, C_2]$ is a trunk, instead of $[L, L_1]$. Once the trunks are chosen, the sets δ_P are

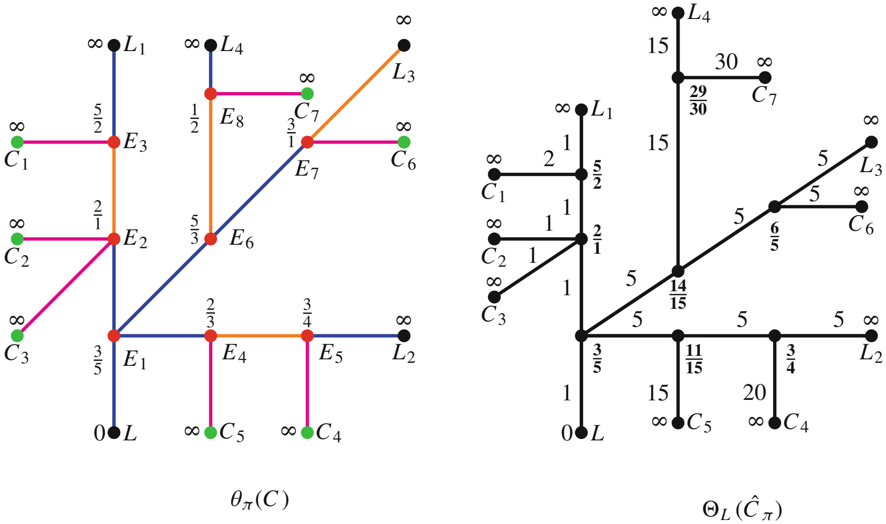


Fig. 1.55 The fan tree $\theta_\pi(C)$ and the corresponding Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$ in Example 1.6.29

determined for every point P of the tree. This allows to compute the slope function $\mathbf{S}_\pi$ by integrating the differential relation $d\mathbf{S}_\pi = \mathbf{i}_L d\mathbf{e}_L$, which is a consequence of Proposition 1.6.28 (2).

Proposition 1.6.28 may be written more explicitly as follows:

Corollary 1.6.31 *Let P be a vertex of $\theta_\pi(C) = \Theta_L(\hat{C}_\pi)$, different from the root L. Assume that when one moves on the segment $[L, P]$ from L to P, one meets successively the vertices $P_1, \ldots, P_k = P$ of $\delta_P \cup \{P\}$. Denote $\mathbf{S}_\pi(P_j) = d_j/c_j$ with coprime $c_j, d_j \in \mathbb{N}^*$, for all $j \in \{1, \ldots, k\}$ (with $c_k = 1$ and $d_k = \infty$ if P is a leaf of the tree). Then:*

1. $\mathbf{i}_L(P) = c_1 \cdots c_{k-1}$.
2. $\mathbf{c}_L(P) = \dfrac{d_1}{c_1} + \dfrac{d_2}{c_1^2 c_2} + \dfrac{d_3}{c_1^2 c_2^2 c_3} + \cdots + \dfrac{d_k}{c_1^2 \cdots c_{k-1}^2 c_k}$.
3. $\mathbf{e}_L(P) = \dfrac{d_1}{c_1} + \dfrac{d_2}{c_1 c_2} + \dfrac{d_3}{c_1 c_2 c_3} + \cdots + \dfrac{d_k}{c_1 \cdots c_k}$.

Example 1.6.32 Let us specialize Corollary 1.6.31 to the case where P is a leaf of $\theta_\pi(C) = \Theta_L(\hat{C}_\pi)$, labeled by a branch C. Therefore the characteristic exponents of a Newton-Puiseux series of C relative to L are:

$$\frac{m_j}{n_1 \cdots n_j} := \frac{d_1}{c_1} + \frac{d_2}{c_1 c_2} + \cdots + \frac{d_j}{c_1 \cdots c_j}, \tag{1.79}$$

for all $j \in \{1, \dots, k\}$. Here the positive integers $(m_1, \dots, m_k)$ and $(n_1, \dots, n_k)$ are chosen such that m_j and n_j are coprime for all $j \in \{1, \dots, k\}$. The relations (1.79) may be reexpressed in the following way:

$$(c_j, d_j) = (n_j, m_j - n_j \cdot m_{j-1}), \tag{1.80}$$

for all $j \in \{1, \dots, k\}$ (with the convention $m_0 := 0$). Sometimes the couples (m_j, n_j) are called the *Puiseux pairs* and the couples (d_j, c_j) are called the *Newton pairs* of the given Newton-Puiseux series. The importance of using both sequences of pairs in the topological study of plane curve singularities was emphasized by Eisenbud and Neumann in their book [34, Page 6]. More details may be found in Weber's survey [132, Section 6.1].

Example 1.6.33 This is a continuation of Example 1.6.32. Consider pairs of coprime integers $(n_j, m_j) \in \mathbb{N}^* \times \mathbb{N}^*$ with $n_j > 1$, for $j = 1, \dots, k$ and the Newton-Puiseux series

$$x^{m_1/n_1} + x^{m_2/(n_1 n_2)} + \cdots + x^{m_k/(n_1 \cdots n_k)},$$

defining a branch C. We can build a toroidal pseudo-resolution π of C with respect to $L = Z(x)$, such that $\hat{C}_\pi = L + C + \sum_{j=1}^{k} L_j$ and the branches $L_1, \dots, L_k$ are defined by the Newton-Puiseux series:

$$0, \quad x^{m_1/n_1}, \quad x^{m_1/n_1} + x^{m_2/(n_1 n_2)}, \quad \dots, \quad x^{m_1/n_1} + x^{m_2/(n_1 n_2)} + \cdots + x^{m_{k-1}/(n_1 \cdots n_{k-1})}.$$

Then the associated lotus is as represented in Fig. 1.38. Using formula (1.80) and the notations introduced in Example 1.5.30, we have:

$$\frac{m_j}{n_j} - m_{j-1} = [p_j, q_j, \dots],$$

for all $j \in \{1, \dots, k\}$. In fact, one gets the same lotus whenever C is an arbitrary branch with the previous characteristic exponents relative to L and the branches L_j are *semiroots* of C (see [99, Corollary 5.6]). This shows that our notion of *completion of a reduced curve singularity C relative to a toroidal pseudo-resolution process is a generalization of the operation which adds to a branch a complete system of semiroots relative to L (see [99, Definition 6.4]).*

1.6.6 Historical Comments

Historical information about the notion of *characteristic exponent* may be found in our paper [44, Introduction, Rem. 2.9].

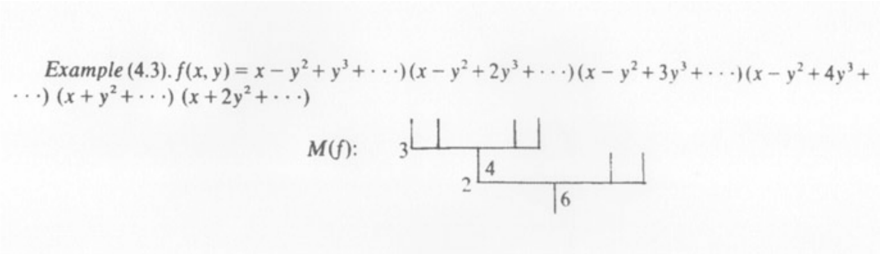


Fig. 1.56 A Kuo-Lu tree

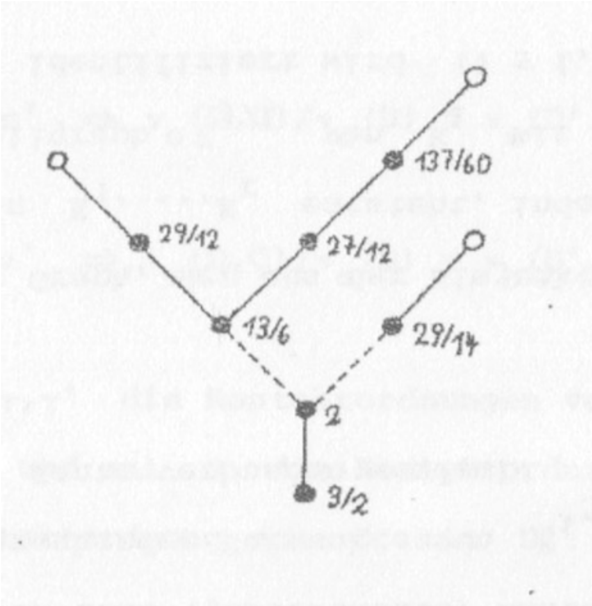


Fig. 1.57 An Eggers tree

In addition to the older Enriques diagrams and dual graphs of exceptional divisors of embedded resolutions, Kuo and Lu associated a third kind of tree to a curve singularity $C = Z(f(x, y))$ in their 1977 paper [75]. An example of such a tree, extracted from their paper, is shown in Fig. 1.57. Their trees were rooted and their sets of leaves were in bijection with the set of Newton-Puiseux series $\eta(x)$ associated with the corresponding plane curve singularity C. They used their trees in order to relate the structure of C to that of its *polar curve* defined by the equation $\dfrac{\partial f}{\partial y} = 0$ (Fig. 1.56).

In his 1983 paper [33], Eggers showed that a kind of Galois quotient of the Kuo-Lu tree of f was more convenient for this purpose. Figure 1.57 shows the first example given in [33]. A variant of the Eggers tree, better suited for computations, was introduced by Wall [130] and presented in more details in his textbook [131, Sections 4.2 and 9.4].

The third author coined in his 2001 thesis [98] the name *Eggers-Wall tree* for Wall's version of Eggers' tree. He proved in [98, Section 4.4] that the Eggers-Wall tree of C relative to generic coordinates could almost always be embedded in the dual graph of the minimal embedded resolution of C as the convex hull of its vertices representing the branches of C. He discovered this fact experimentally, by applying in many examples the first author's algorithm described in her 1996 thesis [42, Section 1.4.6], for the passage from Eggers' tree to the dual graph. Another proof of this embedding result was obtained in terms of certain toroidal-pseudo resolutions introduced by the second author in [52, Section 3.4]. Wall improved the description of this embedding in his 2004 book [131], and Favre and Jonsson explained it differently from their valuative viewpoint in their 2004 book [38, Appendix D2]. Recently, we gave a new viewpoint on this embedding result in [45, Theorem 112], in the framework of Eggers-Wall trees defined relative to arbitrary coordinate systems. It is important to consider the Eggers-Wall tree of C relative to coordinate systems which are not necessarily generic relative to C. Indeed, this freedom is essential when one wants to compare the Eggers-Wall tree of C with that of its strict transform by a blow up or a more complicated toric modification, because after such a modification the natural coordinate x defines the exceptional divisor, and is not necessarily generic with respect to the strict transform. In his paper [100], extracted from his thesis [98], the third author did not consider any genericity hypothesis, in order to extend the definition of this kind of tree to higher dimensional *quasi-ordinary hypersurface singularities*. This generalized notion of Eggers-Wall tree was further developed in connexion with the study of the associated polar hypersurfaces in the 2005 paper [43] of the first and second authors. In turn, the notion of Kuo-Lu tree was extended to quasi-ordinary hypersurface singularities by the first author and Gwoździewicz in their 2015 paper [47] and used again by them in [48], in order to study the structure of higher order polars of such singularities.

The notations for elementary Newton polygons described in Definition 1.6.14 were introduced by Teissier in his 1977 paper [120, Section 3.6], where he restricted them to $a, b \in \mathbb{N}^* \cup \{\infty\}$. Allowing the two numbers in Definition 1.6.14 to be rational is convenient in order to express Newton polygons in terms of Eggers-Wall trees (see Corollary 1.6.17).

Let us consider now the valuative aspects of Eggers-Wall trees. Favre and Jonsson proved in their 2004 book [38] that the set of semivaluations of the local $\mathbb{C}$-algebra $\hat{O}_{S,o}$ which are normalized by the constraint that a defining function x of the smooth germ L has value 1, has a natural structure of rooted real tree, which they called *the valuative tree*. In his 2015 survey [67], Jonsson revisited part of the theory of [38] with a more geometric approach which is valid for algebraically closed fields of arbitrary characteristic. Favre and Jonsson gave several descriptions of its tree structure. In our paper [46, Theorem 8.34] we gave a new description of it, as the universal Eggers-Wall tree of Definition 1.6.12. Namely, we proved that the valuative tree could also be obtained as a projective limit of Eggers-Wall trees. The main point of our proof is that $\Theta_L(C)$ embeds naturally in the valuative tree, for any C. We showed also in [46, Theorem 8.18] that the triple $(\mathbf{i}_L, 1 + \mathbf{e}_L, \mathbf{c}_L)$ is the pullback by this embedding of a triple of three natural functions on the valuative tree: the *multiplicity*, the *log-discrepancy* and the *self-interaction*.

An advantage of the identification of Θ_L with the valuative tree is that it allows to get an interpretation of the points of Θ_L which do not belong to any $\Theta_L(C)$ as special *infinitely singular* semivaluations, in the language of [38] and [67].

Another advantage is obtained when the base algebraically closed field has positive characteristic. Let us *define* the functions $\mathbf{i}_L$, $\mathbf{c}_L$ and $\mathbf{e}_L$ on $\theta_\pi(C)$ by the equalities appearing in Proposition 1.6.28. This provides a *definition* of a notion of Eggers-Wall tree in positive characteristic, where Newton-Puiseux series are not enough for the study of plane curve singularities (see Remark 1.2.17). The approach of Sect. 1.6.3 may be generalized to prove that in restriction to $\theta_\pi(C)$, the multiplicity function relative to L is equal to $\mathbf{i}_L$, the contact complexity function relative to L is equal to $\mathbf{c}_L$ and the log-discrepancy function relative to L is equal to $1 + \mathbf{e}_L$. This abstract Eggers-Wall tree may be associated with the *ultrametric distance* on the branches of C, as described in our paper [45]. It may be seen also as a generalization of the notion of characteristic exponents in positive characteristic introduced in Campillo's book [17], where the author computes these exponents using *Hamburger-Noether expansions* (see [17, Section 3.3]), infinitely near points (see [17, Remark 3.3.8]) or Newton polygons (see [17, Section 3.4]).

Assume now that the germ C is holomorphic. Then the Enriques diagram and the weighted dual graph of the minimal embedded resolution, as well as the Eggers-Wall tree relative to generic coordinates encode the same information, which is equivalent to the embedded topological type of C. Proofs of this fundamental fact may be found in Wall's book [131, Propositions 4.3.8 and 4.3.9].

A basic problem is then to find methods to transform one kind of tree into the two other kinds. Noether described in [90] how to pass from the *characteristic exponents* of an irreducible curve singularity C to the structure of the blow up process leading to an embedded resolution. Enriques and Chisini generalized this approach in [35, Libr. IV, Cap. I] to the case when C is not necessarily irreducible. Namely, they showed how to pass from the characteristic exponents of its branches and the orders of coincidence of pairs of branches in generic coordinates to the associated Enriques diagram.

Zariski and Lejeune-Jalabert proved by different methods in their 1971 paper [137] and 1972 thesis [77] respectively, that the characteristic exponents of the branches of C and the intersection numbers of its pairs of branches determine the embedded topological type of C and the combinatorics of its minimal embedded resolution. This may be seen as a proof of the fact that the weighted dual graph of the minimal embedded resolution is equivalent to the generic Eggers-Wall tree. Methods to pass from the knowledge of the characteristic exponents and intersection numbers to the dual graph were explained by Eisenbud and Neumann [34, Appendix to Ch. 1], Brieskorn and Knörrer [15, Section 8.4], Michel and Weber [86], de Jong and Pfister [66, Section 5.4] and an algorithm was described by the first author in [42, Sect. 1.4.6].

Let us mention now several other trees which were associated to plane curve singularities.

As explained in Sect. 1.4.5, the changes of variables considered by Puiseux (called sometimes *Newton maps*) were compositions of affine and of toric ones, which in general were not birational. Nevertheless, an algorithm of abstract resolution and of computation of Newton-Puiseux series may be developed also using them. A variant of the fan trees, adapted to this context and called *Newton trees*, was used by Cassou-Noguès in her papers mentioned in Sect. 1.4.5, written alone or in collaboration. The Newton trees encode also the toroidal pseudo-resolution processes described in the paper [21] of Cassou-Noguès and Libgober. We refer the reader especially to the papers [20] and [22] for more details about this approach. The changes of coordinates (1.71), which are very similar to Newton maps, were also used in the paper [72] of Kennedy and McEwan to study the monodromy of holomorphic plane curve singularities.

Newton maps and Newton trees have been used to study the singularities of quasi-ordinary hypersurfaces by Artal, Cassou-Noguès, Luengo and Melle Hernández (see for instance [10] and [11]). In their 2014 paper [55], the second author and González Villa compared the Newton maps with the toric morphisms appearing in a toroidal pseudo-resolution of an irreducible germ of quasi-ordinary hypersurface.

Newton trees are algebraic variants of the *splice diagrams* associated by Eisenbud and Neumann in their 1986 book [34] to any oriented graph link in an integral homology sphere, extending a graphical convention introduced by Siebenmann in his 1980 paper [114]. In our recent paper [46, Section 5], we explained how to pass from the Eggers-Wall tree of a holomorphic plane curve singularity C relative to a smooth branch L to the splice diagram of the oriented link of $L + C$ in $\mathbb{S}^3$.

In his 1993 papers [69] and [70], Kapranov associated a version of Kuo and Lu's trees to finite sets of formal power series with complex and real coefficients respectively. He called them *Bruhat-Tits trees*.

A version of Kuo and Lu's trees was used recently by Ghys in his book [50] about the topology of *real* plane curve singularities. He associated two such trees, one for $x > 0$ and another one for $x < 0$ to any germ whose branches are smooth and transversal to the reference branch $x = 0$, and studied their relation, describing all the possible couples of such trees. In a theorem proved with Christopher-Lloyd Simon (see [50, Page 266]), Ghys extended this analysis to all plane curve singularities with only real branches. For this more general problem, it was not any more a variant of Kuo and Lu's tree which was crucial, but a real version of the dual graph of the associated minimal resolution. A different real version of the dual resolution graph was introduced before by Castellini in [25, Chap. 3].

Ghys' version of Kuo and Lu's trees was also used by Sorea in her study [116] of curve singularities defined over $\mathbb{R}$ but without any real branch, that is, singularities of real analytic functions $f(x, y)$ in the neighborhood of a local maximum or minimum. Those trees were related in this work with another kind of tree, defined using Morse theory, the so-called *Poincaré-Reeb tree* of the function f relative to x.

Versions of our fan tree were considered by Weber in his 2008 survey [132] about the embedded topological type of holomorphic plane curve singularities, based on the earlier 1985 preprint [86] of Michel and Weber, which contained also many examples. The reading of Weber's survey [132] should facilitate the interpretations of the objects manipulated in this paper in terms of the embedded topological type of C.

1.7 Overview and Perspectives

We begin this final section by an overview of the content of the paper. Then we formulate a few remarks about perspectives of development of the use of lotuses in the study of singularities. The final Sect. 1.7.3 contains a list of notations used in this paper.

1.7.1 Overview

In this subsection we give an overview of the construction of the fan tree and of the associated lotus from the Newton fans generated by a toroidal pseudo-resolution process of a plane curve singularity. It helps us to understand the relations between Newton polygons, Newton-Puiseux series, iterations of blow ups, final exceptional divisor and the associated Enriques diagrams, dual graphs and Eggers-Wall trees.

We invite the reader to look at Fig. 1.58, which combines Figs. 1.35 and 1.37, but without their labels. Let us recall briefly the names and main properties of the

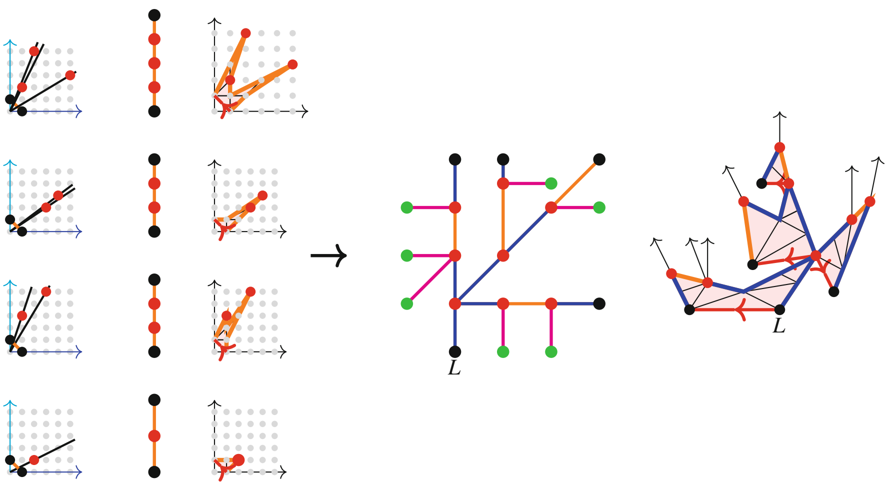


Fig. 1.58 Overview of the constructions of the paper

objects presented in this drawing, which is our way to encode the combinatorics of Algorithm 1.4.22, and how they allow to visualize the relations between Enriques diagrams, dual graphs and Eggers-Wall trees (see Theorem 1.5.29):

1. Given a curve singularity C embedded in a smooth germ of surface S, study it using a *cross* (L, L') (see Definition 1.3.31).
2. Construct the *Newton fan* $\mathcal{F}_{L,L'}(C)$ from the associated *Newton polygon* $\mathcal{N}_{L,L'}(C)$ (see Definitions 1.4.2 and 1.4.14).
3. Draw the *trunk* $\theta_{\mathcal{F}_{L,L'}(C)}$ (see Definition 1.4.32) and the *lotus* $\Lambda(\mathcal{F}_{L,L'}(C))$ (see Definitions 1.5.4 and 1.5.5) of the Newton fan.
4. As a simplicial complex, the lotus of a Newton fan is determined by the continued fraction expansions of the slopes of the fan's rays (see Sect. 1.5.2).
5. Make the *Newton modification* (see Definition 1.4.14) determined by the Newton fan and look at the germs of the strict transform of C at all its intersection points with the exceptional divisor. All those points are smooth on the reduced total transform of $L + L'$. For each such germ of the strict transform of C, complete locally the exceptional divisor into a cross.
6. Each new cross allows to construct again a trunk and a lotus associated to the corresponding germ of the strict transform of C. Combining the corresponding Newton modifications, one gets a new level of Newton modifications.
7. One iterates these constructions until reaching a toroidal surface Σ (see Definition 1.3.29) on which the total transform of C and of all the crosses used during the process is an abstract normal crossings curve, forming the boundary divisor $\partial\Sigma$ of a *toroidal pseudo-resolution* π of C (see Definition 1.4.15). The map π is also a toroidal pseudo-resolution of the *completion* $\hat{C}_\pi = \pi(\partial\Sigma)$ of C relative to π (see Definition 1.4.15), which is a curve singularity containing the branches of C and all the branches whose strict transforms are chosen to define crosses at certain steps of Algorithm 1.4.22.
8. In order to get a global combinatorial view, one constructs the associated *fan tree* $(\theta_\pi(C), \mathbf{S}_\pi)$ (see Definition 1.4.33), by gluing the trunks generated by the toroidal pseudo-resolution process. The function $\mathbf{S}_\pi : \theta_\pi(C) \to [0, \infty]$ is called the *slope function*.
9. The fan tree does not allow to visualize the decomposition of the regularization π^{reg} of π (see Proposition 1.4.29) into blow ups of points. In order to get such a vision, one constructs the *lotus* $\Lambda_\pi(C)$ of the process (see Definition 1.5.26) by gluing the Newton lotuses (see Definition 1.5.4) of the strict transforms of C relative to all the crosses used during the process.
10. The edges of the lotus correspond bijectively to the crosses created during the toroidal embedding resolution process by blow ups of points (see Theorem 1.5.29 (6)). Therefore, one may see the lotus as the space-time of the evolution of the dual graphs of the toroidal surfaces appearing during this process.
11. The graph of the proximity binary relation (see Definition 1.4.31) on the constellation which is blown up is the full subgraph of the 1-skeleton of the lotus $\Lambda_\pi(C)$ on its set of non-basic vertices (see Theorem 1.5.29 (7)).

12. The Enriques diagram (see Definition 1.4.31) of the constellation of infinitely near points blown up in order to decompose π^{reg}, which are the base points of the crosses appearing in the algorithm, is isomorphic with the Enriques tree (see Definition 1.5.26) of the lotus $\Lambda_\pi(C)$.
13. There is a second way of visualizing the Enriques diagram, using a *truncated lotus* $\Lambda_\pi^{tr}(C)$ (see Sect. 1.5.5).
14. The fan tree $\theta_\pi(C)$ is homeomorphic with the lateral boundary $\partial_+\Lambda_\pi(C)$ (see Definition 1.5.26) of the lotus generated by running Algorithm 1.4.22.
15. The lateral boundary $\partial_+\Lambda_\pi(C)$ is isomorphic with the dual graph (see Definition 1.3.22) of the boundary divisor $\partial\Sigma$. There is a simple combinatorial rule for reading on the lotus the self-intersection numbers of the components of the exceptional divisor of the modification π^{reg} (see Theorem 1.5.29 (5)).
16. The fan tree $\theta_\pi(C)$ is also isomorphic with the Eggers-Wall tree $\Theta_L(\hat{C}_\pi)$ (see Definition 1.6.3) of the completion of C relative to the toroidal modification π (see Theorem 1.6.27). The triple of functions (*index* $\mathbf{i}_L$*, exponent* $\mathbf{e}_L$*, contact complexity* $\mathbf{c}_L$) defined on $\Theta_L(\hat{C}_\pi)$ is determined by the *slope function* $\mathbf{S}_\pi$ on the fan tree through explicit formulae (see Proposition 1.6.28).
17. If (L, L') is a cross on S, then the Eggers-Wall tree $\Theta_L(C + L')$ determines the Newton polygon $\mathcal{N}_{L,L'}(C)$ (see Corollary 1.6.17).

1.7.2 Perspectives

In this subsection we give a few perspectives on possible uses of lotuses. We believe that the lotuses of plane curve singularities may be useful in the following research topics:

1. *In the study of the topology of δ-constant deformations of such singularities.* As mentioned in Sect. 1.6.6, Castellini's work [25] gives a first step in this direction. An important advantage of lotuses in this context is that the lotuses of the singularities appearing in the deformations constructed in [25] by A'Campo's method embed in the lotus of the original singularity. This embedding relation is much more difficult to express in terms of classical tree invariants of plane curve singularities. A crucial question is to understand whether this embedding property is specific to A'Campo type deformations, or if it extends to other kinds of δ-constant deformations.
2. *In the analogous study for* real *plane curve singularities*. One should probably describe real variants of the lotuses, embedded canonically up to isotopy in an oriented real plane. Again, Castellini's work [25, Sect. 3.3.2] gives a first step in this direction.
3. *In the extension of the distributive lattice structures described by Pe Pereira and the third author in [96] to arbitrary finite constellations, and in the application of those structures to the problem of adjacency of plane curve singularities*. The natural *operad structure* on the set of finite lotuses associated to toroidal pseudo-

resolution processes (defined by gluing the base of one lotus to an edge of the lateral boundary of another lotus) could be also useful in this direction.

4. *In the study of complex surface singularities through the Hirzebruch-Jung method* (see [103]). This method starts from a finite projection to a germ of smooth surface, and considers then an embedded resolution of the discriminant curve. The lotuses of such discriminant curves could be used as supports for encoding information about the initial finite projection, from which one could read invariants of the surface singularity.

1.7.3 List of Notations

In order to help browsing through the text, we list the notations used for the main objects met in it:

$\left\{\frac{a}{b}\right\}$	Elementary Newton polygon (see Definition 1.6.14).
$[a_1, \ldots, a_k]$	Continued fraction with terms $a_1, \ldots, a_k$ (see Definition 1.5.17).
$c_m(f)$	Coefficient of the monomial χ^m in the series f (see Definition 1.4.1).
$\mathbf{c}_L$	Contact complexity function (see Definition 1.6.9).
$C_{L,L'}$	Strict transform of C by the Newton modification $\psi^C_{L,L'}$ (see Definition 1.4.14).
$\hat{C}_\pi$	Completion of C relative to the toroidal pseudo-resolution π (see Definition 1.4.15).
$\mathrm{Conv}(Y)$	Convex hull of a subset Y of a real affine space.
χ^m	Monomial with exponent $m \in M$ (see the beginning of Sect. 1.3.2).
∂X	Toric boundary of the toric variety X (see Definition 1.3.18), or toroidal boundary of the toroidal variety X (see Definition 1.3.29).
$\partial_+ \Lambda_\pi(C)$	Lateral boundary of the lotus $\Lambda_\pi(C)$ (see Definition 1.5.5).
$\mathbf{e}_L$	Exponent function (see Definition 1.6.3 and Notations 1.6.7).
f_K	Restriction of f to the compact edge K of its Newton polygon (see Definition 1.4.2).
$\mathcal{F}(f)$	Newton fan of the non-zero series $f \in \mathbb{C}[[x, y]]$ (see Definition 1.4.9).
$\mathcal{F}_{L,L'}(C)$	Newton fan of C relative to the cross (L, L') (see Definition 1.4.14).
$\mathcal{F}^{reg}$	Regularization of the fan $\mathcal{F}$ (see Definition 1.3.8).
$\Gamma(C)$	Enriques diagram of the finite constellation C (see Definition 1.4.31).
$H_{f,\rho}$	Supporting half-plane of the Newton polygon $\mathcal{N}(f)$ determined by the ray $\rho \subset \sigma_0$ (see Proposition 1.4.7).
$\mathbf{i}_L$	Index function (see Definition 1.6.3 and Notations 1.6.7).

$k_x(\xi, \xi')$	Order of coincidence of two Newton-Puiseux series (see Definition 1.6.2).
$k_x(C, C')$	Order of coincidence of two distinct branches, relative to a local coordinate system (x, y) (see Definition 1.6.2).
$l_{\mathbb{Z}}$	Integral length (see Definition 1.3.1).
(L, L')	Cross on a germ of smooth surface (see Definition 1.3.31).
$\Lambda(\mathcal{F})$	Lotus of the Newton fan $\mathcal{F}$ (see Definition 1.5.4).
$\Lambda(\lambda_1, \dots, \lambda_r)$	Lotus associated to the finite set $\{\lambda_1, \dots, \lambda_r\} \subset \mathbb{Q}_+ \cup \{\infty\}$ (see Definition 1.5.4).
$\Lambda_\pi(C)$	Lotus of the toroidal pseudo-resolution π of C (see Definition 1.5.26).
$\Lambda_\pi^{trunc}(C)$	Truncation of the lotus $\Lambda_\pi(C)$ (see Definition 1.5.35).
$m_o(C)$	Multiplicity of the plane curve singularity C at the point o (see Definition 1.2.5).
$M_{L,L'}$	Monomial lattice associated to the cross (L, L'), (see Definition 1.3.32).
$\mathbb{N}$	Set of non-negative integers.
$\mathbb{N}^*$	Set of positive integers.
$N_{L,L'}$	Weight lattice associated to the cross (L, L') (see Definition 1.3.32).
$\mathcal{N}(f)$	Newton polygon of the non-zero series $f \in \mathbb{C}[[x, y]]$ (see Definition 1.4.2).
$\mathcal{N}_{L,L'}(C)$	Newton polygon of C relative to the cross (L, L') (see Definition 1.4.14).
O_ρ	Toric orbit associated to the cone ρ of a fan (see the relation (1.22)).
$\hat{O}_{S,o}$	Completed local ring of the complex surface S at the point o (see Definition 1.2.5).
$\pi^*(C)$	Total transform of a plane curve singularity C by a modification π (see Definition 1.2.31).
$\psi_\sigma^{\mathcal{F}}$	Toric morphism from $X_{\mathcal{F}}$ to X_σ associated to any fan $\mathcal{F}$ which subdivides the cone σ (see relation (1.25)).
$\psi_{L,L'}^C$	Newton modification defined by C relative to the cross (L, L') (see Definition 1.4.14).
$\mathbb{R}_+$	Set of non-negative real numbers.
$\mathcal{S}(f)$	Support of the power series $f \in \mathbb{C}[[x, y]]$ (see Definition 1.4.1).
$\mathbf{S}_\pi$	Slope function of the toroidal pseudo-resolution π of C (see Definition 1.4.33).
σ_0	Regular cone generated by the canonical basis of the lattice $\mathbb{Z}^2$.
$\sigma_0^{L,L'}$	Regular cone generated by the canonical basis of the lattice $N_{L,L'}$ (see Definition 1.3.32).
t^w	One parameter subgroup of the algebraic torus $\mathcal{T}_N$, corresponding to the weight vector $w \in N$ (see the beginning of Sect. 1.3.2).

$\mathcal{T}_N$	Complex algebraic torus with weight lattice N (see formula (1.16)).
trop^f	Tropicalization of the non-zero power series $f \in \mathbb{C}[[x, y]]$ (see Definition 1.4.4).
$\mathrm{trop}^C_{L,L'}$	Tropical function of the curve singularity C relative to the cross (L, L') (see Definition 1.4.14).
$\theta(\mathcal{F})$	Trunk of the fan $\mathcal{F}$ (see Definition 1.4.32).
$\theta_\pi(C)$	Fan tree of the toroidal pseudo-resolution π of C (see Definition 1.4.33).
$\Theta_L(C)$	Eggers-Wall tree of the plane curve singularity C relative to the smooth branch L (see Definition 1.6.3 and Notations 1.6.7).
Θ_L	Universal Eggers-Wall tree (see Definition 1.6.12).
X_σ	Affine toric variety defined by the fan consisting of the faces of the cone σ (see Definition 1.3.14).
$X_{\mathcal{F}}$	Toric variety defined by the fan $\mathcal{F}$ (see Definition 1.3.15).
$\uplus$	Operation of the monoid of abstract lotuses (see formula (1.50)).
$\wedge$	Operation on the set $\mathbb{Q}_+^*$ allowing to describe the intersection of Newton lotuses (see formula (1.48)).
$Z(f)$	Zero-locus of a holomorphic function f or of a formal germ $f \in \hat{O}_{S,o}$.
$\mathcal{Z}_x(C)$	Set of Newton-Puiseux roots of a plane curve singularity C relative to a local coordinate system (x, y) (see Definition 1.6.2).

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