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Topological type of discriminants of some special families

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Abstract
We will describe the topological type of the discriminant curve of the morphism (ℓ, f), where ℓ is a smooth curve and f is an irreducible curve (branch) of multiplicity less than five or a branch such that the difference between its Milnor number and Tjurina number is less than 3. We prove that for a branch of these families, the topological type of the discriminant curve is determined by the semigroup, the Zariski invariant and at most two other analytical invariants of the branch.

1 Introduction

Let $f(x, y) \in \mathbb{C}\{x, y\}$ be irreducible. The germ of the irreducible analytic curve (*branch*) of equation $f(x, y) = 0$ is denoted by $C \equiv f(x, y) = 0$. Observe that the curves $f(x, y) = 0$ and $u(x, y) f(x, y) = 0$ are the same, for any unit $u(x, y) \in \mathbb{C}\{x, y\}$. The *multiplicity* of C, denoted by $m(C)$, is by definition the order of the power series $f(x, y)$. Suppose that C has multiplicity n. We will say that C is *singular* if $n > 1$. Otherwise C is a *smooth* curve. The *initial form* of $f(x, y)$ is the sum of all terms of $f(x, y)$ of degree n. Since f is irreducible, its initial form is a power of a linear form. After a linear change of coordinates, if necessary,

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we can suppose that the initial form of $f(x, y)$ is y^n. Suppose that C has multiplicity $n > 1$. We denote by $\mathbb{N}^*$ the set of positive integers. By Newton's theorem [7, Theorem 3.8] there is $\alpha(x^{1/n}) \in \mathbb{C}\{x\}^* = \bigcup_{m \in \mathbb{N}^*} \mathbb{C}\{x^{1/m}\}$ with $\alpha(0) = 0$ such that $f(x, \alpha(x^{1/n})) = 0$ and we say that $\alpha(x^{1/n}) \in \mathbb{C}\{x\}^*$ is a *Newton–Puiseux root* of C. Let us denote by $\mathrm{Zer}(f)$ the set of Newton–Puiseux roots of C. Let $\alpha(x^{1/n}) \in \mathrm{Zer}(f)$. After Puiseux's theorem [7, Corollary 3.12] we have that $\mathrm{Zer}(f) = \left\{\alpha_j := \alpha(\omega^j x^{1/n})\right\}_{j=1}^n$, where ω is an nth-primitive root of unity. Hence

$$f(x, y) = u(x, y) \prod_{j=1}^n \left(y - (\alpha(\omega^j x^{1/n})) \right), \tag{1.1}$$

where $u \in \mathbb{C}\{x, y\}$ is a unit. After a change of coordinates, if necessary, we can write $\alpha(x) = \sum_{i \geq s_1} a_i x^{i/n}$, where $s_1 > n$ and $s_1 \not\equiv 0 \bmod n$.

If we put $x = t^n$, where t is a new variable, the Newton–Puiseux root $\alpha(x^{1/n})$ can be written as

$$\begin{cases} x(t) = t^n, \\ y(t) = \sum_{i \geq s_1} a_i t^i, \end{cases}$$

that we will call the *Puiseux parametrisation* of C.

There are $g \in \mathbb{N}$ and a sequence $\left(\beta_0 = n < \beta_1 = s_1 < \beta_2 < \cdots < \beta_g\right)$ of nonnegative integers such that

$$\{\mathrm{ord}(\alpha_i - \alpha_j) : \alpha_i, \alpha_j \in \mathrm{Zer}(f), \ i \neq j\} = \left\{ \frac{\beta_l}{\beta_0} : 1 \leq l \leq g \right\} \subseteq \mathbb{Q} \backslash \mathbb{Z}. \tag{1.2}$$

The sequence $(\beta_0, \cdots, \beta_g) \subseteq \mathbb{N}$ is called the sequence of *characteristic exponents* of C. The number g is a topological invariant called the *genus* of the branch C.

Consider the set

$$S(C) := \{i_0(f, h) : h \in \mathbb{C}\{x, y\}, \ h \not\equiv 0 \bmod f\},$$

where $i_0(f, h) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f, h)$ is the *intersection number* (or *intersection multiplicity*) of $f(x, y) = 0$ and $h(x, y) = 0$ at the origin. It is well known that $S(C)$ is a semigroup called the *semigroup of values* of the branch C. The complement of $S(C)$ in $\mathbb{N}$ is finite. The *conductor* of $S(C)$ is by definition the smallest natural number $c \in \mathbb{N}$ such that every natural number $N \in \mathbb{N}$ with $N \geq c$ is an element of $S(C)$.

The semigroup $S(C)$ admits a minimal system of generators $(s_0, s_1, \ldots, s_g)$, where $s_{i-1} < s_i$, g is the genus of C, $s_0 = n = i_0(f, x)$ and $s_1 = m =: i_0(f, y)$. It is a well-known property of $S(C)$ [7, p. 88, (6.5)] that $e_k := \gcd(s_0, \ldots, s_k) = \gcd(\beta_0, \ldots, \beta_k)$ for $0 \leq k \leq g$ and $e_{k-1}s_k < e_k s_{k+1}$ for $1 \leq k \leq g - 1$.

If $n > 2$, then we have $c \geq s_1 + 1$. Let q be the number of natural numbers between s_1 and c which are not in $S(C)$. We can verify (see [16, p. 21]) that $q = \frac{c}{2} - s_1 + \left[\frac{s_1}{s_0}\right] + 1$, for $s_0 = n > 2$, where $[z]$ denotes the integral part of $z \in \mathbb{R}$.

Let $f, h \in \mathbb{C}\{x, y\}$ be irreducible power series. Using the Halphen–Zeuthen formula we get

$$i_0(f, h) = \sum_{i,j} \mathrm{ord}(\gamma_j - \alpha_i), \tag{1.3}$$

where $\mathrm{Zer}\, f = \{\alpha_i\}_i$ and $\mathrm{Zer}\, h = \{\gamma_j\}_j$.

Two branches C and D have the *same topological type* (or they are *equisingular*) if they are topologically equivalent as embedded curves in $\mathbb{C}^2$. It is well known [16, Chapter II]

that two branches are equisingular if and only if they have the same semigroup of values or equivalently they have the same characteristic exponents. Denote by $\mathcal{E}(C)$ the set of branches which are equisingular to C. In the set $\mathcal{E}(C)$ we define the next equivalence relation: two branches D_1 and D_2 in $\mathcal{E}(C)$ are *analytically equivalent*, and we will denote this by $D_1 \cong D_2$, if there exists an analytic isomorphism $T : U_1 \longrightarrow U_2$ such that U_i are neighbourhoods of the origin, D_i is defined in U_i, $1 \le i \le 2$ and $T(D_1 \cap U_1) = D_2 \cap U_2$.

The *moduli space* of the equisingularity class $\mathcal{E}(C)$ is the quotient space $\mathcal{E}(C)/\cong$. Let $v_1 < v_2 < \cdots < v_q$ be the integers of the set $\{s_1 + 1, \ldots, c - 1\}$ which are not in $S(C)$. Zariski proved [16, Proposition 1.2, Chapter III] that there exists a branch $\overline{C}$ analytically equivalent to C, parametrized as follows:

$$\begin{cases} \bar{x} = t^n, \\ \bar{y} = t^{s_1} + \sum_{i=1}^{q} a_i t^{v_i}. \end{cases} \tag{1.4}$$

Put $\Omega := \{\omega = g(x, y)dx + h(x, y)dy : g, h \in \mathbb{C}\{x, y\}\}$. If $(x(t), y(t))$ is a Puiseux parametrisation of C, we put

$$\nu(\omega) := \text{ord}\left(f(x(t), y(t))x'(t) + g(x(t), y(t))y'(t)\right) + 1.$$

Let $\Lambda := \{\nu(\omega) : \omega \in \Omega\}$. If $\Lambda \backslash S(C) \neq \emptyset$, then the number $\lambda := \min\left(\Lambda \backslash S(C)\right) - v_0$ is an analytical invariant of C called the *Zariski invariant*.
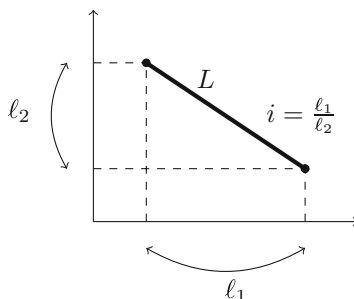
Using [16, Lemma 2.6, Chapter IV], we can rewrite the parametrization (1.4) in the next form:

$$\begin{cases} \bar{x} = t^n, \\ \bar{y} = t^{s_1} + at^{\lambda} + \text{a finite sum of terms } a_i t^{v_i}, \end{cases}$$

where $a \neq 0$, $v_i > \lambda > s_1$.

The Normal Forms Theorem (see [8, Theorem 1]) states that the plane branch C with semigroup of values $S(C) = \langle s_0, \ldots, s_g \rangle$ and value set of differentials Λ is either analytically equivalent to a branch with Puiseux parametrization (t^{s_0}, t^{s_1}) or Puiseux parametrization $(t^{s_0}, t^{s_1} + t^{\lambda} + \sum_{i>\lambda, i \notin \Lambda - s_0} a_i t^i)$.

Let $f(x, y) = \sum_{i,j} a_{ij} x^i y^j \in \mathbb{C}\{x, y\}$. The *support* of f is $\text{supp}(f) := \{(i, j) \in \mathbb{N}^2 : a_{ij} \neq 0\}$. The *Newton polygon* of f, denoted by $\mathcal{N}(f)$, is by definition the convex hull of $\text{supp}(f) + \mathbb{R}^2_{\geq 0}$. Observe that $\mathcal{N}(f) = \mathcal{N}(uf)$ for any unit $u \in \mathbb{C}\{x, y\}$. Nevertheless, the Newton polygon depends on coordinates. The *inclination* of any compact face L of $\mathcal{N}(f)$ is by definition the quotient of the length of the projection of L over the horizontal axis by the length of its projection over the vertical axis. The Newton polygon of f gives information on the Newton–Puiseux roots of $f(x, y) = 0$. More precisely, if L is a compact face of $\mathcal{N}(f)$ of inclination i and the length of its projection over the vertical axis is ℓ_2, then f has ℓ_2 Newton–Puiseux roots of order i (see [3, Lemme 8.4.2]).

We say that $f(x, y) \in \mathbb{C}\{x, y\}$ is *nondegenerate* in the sense of Kouchnirenko with respect to the coordinates (x, y), if for any compact edge L of $\mathcal{N}(f)$ the polynomial $f_L(x, y) := \sum_{(i,j) \in L \cap \text{supp}(f)} a_{ij} x^i y^j$ does not have critical points outside the axes $x = 0$ and $y = 0$, or equivalently, the polynomial $F_L(z) := \frac{f_L(1,z)}{z^{j_0}}$ has no multiple roots, where $j_0 := \min\{j \in \mathbb{N} : (i, j) \in L\}$. Since $\mathcal{N}(f) = \mathcal{N}(uf)$, for any unit $u \in \mathbb{C}\{x, y\}$, the notion of nondegeneracy is extended to curves. The topological type of nondegenerate plane curves are completely determined by their Newton polygons (see [13, Proposition 4.7] and [6, Theorem 3.2]).
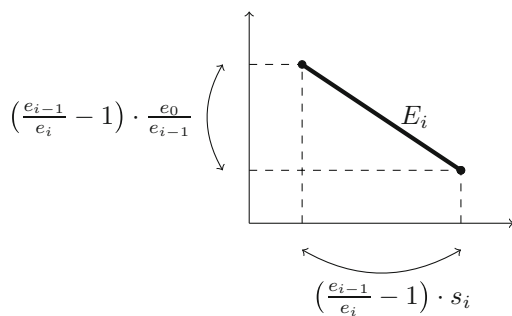
Let $\ell(x, y) = 0$ be a smooth curve and let $f(x, y) = 0$ define an isolated singularity at $0 \in \mathbb{C}^2$. Assume that $\ell(x, y)$ does not divide $f(x, y)$ and consider the morphism

$$(\ell, f) \colon (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0),$$
$$(x, y) \longmapsto (u, v) := (\ell(x, y), f(x, y)). \tag{1.5}$$

There are two curves associated with (ℓ, f): the *polar curve* $\frac{\partial \ell}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial \ell}{\partial y} \frac{\partial f}{\partial x} = 0$ and its direct image $D(u, v) = 0$ which is called the *discriminant curve* of the morphism (ℓ, f).

The topological type of the polar curve depends on the analytical type of $\ell(x, y) = 0$ and $f(x, y) = 0$. In [9] the authors completely determine the topological type of the generic polar curve when the multiplicity of $f(x, y) = 0$ is less than 5.

The Newton polygon of $D(u, v)$ in the coordinates (u, v) is called the *jacobian Newton polygon* of the morphism (ℓ, f). This notion was introduced by Teissier [14], who proved that the inclinations of this jacobian polygon are topological invariants of (ℓ, f) called *polar invariants*. After Merle [12], when f is irreducible with semigroup of values $S(f) = \langle s_0, s_1, \ldots, s_g \rangle$ then the jacobian Newton polygon of (ℓ, f) has g compact edges $\{E_i\}_{i=1}^g$. The length of the projection of E_i on the vertical axis is $\left(\frac{e_{i-1}}{e_i} - 1\right) \cdot \frac{e_{i-1}}{e_0}$. The length of the projection of E_i on the horizontal axis is $\left(\frac{e_{i-1}}{e_i} - 1\right) \cdot s_i$. Hence the inclinations (quotient between the length of the horizontal projection and the length of the vertical projection) of the compact edges of the jacobian polygon are $s_1 < \frac{e_1}{e_0} s_2 < \frac{e_2}{e_0} s_3 < \cdots < \frac{e_{g-1}}{e_0} s_g$.

In [5] the authors study the pairs (ℓ, f) for which the discriminant curve is nondegenerate in the Kouchnirenko sense. In particular, when f is irreducible, after [5, Corollary 4.4] the discriminant curve $D(u, v) = 0$ is nondegenerate if and only if the multiplicity of $f(x, y) = 0$ equals 2 or equals 4 and the genus equals 2. Otherwise the discriminant curve is degenerate. Our aim in this paper will be to describe the topological type of the discriminant curve $D(u, v) = 0$ of the morphism (ℓ, f), where f is irreducible and belongs to some special families, as for example, branches of multiplicity less than 5, branches C such that the difference between its *Milnor number* $\mu(C)$ and *Tjurina number* $\tau(C)$ is less than 3, with $\mu(C) = i_0\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \dim$, not $\text{im}_\mathbb{C} \mathbb{C}\{x, y\}/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ and

$\tau(C) = \dim_{\mathbb{C}}\mathbb{C}\{x, y\}/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, where $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ (resp. $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f\right)$) denotes the ideal of $\mathbb{C}\{x, y\}$ generated by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (resp. by $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and f).

For these families of plane branches we determine the topological type of their discriminant curves, in the spirit of [9]. We prove that the topological type of the discriminant curve $D(u, v) = 0$ is determined, at most, by the semigroup of values $S(f)$, the Zariski invariant and two other analytical invariants of the curve $f(x, y) = 0$. In all cases we explicitly determine such analytical invariants. Hence, in order to describe the topological type of the discriminant curve of a branch, it is necessarily the same number of analytical invariants of the initial branch, as it happens for its generic polar curves (see [9]). Finally in Sect. 5 we summarize the different topological types of the discriminant curve in some tables.

2 Equation of the discriminant curve

An analytic change of coordinates does not affect the discriminant curve of the morphism defined in (1.5) (see for example [2, Section 3]). Hence in what follows we assume that $\ell(x, y) = x$. Then $\frac{\partial f}{\partial y} = 0$ is the polar curve of the morphism (x, f).

In this paper we will determine the topological type of the discriminant curve of the morphism (1.5) for $\ell(x, y) = x$ and $f(x, y) \in \mathbb{C}\{x, y\}$ irreducible belonging to some special families. The corresponding study relative to the polar curves was done in [10] and [11], where the authors characterize the equisingularity classes of irreducible plane curve germs whose general members have nondegenerate general polar curves. In addition, they give explicit Zariski open sets of curves in these equisingularity classes whose general polars are nondegenerate and describe their topology.

Suppose that the Newton–Puiseux factorizations of $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are of the form

$$f(x, y) = u_1(x, y) \prod_{i=1}^{n} [y - \alpha_i(x)], \tag{2.1}$$

$$\frac{\partial f}{\partial y}(x, y) = u_2(x, y) \prod_{j=1}^{n-1} [y - \gamma_j(x)], \tag{2.2}$$

where $u_1(x, y), u_2(x, y) \in \mathbb{C}\{x, y\}$ are units, $n = \text{ord} f$, $\text{Zer}(f) = \{\alpha_i(x)\}_i$ and $\text{Zer}\left(\frac{\partial f}{\partial y}\right) = \{\gamma_j(x)\}_j$. If f is irreducible of order n then n is the smallest natural number such that $\{\alpha_i(x)\}_i \subset \mathbb{C}\{x^{1/n}\}$. Moreover, if we fix $\alpha_i(x^{1/n})$ then $\alpha_j(x^{1/n}) = \alpha_i(\omega x^{1/n})$ for any $1 \leq j \leq n$, where w is an nth root of unity.

Following [4, Lemma 5.4] the discriminant curve of the morphism (x, f) can be written as

$$D(u, v) = \prod_{j=1}^{n-1} (v - f(u, \gamma_j(u))). \tag{2.3}$$

3 Discriminants of branches of small multiplicities

In this section we determine the topological type of the discriminant of the morphism given in (1.5), where $C \equiv f(x, y) = 0$ has small multiplicity. For this we will make use the results

of [16] and the analytic classification of plane branches of multiplicity less than or equal to 4, given in [8].

3.1 Discriminants of branches of multiplicity 2

Let $C \equiv f(x, y) = 0$ be a branch of multiplicity 2. The minimal system of generators of the semigroup of C is $\langle 2, s_1 \rangle$, where s_1 is an odd natural number. By [16, Chapitre V] the moduli space of branches of multiplicity 2 have a unique point whose parametrization is given by (t^2, t^{s_1}), that is the branch $y^2 - x^{s_1} = 0$. Then $f_y(x, y) = 2y$ whose Newton–Puiseux root is $y = 0$. Hence, after (2.3) we have $D(u, v) = v - f(u, 0) = v + u^{s_1}$, that is the discriminant curve is smooth. The Newton polygon of $D(u, v)$ has only one compact edge. The univariate polynomial associated with this edge is $z + 1$, so $D(u, v)$ is nondegenerate.

3.2 Discriminants of branches of multiplicity 3

Let $C \equiv f(x, y) = 0$ be a branch of multiplicity 3. The minimal system of generators of the semigroup of C is $\langle 3, s_1 \rangle$, where $s_1 \in \mathbb{N}$ is such that $s_1 \not\equiv 0 \bmod 3$. By [16, Chapitre V] the moduli space of branches of multiplicity 3 is completely determined by the semigroup of the branch and its Zariski invariant λ. The corresponding normal forms are as follows:

$$x = t^3, \quad y = t^{s_1} + a t^\lambda,$$

where $a = 0$ when $\lambda = 0$ or if $\lambda \neq 0$ then we have $a = 1$ and

$$\lambda = \begin{cases} 3e + 3k + 4 & \text{when } s_1 = 3e + 2 \\ 3e + 3k + 2 & \text{when } s_1 = 3e + 1, \end{cases} \quad \text{where } 0 \leq k \leq e - 2. \tag{3.1}$$

Observe that if $\lambda \neq 0$ then $\frac{s_1 + \lambda}{3}$ is a natural number greater than 2.

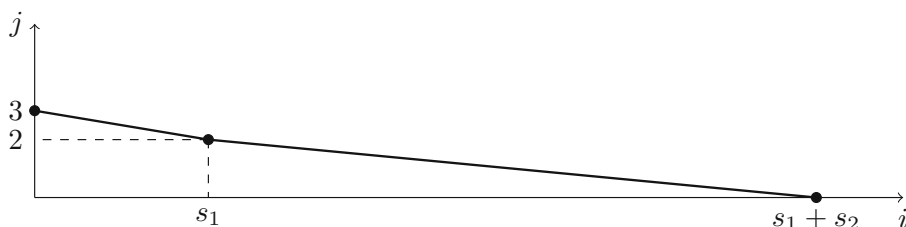**Proposition 3.1** *Let $C \equiv f(x, y) = 0$ be a branch of semigroup $\langle 3, s_1 \rangle$ and let the Zariski invariant equal λ. The discriminant curve $D(u, v) = 0$ is degenerate and its topological type is determined by $(3, s_1, \lambda)$ in the following way:*

(1) If $\lambda = 0$ then the discriminant is the double smooth branch $(v + u^{s_1})^2 = 0$.
(2) If $\lambda \neq 0$ then

    (a) when $\gcd(2, s_1 + \lambda) = 2$, the discriminant curve is the union of two smooth branches $D_i(u, v) = 0$, $1 \leq i \leq 2$, with intersection number $i_0(D_1, D_2) = \frac{s_1 + \lambda}{2}$.
    (b) When $\gcd(2, s_1 + \lambda) = 1$, the discriminant is a branch of semigroup $\langle 2, s_1 + \lambda \rangle$.

Proof Suppose $\lambda = 0$. The implicit equation of the normal form is $f(x, y) = y^3 - x^{s_1} = 0$. Then $f_y(x, y) = 3y^2$ whose Newton–Puiseux root is $y = 0$, with multiplicity 2. Hence, after (2.3) we have $D(u, v) = (v - f(u, 0))^2 = (v + u^{s_1})^2$, that is the discriminant is a double smooth branch.

Suppose now $\lambda \neq 0$. The normal forms are $x = t^3$, $y = t^{s_1} + t^\lambda$, with λ as in (3.1). After (1.1) the implicit equation is $f(x, y) = y^3 - 3x^{\frac{s_1 + \lambda}{3}} y - (x^{s_1} + x^\lambda)$. Then $f_y(x, y) = 3y^2 - 3x^{\frac{s_1 + \lambda}{3}}$, whose Newton–Puiseux roots are $\gamma_i(x) = \pm x^{\frac{s_1 + \lambda}{6}}$ for $1 \leq i \leq 2$. Using (2.3) we have

$$D(u, v) = \left(v + u^{s_1} - 2u^{\frac{s_1 + \lambda}{2}} + u^\lambda \right) \left(v + u^{s_1} + 2u^{\frac{s_1 + \lambda}{2}} + u^\lambda \right).$$

Fig. 1 Jacobian Newton polygon of a branch with semigroup $\langle 4, s_1, s_2 \rangle$

Hence $\mathrm{Zer}(D) = \left\{ \eta_1 := -u^{s_1} - 2u^{\frac{s_1+\lambda}{2}} - u^\lambda, \eta_2 := -u^{s_1} + 2u^{\frac{s_1+\lambda}{2}} - u^\lambda \right\}$ and $\mathrm{ord}(\eta_1 - \eta_2) = \frac{s_1+\lambda}{2}$. We distinguish two cases: if $\gcd(2, s_1 + \lambda) = 2$ then by (1.2) we conclude that $D(u, v) = 0$ has two smooth branches of equations $D_1(u, v) := v + u^{s_1} - 2u^{\frac{s_1+\lambda}{2}} + u^\lambda$ and $D_2(u, v) = v + u^{s_1} + 2u^{\frac{s_1+\lambda}{2}} + u^\lambda$ such that, after the Halphen–Zeuthen formula, the intersection multiplicity is $i_0(D_1, D_2) = \frac{s_1+\lambda}{2}$. If $\gcd(2, s_1 + \lambda) = 1$ then the discriminant curve $D(u, v) = 0$ is a singular branch of the semigroup $\langle 2, s_1 + \lambda \rangle$.

The Newton polygon of $D(u, v)$ is elementary (it has only one compact edge) and the univariate polynomial associated with its compact edge is $(z + 1)^2$. Hence the discriminant $D(u, v) = 0$ is degenerate. $\qquad\square$

Corollary 3.2 *If C is a branch of multiplicity 2 or 3 and nonzero Zariski invariant, then the discriminant curve $D(u, v) = 0$ has no multiple irreducible branches.*

Corollary 3.2 does not hold for branches of multiplicity 4 as the proof of Proposition 3.5 shows.

3.3 Discriminants of branches of multiplicity 4

Let $C \equiv f(x, y) = 0$ be a branch of multiplicity 4. The branch C may have genus 1 or 2.

Proposition 3.3 *Let $C \equiv f(x, y) = 0$ be a branch of the semigroup $\langle 4, s_1, s_2 \rangle$. Then the discriminant curve $D(u, v) = 0$ is nondegenerate. Moreover, $D(u, v) = D_1(u, v) D_2(u, v)$, where D_1 is a smooth branch, D_2 is a singular branch of semigroup $\langle 2, s_2 \rangle$ and the intersection multiplicity between the two branches is $i_0(D_1, D_2) = 2s_1$.*

Proof For genus 2, and after the second part of [5, Corollary 4.4] we get that $D(u, v) = 0$ is nondegenerate and we can determine its topological type from its Newton polygon (see [13, Proposition 4.7] and [6, Theorem 3.2]), which is the jacobian Newton polygon of (x, f) (see Fig. 1).

Since s_2 is an odd natural number, $\gcd(s_1 + s_2, 2) = 1$. Moreover, after the properties of $S(f)$, we get $e_0 s_1 = 4s_1 < 2s_2 = e_1 s_2$, hence $s_2 > 2s_1$ and $D(u, v) = D_1(u, v) D_2(u, v)$, where $D_1(u, v) = 0$ is a smooth curve admitting as parametrization $(t, t^{s_1} + \cdots)$ and $D_2(u, v) = 0$ is a singular curve of genus 1 which semigroup of values is $\langle 2, s_2 \rangle$. Finally, after the Halphen–Zeuthen formula, the intersection multiplicity between the two branches is $i_0(D_1, D_2) = 2s_1$. $\qquad\square$

Suppose now that the branch C has genus 1 and semigroup of values equals $\langle 4, s_1 \rangle$. By [8], the moduli space of branches of multiplicity 4 and genus 1 has five families of normal forms:

$$\text{NF } 4.i : \quad x = t^4, \quad y_i = \sigma_i(t), \quad 1 \le i \le 5, \tag{3.2}$$

where λ_i is the Zariski invariant of the ith normal form family. More precisely, we have

(1) $(\lambda_1, \sigma_1(t)) = (0, t^{s_1})$,
(2) If $2 \le i \le 4$ then $\lambda_i = 2s_1 - 4j$ for $2 \le j \le [\frac{s_1}{4}]$ and

$$\sigma_2(t) = t^{s_1} + t^{\lambda_2} + a_k t^{3s_1 - (4[\frac{s_1}{4}]+j+1-k)} + \cdots + a_{j-[\frac{s_1}{4}]-2} t^{3s_1 - 4([\frac{s_1}{4}]+3-k)},$$

with $a_k \neq 0$, and $1 \le k \le [\frac{s_1}{4}] - j$.

$$\sigma_3(t) = t^{s_1} + t^{\lambda_3} + \frac{3s_1 - 4j}{2s_1} t^{3s_1 - 8j} + a_{[\frac{s_1}{4}]-j+2} t^{3s_1 - 4(2j-1)} + \cdots$$
$$+ a_{[\frac{s_1}{4}]} t^{3s_1 - (j+1)}.$$

$$\sigma_4(t) = t^{s_1} + t^{\lambda_4} + a_{[\frac{s_1}{4}]-j+1} t^{3s_1 - 8j} + a_{[\frac{s_1}{4}]-j+2} t^{3s_1 - 4(2j-1)} + \cdots$$
$$+ a_{[\frac{s_1}{4}]-1} t^{3s_1 - 4(j+2)},$$

where $a_{[\frac{s_1}{4}]-j+1} \neq \frac{3s_1 - 4j}{2s_1}$.
(3) If $i = 5$ then $\lambda_5 = 3s_1 - 4j$ for $2 \le j \le [\frac{s_1}{2}]$ and

$$\sigma_5(t) = t^{s_1} + t^{3s_1 - 4j} + a_k t^{2s_1 - 4(j - [\frac{s_1}{4}]-k)} + a_{k+s} t^{2s_1 - 4(j - [\frac{s_1}{4}]k-s)}$$
$$+ \cdots$$

After the Newton–Puiseux Theorem, the ith normal form has the equation

$$f_i(x, y) = \prod_{\omega^4 = 1} (y - \sigma_i(\omega x^{1/4})), \tag{3.3}$$

where ω is a 4th primitive root of unity.

Hence, we obtain the implicit equation for each normal form family:

NF 4.1: $f_1(x, y) = y^4 - x^{s_1}$.
NF 4.i: For $2 \le i \le 4$ we get

$$f_i(x, y) = y^4 + P_i(x)y^2 + Q_i(x)y + x^{s_1} u(x), \tag{3.4}$$

where $u(x) \in \mathbb{C}\{x\}$ is a unit (that is $u(0) \neq 0$), $\lambda_i = 2s_1 - 4j$ for $2 \le j \le [\frac{s_1}{4}]$, $Q_i(x) = -4x^{s_1 - j} + \cdots$ and

$$P_i(x) = \begin{cases} -4a_k x^{s_1-([\frac{s_1}{4}]+j+1-k)} + \cdots & \text{for } i = 2 \\[2mm] bx^{s_1-2j} + \cdots & \text{for } i = 3 \text{ where } b = \frac{-8(s_1-j)}{s_1}. \\[2mm] (-2 - 4a_{[\frac{s_1}{4}]-j+1})x^{s_1-2j} + \cdots & \text{for } i = 4 \text{ and } -2 - 4a_{[\frac{s_1}{4}]-j+1} \neq 0 \\[2mm] 0 & \text{for } i = 4, \ -2 - 4a_{[\frac{s_1}{4}]-j+1} = 0 \\ & \text{and } a_{[\frac{s_1}{4}]-j+2} = \cdots = a_{[\frac{s_1}{4}]-1} = 0 \\[2mm] -4a_{[\frac{s_1}{4}]-j+k_0}x^{s_1-(2j-(k_0-1))} + \cdots & \text{for } i = 4, \ -2 - 4a_{[\frac{s_1}{4}]-j+1} = 0 \\ & \text{and } a_l \neq 0 \text{ for some } l = [\frac{s_1}{4}] - j + 2, \ldots, [\frac{s_1}{4}] - 1 \end{cases}$$

where $2 \leq k_0 \leq j - 1$ such that

$$k := \min\left\{l : a_l \neq 0, \ \left[\frac{s_1}{4}\right] - j + 2 \leq l < \left[\frac{s_1}{4}\right]\right\} = \left[\frac{s_1}{4}\right] - j + k_0.$$

Observe that

$$\operatorname{ord} P_i(x) \geq \frac{2}{3}(s_1 - j). \tag{3.5}$$

NF 4.5: $f_5(x, y) = y^4 + P(x)y^2 + Q(x)y + R(x)$, where

$$P(x) = -4x^{s_1-j} - 2a_k^2 x^{s_1-2(j-[\frac{s_1}{4}]-k)} - 4a_k a_{k+s} x^{s_1-2(j-s_1-[\frac{s_1}{4}]-k)+s}$$
$$-2a_{k+s}^2 x^{s_1-2(j-[\frac{s_1}{4}]-k-s)} + \cdots \tag{3.6}$$

$$Q(x) = -4a_k x^{s_1-j+[\frac{s_1}{4}]+k} - 4a_{k+s}x^{s_1-j-[\frac{s_1}{4}]+k+s}$$
$$-4a_k x^{2(s_1-j)-(j-[\frac{s_1}{4}]-k)} - 4a_{k+s}x^{2(s_1-j)-(j-[\frac{s_1}{4}]-k-s)} + \cdots) \tag{3.7}$$

$$R(x) = -x^{s_1} + 2x^{2(s_1-j)} - x^{3s_1-4j} + a_k x^{2s_1-4(j-[\frac{s_1}{4}]-k)} + \cdots \tag{3.8}$$

for $2 \leq j \leq [\frac{s_1}{2}]$.

Proposition 3.4 *Let* $C \equiv f(x, y) = 0$ *be a branch belonging to the family NF 4.i, for* $1 \leq i \leq 4$. *Then the discriminant curve* $D(u, v) = 0$ *is degenerate and its topological type is determined by the semigroup* $S(f) = \langle 4, s_1 \rangle$ *and the Zariski invariant* λ_i *of* C. *Moreover,*

(1) *if* $\lambda_i = 0$ *then the discriminant curve is the triple smooth branch* $(v + u^{s_1})^3 = 0$;

(2) *if* $\lambda_i \neq 0$ *then* $\lambda_i = 2s_1 - 4j$ *for* $2 \leq j \leq [\frac{s_1}{4}]$ *and*

   (a) *when* $\gcd(3, 2s_1 + \lambda_i) = 1$, *the discriminant curve* $D(u, v) = 0$ *is a branch with semigroup* $\langle 3, 2s_1 + \lambda_i \rangle$;

   (b) *when* $\gcd(3, 2s_1 + \lambda_i) = 3$, *the discriminant curve is the union of three smooth branches* $D_i(u, v) = 0$, $1 \leq i \leq 3$ *with intersection number* $i_0(D_l, D_r) = \frac{2s_1+\lambda_i}{3}$ *for* $l \neq r$.

Proof Suppose $\lambda_i = 0$. By (3.3) the implicit equation of the normal form is $f_1(x, y) = y^4 - x^{s_1}$. Then $\frac{\partial f_1}{\partial y}(x, y) = 4y^3$ whose Newton–Puiseux root is $y = 0$, with multiplicity three. Hence, after (2.3) we have $D(u, v) = (v - f_1(u, 0))^3 = (v + u^{s_1})^3$, that is the discriminant curve is a triple smooth branch.

Suppose now $\lambda_i \neq 0$. Then $\lambda_i = 2s_1 - 4j$ for $2 \leq j \leq [\frac{s_1}{4}]$ and the normal form of C is $x = t^4$, $y = \sigma_i(t)$ for $2 \leq i \leq 4$.

From the implicit equations $f_i(x, y)$, $2 \leq i \leq 4$, given in (3.4) and from inequality (3.5) we get that the Newton polygon of $\frac{\partial f_i}{\partial y}(x, y)$ has only one compact edge whose vertices are

$(0, 3)$ and $(s_1 - j, 0)$. All the parametrisations of $\frac{\partial f_i}{\partial y}(x, y) = 0$, for $2 \leq i \leq 4$, have the same order and we can write them as $\gamma_r(u) = \varepsilon u^{\frac{s_1-j}{3}} + \cdots$, where ε is a 3rd root of unity. From (2.3) and for a fixed $i \in \{2, 3, 4\}$, we have

$$D(u, v) = \prod_{\frac{\partial f_i}{\partial y}(\gamma_r)=0} (v - f_i(u, \gamma_r(u))),$$

and considering the development of $f_i(u, \gamma_r(u))$ we obtain

$$D(u, v) = \prod_{\varepsilon^3=1} \left(v + u^{s_1} + 3\varepsilon u^{4(\frac{s_1-j}{3})} + \cdots\right) = \prod_{\varepsilon^3=1} \left(v + u^{s_1} + 3\varepsilon u^{(\frac{2s_1+\lambda_i}{3})} + \cdots\right),$$

so $\mathrm{Zer}(D) = \left\{\eta_i(\varepsilon) := -u^{s_1} - 3\varepsilon u^{(\frac{2s_1+\lambda_i}{3})} + \cdots\right\}_{\varepsilon^3=1}$ and $\mathrm{ord}(\eta_l - \eta_r) = \frac{2s_1+\lambda_i}{3}$, for $1 \leq l \neq r \leq 3$. The topological type of $D(u, v) = 0$ is determined by the semigroup of the branch $C \equiv f_i(x, y) = 0$ and its Zariski analytical invariant $\lambda_i = 2s_1 - 4j$ for $2 \leq j \leq [\frac{s_1}{4}]$. We distinguish two cases: if 3 and $2s_1 + \lambda_i$ are coprime then the discriminant $D(u, v) = 0$ is a branch of the semigroup $\langle 3, 2s_1 + \lambda_i \rangle$. Otherwise, by (1.2), we conclude that the discriminant curve is the union of three different smooth branches $D_l(u, v) = 0$ with intersection multiplicity $i_0(D_l, D_r) = \frac{2s_1+\lambda_i}{3}$ for $l \neq r$.

In all cases the Newton polygon of $D(u, v)$ is elementary with vertices $(0, 3)$ and $(3s_1, 0)$. The polynomial associated with its compact edge is $(z + 1)^3$, so the discriminant curve $D(u, v) = 0$ is degenerate.   □

Proposition 3.5 *Let $C \equiv f(x, y) = 0$ be a branch belonging to the family NF 4.5. Then the discriminant curve $D(u, v) = 0$ is degenerate and its topological type is determined by the semigroup $S(f) = \langle 4, s_1 \rangle$, the Zariski invariant λ_5 and at most two other analytical invariants of C.*

Proof The implicit equation of C has the form $f_5(x, y) = y^4 + P(x)y^2 + Q(x)y + R(x)$, where $P(x)$, $Q(x)$, $R(x)$ are as in (3.6), (3.7) and (3.8). Hence $\frac{\partial f_5}{\partial y}(x, y) = 4y^3 + 2P(x)y + Q(x)$.

We distinguish the following cases:

Case A. If $a_i = 0$ in $\sigma_5(t)$ for all i, then we have that $\frac{\partial f_5}{\partial y}(x, y) = 4y^3 - 8x^{s_1-j}y = 4y(y^2 - 2x^{s_1-j})$ whose Newton–Puiseux roots are $\left\{0, \pm\sqrt{2}x^{\frac{s_1-j}{2}}\right\}$. Therefore $f_5(u, 0) = R(u) = -u^{s_1} + 2u^{2(s_1-j)} - u^{3s_1-4j}$ and $f_5(u, \pm\sqrt{2}u^{\frac{s_1-j}{2}}) = -u^{s_1} - 2u^{2(s_1-j)} - u^{3s_1-4j}$. From (2.3) we conclude that the discriminant curve is the union of three smooth curves $D_l(u, v) = 0$, where $D_1(u, v) = v + u^{s_1} - 2u^{2(s_1-j)} + u^{3s_1-4j}$, $D_2(u, v) = D_3(u, v) = v + u^{s_1} + 2u^{2(s_1-j)} + u^{3s_1-4j}$, and $i_0(D_1, D_l) = 2(s_1 - j)$ for $l \in \{2, 3\}$.

Case B. If $a_k \neq 0$ and $a_{k+l} = 0$ in $\sigma_5(t)$ for $l > 0$, then

$$\frac{\partial f_5}{\partial y}(x, y) = 4y^3 + 2(-4x^{s_1-j} - 2a_k^2 x^{s_1-2(j-[\frac{s_1}{4}]-k)} + \cdots)y$$

$$- 4a_k \left(x^{s_1-j+[\frac{s_1}{4}]+k} + x^{2(s_1-j)-(j-[\frac{s_1}{4}]-k)} + \cdots\right).$$

So the Newton polygon of $\frac{\partial f_5}{\partial y}(x, y)$ depends on the position of the point $M = (s_i - j, 1)$ with respect to the line passing by $E = (0, 3)$ and $F = (s_1 - j + [\frac{s_1}{4}] + k, 0)$. We have three subcases:

B.1. If $\frac{2}{s_1-j} < \frac{1}{[\frac{s_1}{4}]+k}$ then $\mathcal{N}(\frac{\partial f_5}{\partial y})$ has only one compact edge of vertices E and F. Hence the order of the Newton–Puiseux roots $\{\gamma_i\}_{i=1}^3$ of $\frac{\partial f_5}{\partial y}(x, y) = 0$ equals $\frac{s_1-j+[\frac{s_1}{4}]+k}{3}$. After (2.3) we obtain

$$D(u, v) = \prod_{w_i^3=1}\left(v - 3a^{\frac{4}{3}}w_i u^{4\left(\frac{s_1-j+[\frac{s_1}{4}]+k}{3}\right)}\right) + u^{s_1} + \cdots),$$

for some nonzero complex number a and where w_i is a cubic root of unity. If $\gcd(3, (s_1 - j + [\frac{s_1}{4}] + k)) = 1$ then the discriminant curve is irreducible with semigroup $\langle 3, 4(s_1 - j + [\frac{s_1}{4}]+k)\rangle$. In the other case, we get $\gcd(3, (s_1 - j + [\frac{s_1}{4}]+k)) = 3$ and the discriminant curve is the union of three smooth curves $D_l(x, y) = 0$ such that $i_0(D_l, D_r) = \frac{4(s_1-j+[\frac{s_1}{4}]+k)}{3}$.

B.2. If $\frac{2}{s_1-j} > \frac{1}{[\frac{s_1}{4}]+k}$ then $\mathcal{N}(\frac{\partial f_5}{\partial y})$ has two compact edges of vertices M, E and F. By (2.3) we get

$$D(u, v) = (v + u^{s_1} - 2u^{2(s_1-j)} + \cdots)(v + u^{s_1} + 2u^{2(s_1-j)} \pm 4\sqrt{2}a_k u^{\frac{3}{2}(s_1-j)+[\frac{s_1}{4}]+k} + \cdots).$$

If $s_1 - j$ is odd then the discriminant curve is the union of a smooth branch $D_1(u, v) = 0$ and a singular branch $D_2(x, y) = 0$ with semigroup $\langle 2, 3(s_1 - j) + 2([\frac{s_1}{4}] + k)\rangle$. Moreover, $i_0(D_1, D_2) = 4(s_1 - j)$. Otherwise, if $s_1 - j$ is even then the discriminant curve is the union of three smooth branches $D_l(x, y) = 0$ such that $i_0(D_1, D_l) = 2(s_1 - j)$ for $2 \leq l \leq 3$ and $i_0(D_2, D_3) = \frac{3}{2}(s_1 - j) + [\frac{s_1}{4}] + k$.

B.3. If $\frac{2}{s_1-j} = \frac{1}{[\frac{s_1}{4}]+k}$ then $\mathcal{N}(\frac{\partial f_5}{\partial y})$ has only one compact edge of vertices E and F and M is an interior point of this edge. The polynomial in one variable associated with the compact edge of $\mathcal{N}(\frac{\partial f_5}{\partial y})$ is $p(z) = z^3 - 2z - a_k$. After the z-discriminant of $p(z)$ we have that the roots of $p(z)$ are simple if and only if $a_k \neq \pm\left(\frac{4\sqrt{6}}{9}\right)$. We will study both cases:

B.3.1 Suppose that $a_k \neq \pm\left(\frac{4\sqrt{6}}{9}\right)$. Denote by z_i the three different roots of the polynomial $p(z)$. For any $\gamma_i(x) \in \mathrm{Zer}(\frac{\partial f_5}{\partial y})$ we have

$f(u, \gamma_i(u)) = -u^{s_1} + q(z_i)u^{2(s_1-j)} + \cdots$, where $q(z) = z^4 - 4z^2 - 4a_kz + 2$.

Since $z^3 - 2z = a_k$, we get $q(z) = -3z^4 + 4z^2 + 2$ and if $z_r \neq z_l$ then $q(z_r) \neq q(z_l)$. Hence $D(u, v) = \prod_i(z + u_i^s + q(z_i)u^{2(s_1-j)} + \cdots)$ and the discriminant curve $D(u, v) = 0$ is the union of three smooth branches $D_l(x, y) = 0$ such that $i_0(D_l, D_r) = 2(s_1 - j)$.

B.3.2 Suppose that $a_k = \frac{4\sqrt{6}}{9}$. The polynomial $p(z)$ has $z_1 = 2\sqrt{\frac{2}{3}}$ as a simple root and $z_2 = -\sqrt{\frac{2}{3}}$ as a double root. If $\gamma_i \in \mathrm{Zer}(\frac{\partial f_5}{\partial y})$ corresponds to z_i then

$$f_5(u, \gamma_1(u)) = u^{s_1} + q(z_1)u^{2(s_1-j)} + \cdots,$$
$$f_5(u, \gamma_2(u)) = u^{s_1} + q(z_2)u^{2(s_1-j)} + r_2 u^{2(s_1-j)+\frac{s_1-2j}{2}} + \cdots,$$
$$f_5(u, \gamma_3(u)) = u^{s_1} + q(z_2)u^{2(s_1-j)} + r_3 u^{2(s_1-j)+\frac{s_1-2j}{2}} + \cdots,$$

where $r_2, r_3 \in \mathbb{C}$ are different. Observe that $q(z_1) \neq q(z_2)$. Hence, if $s_1 - 2j$ is odd then the discriminant curve is the union of a smooth branch $D_1(u, v) = 0$ and a singular branch $D_2(u, v) = 0$ with semigroup $\langle 2, 5s_1 - 6j\rangle$, where $i_0(D_1, D_2) = 4(s_1 - j)$. Otherwise, if $s_1 - 2j$ is even then the discriminant curve is the union of three smooth branches $D_l(u, v) = 0$ such that $i_0(D_1, D_l) = 2(s_1 - j)$, for $2 \leq l \leq 3$ and $i_0(D_2, D_3) = \frac{5s_1-7j}{2}$.

B.3.3 Suppose that $a_k = -\frac{4\sqrt{6}}{9}$. The polynomial $p(z)$ has $2\sqrt{\frac{2}{3}}$ as a double root and $-\sqrt{\frac{2}{3}}$ as a simple root. After a similar procedure we conclude, in this case, that the topological type of the discriminant curve is as in B.3.2.

Case C If $a_k \neq 0 \neq a_{k+s}$, then we have $\frac{\partial f_5}{\partial y}(x, y) = 4y^3 + 2P(x)y + Q(x)$. The Newton polygon of $\frac{\partial f_5}{\partial y}(x, y)$ depends on the position of the point $M = (s_i - j, 1)$ with respect to the line passing through $E = (0, 3)$ and $F = (s_1 - j + [\frac{s_1}{4}] + k, 0)$. We have the following subcases:

C.1. If $\frac{2}{s_1-j} < \frac{1}{[\frac{s_1}{4}]+k}$ then the topological type of the discriminant curve $D(u, v) = 0$ is as in In emphatic way.

C.2. If $\frac{2}{s_1-j} > \frac{1}{[\frac{s_1}{4}]+k}$ then the topological type of de discriminant curve is as in In emphatic way.

C.3. If $\frac{2}{s_1-j} = \frac{1}{[\frac{s_1}{4}]+k}$ then the Newton polygon of $\frac{\partial f_5}{\partial y}(x, y)$ has only one compact edge containing the points E, F, M, as in In emphatic way. The polynomial associated with this compact edge is $p(z) = z^3 - 2z - a_k$ whose roots are simple if and only if $a_k \neq \pm(\frac{4\sqrt{6}}{9})$. Let us study the different cases:

C.3.1. If $a_k \neq \pm(\frac{4\sqrt{6}}{9})$ then the topological type of the discriminant curve $D(u, v) = 0$ is as in In emphatic way.

C.3.2. Suppose that $a_k = \pm(\frac{4\sqrt{6}}{9})$. The polynomial $p(z)$ has $z_1 = 2\sqrt{\frac{2}{3}}$ as a simple root and $z_2 = z_3 = -\sqrt{\frac{2}{3}}$ as a double root. If $\gamma_i \in \text{Zer}(\frac{\partial f_5}{\partial y})$ corresponds to z_i then we can write

$$\gamma_i = z_i u^{[\frac{s_1}{4}]+k} + \cdots \tag{3.9}$$

Hence

$$\delta_1 := f(u, \gamma_1(u)) = -u^{s_1} + q(z_1)u^{2(s_1-j)} + \cdots ,$$
$$\delta_2 := f(u, \gamma_2(u)) = -u^{s_1} + q(z_2)u^{2(s_1-j)} + \cdots ,$$
$$\delta_3 := f(u, \gamma_3(u)) = -u^{s_1} + q(z_2)u^{2(s_1-j)} + \cdots ,$$

where $\delta_2, \delta_3 \in \text{Zer} D(u, v)$ are different. Observe that $q(z_1) \neq q(z_2)$. So $\text{ord}(\delta_1 - \delta_l) = 2(s_1 - j)$ for $2 \leq l \leq 3$. Let us determine $\text{ord}(\delta_2 - \delta_3)$. For that we need to find new terms in δ_2 and δ_3. We apply the Newton procedure for $\frac{\partial f_5}{\partial y}$: let y_1 be a new variable. Substituting $(x, y) := (x, x^{[\frac{s_1}{4}]+k}(z_2 + y_1))$ in $\frac{\partial f_5}{\partial y}(x, y)$ we get

$$\frac{\partial f_5}{\partial y}(x, x^{[\frac{s_1}{4}]+k}(z_2 + y_1)) = x^{3([\frac{s_1}{4}]+k)}g(x, y_1),$$

with

$$g(x, y_1) = 4\left[\left(\frac{2}{3}\right)z_2 + 2y_1 + 3z_2y_1^2 + y_1^3\right] - (4a_k + 4a_{k+s}x^s + 4a_kx^{s_1-2j} + \cdots)$$
$$-[8 + 4a_k^2x^{s_1-2j} + 8a_ka_{k+s}x^{s_1-2j+s} + 4a_{k+s}^2x^{s_1-2j+2s} + \cdots]y_1$$
$$-[8 + 4a_k^2x^{s_1-2j} + 8a_ka_{k+s}x^{s_1-2j+s} + 4a_{k+s}^2x^{s_1-2j+2s} + \cdots]z_2$$
$$= 4\left[3z_2y_1^2 + y_1^3\right] - (4a_{k+s}x^s + 4a_kx^{s_1-2j} + \cdots)$$
$$-[4a_k^2x^{s_1-2j} + 8a_ka_{k+s}x^{s_1-2j+s} + 4a_{k+s}^2x^{s_1-2j+2s} + \cdots]y_1$$
$$-[4a_k^2x^{s_1-2j} + 8a_ka_{k+s}x^{s_1-2j+s} + 4a_{k+s}^2x^{s_1-2j+2s} + \cdots]z_2, \tag{3.10}$$

where the last equality follows from $p(z_2) = 0$.

Hence, in the next step of the Newton procedure it is enough to consider the polynomial

$$H(x, y_1) = 12z_2 y_1^2 - 4a_k^2 x^{s_1-2j} y_1 + (-4z_2 a_k^2 - 4a_k)x^{s_1-2j} - 4a_{k+s}x^s. \quad (3.11)$$

The topological type of the discriminant curve will depend on the relation between s and $s_1 - 2j$:

C.3.2.1 Suppose that $s_1 - 2j > s$. We obtain

$$\gamma_2(u) = -\frac{\sqrt{6}}{3}u^{\frac{s_1-j}{2}} + \frac{\sqrt{-a_{k+s}}}{\sqrt[4]{6}}u^{\frac{s_1-j}{2}+\frac{s}{2}} + \cdots,$$

$$\gamma_3(u) = -\frac{\sqrt{6}}{3}u^{\frac{s_1-j}{2}} - \frac{\sqrt{-a_{k+s}}}{\sqrt[4]{6}}u^{\frac{s_1-j}{2}+\frac{s}{2}} + \cdots.$$

Hence

$$f_5(u, \gamma_2(u)) = -u_1^s + q(z_2)u^{2(s_1-j)} + p(z_2)u^{2(s_1-j)+\frac{s}{2}} + l(a_{k+s})u^{3s_1-4j}$$
$$+ \frac{8}{3}ca_{k+s}u^{2(s_1-j)+\frac{3}{2}s} + \cdots,$$

and

$$f_5(u, \gamma_3(u)) = -u_1^s + q(z_2)u^{2(s_1-j)} - p(z_2)u^{2(s_1-j)+\frac{s}{2}} + l(a_{k+s})u^{3s_1-4j}$$
$$- \frac{8}{3}ca_{k+s}u^{2(s_1-j)+\frac{3}{2}s} + \cdots,$$

where $c := \frac{\sqrt{-a_{k+s}}}{\sqrt[4]{6}}$, and $l(z) = -4z_2 z - 1$. As a consequence, when s is odd then the discriminant curve $D(u, v) = 0$ is the union of a smooth branch $D_1(u, v) = 0$ and a singular branch with semigroup $\langle 2, 4(s_1-j)+3s \rangle$, with $i_0(D_1, D_2) = 4(s_1-j)$. Otherwise, if s is even then the discriminant curve $D(u, v) = 0$ is the union of three smooth braches $D_l(u, v) = 0$ such that $i_0(D_1, D_l) = 2(s_1 - j)$ for $2 \leq l \leq 3$, and... and $i_0(D_2, D_3) = 2(s_1 - j) + \frac{3}{2}s$.

C.3.2.2 Suppose that $s_1 - 2j < s$. After (3.11) and for the next step of the Newton procedure we only need the polynomial

$$\bar{H}(Z) = 12z_2 Z^2 - 4(z_2 a_k^2 + a_k),$$

whose roots are $\pm\frac{\sqrt{8}}{9}$. Let $d = \frac{\sqrt{8}}{9}$. We obtain

$$\gamma_2(u) = z_2 u^{[\frac{s_1}{4}]+k} + du^{[\frac{s_1}{4}]+k+\frac{s_1-2j}{2}} + \cdots, \quad (3.12)$$

$$\gamma_3(u) = z_2 u^{[\frac{s_1}{4}]+k} - du^{[\frac{s_1}{4}]+k+\frac{s_1-2j}{2}} + \cdots. \quad (3.13)$$

Hence

$$f_5(u, \gamma_2(u)) = -u_1^s + q(z_2)u^{2(s_1-j)} + p(z_2)u^{2(s_1-j)+\frac{s}{2}} + l(a_k)u^{3s_1-4j}$$
$$+ \frac{80}{81}z_2 du^{\frac{7s_1-10j}{2}} + \cdots,$$

and

$$f_5(u, \gamma_3(u)) = -u_1^s + q(z_2)u^{2(s_1-j)} - p(z_2)u^{2(s_1-j)+\frac{s}{2}} + l(a_k)u^{3s_1-4j}$$
$$- \frac{80}{81}z_2 du^{\frac{7s_1-10j}{2}} + \cdots$$

Since 4 and s_1 are coprime then $7s_1 - 10$ is odd and the discriminant curve $D(u, v) = 0$ is the union of a smooth branch $D_1(u, v) = 0$ and a singular branch with semigroup $\langle 2, 7s_1 - 10j \rangle$, where $i_0(D_1, D_2) = 4(s_1 - j)$.

C.3.2.3 Suppose that $s_1 - 2j = s$. After (3.11), in order to obtain the next term in the power series γ_i, it is enough to consider the polynomial

$$\bar{H}(Z) = 12z_2 Z^2 - 4(z_2 a_k^2 + a_{k+s} + a_k).$$

The topological type of the discriminant will depend on the value of a_{k+s}:

C.3.2.3.1 For $a_{k+s} \neq -4\frac{\sqrt{6}}{81}$, we get

$$\bar{H}(Z) = 12z_2 Z^2 - 4\left(a_{k+s} + 4\frac{\sqrt{6}}{81}\right),$$

whose roots are $\pm b := \pm\left(\frac{\sqrt{-d_{k+s}}}{\sqrt[4]{6}}\right)$, where $d_{k+s} = a_{k+s} + 4\frac{\sqrt{6}}{81}$. Hence

$$\gamma_2(u) = z_2 u^{\left[\frac{s_1}{4}\right]+k} + bu^{\left[\frac{s_1}{4}\right]+k+\frac{s}{2}} + \cdots,$$

$$\gamma_3(u) = z_2 u^{\left[\frac{s_1}{4}\right]+k} - bu^{\left[\frac{s_1}{4}\right]+k+\frac{s}{2}} + \cdots.$$

We conclude that

$$f_5(u, \gamma_2(u)) = -u_1^s + q(z_2)u^{2(s_1-j)} + l(a_{k+s})u^{3s_1-4j} + \frac{80}{81}z_2 u^{2s_1-j+\frac{3}{2}s} + \cdots,$$

and

$$f_5(u, \gamma_3(u)) = -u_1^s + q(z_2)u^{2(s_1-j)} + l(a_{k+s})u^{3s_1-4j} - \frac{80}{81}z_2 u^{2s_1-j+\frac{3}{2}s} + \cdots.$$

So, if s is odd then $D(u, v) = 0$ is the union of a smooth branch $D_1(u, v) = 0$ and a singular branch of semigroup $\langle 2, 7s_1 - 10j \rangle$, with $i_0(D_1, D_2) = 4(s_1 - j)$. In the other case, if s is even then $D(u, v) = 0$ is the union of three smooth branches $D_l(u, v) = 0$ such that $i_0(D_1, D_l) = 2(s_1 - j)$ for $2 \leq l \leq 3$, and ..., and $i_0(D_2, D_3) = \frac{7s_1 - 10j}{2}$.

C.3.2.3.2 For $a_{k+s} = -4\frac{\sqrt{6}}{81}$, after (3.10), in order to obtain the next term in the power series γ_i, it is enough to consider the polynomial

$$\bar{H}(Z) = 12z_2 Z^2 - 4a_k^2 Z - 4a_{k+s}(1 + 2a_k z_2),$$

whose roots t_1, t_2 are simple. Hence

$$\gamma_2(u) = z_2 u^{\left[\frac{s_1}{4}+k\right]+k} + t_1 u^{\left[\frac{s_1}{4}+k\right]+k+s} + \cdots,$$

and

$$\gamma_3(u) = z_2 u^{\left[\frac{s_1}{4}+k\right]+k} - t_2 u^{\left[\frac{s_1}{4}+k\right]+k+s} + \cdots.$$

We conclude that

$$f_5(u, \gamma_2(u)) = -u_1^s + q(z_2)u^{2(s_1-j)} - u^{2(s_1-j)+s} + h(t_1)u^{2(s_1-j)+s2} + \cdots,$$

and

$$f_5(u, \gamma_3(u)) = -u_1^s + q(z_2)u^{2(s_1-j)} - u^{2(s_1-j)+s} + h(t_2)u^{2(s_1-j)+s2} + \cdots,$$

where $h(z) = -4a_{k+s}z + a_k$. The discriminant curve $D(u, v) = 0$ is the union of three smooth branches $D_l(u, v) = 0$ such that $i_0(D_1, D_l) = 2(s_1 - j)$ for $2 \leq l \leq 3$ and... and $i_0(D_2, D_3) = 4s_1 - 6j$. $\square$

From the above computations we obtain

Theorem 3.6 *Let $f(x, y) = 0$ be a plane branch of multiplicity n. Then the following hold.*

(1) If $n = 2$ then the discriminant curve $D(u, v) = 0$ is nondegenerate.
(2) If $n = 3$ then the discriminant curve $D(u, v) = 0$ is degenerate and its topological type depends on the semigroup $S(f)$ and the Zariski invariant (when it is not zero).
(3) If $n = 4$ then the discriminant curve $D(u, v) = 0$ is degenerate and its topological type depends on the semigroup $S(f)$, the Zariski invariant (when it is not zero) and at most two other analytical invariants.

Remark 3.7 In [8], Hefez and Hernandes give explicit normal forms for branches of multiplicity 2, 3 or 4. Thanks to that, the description of the topological type of the discriminant curve of such a branch was possible. The normal forms for branches of multiplicity greater than 4 are not completely determined.

4 Discriminant of branches C with $\mu(C) - \tau(C) \leq 2$

Let $C : f(x, y) = 0$ be a plane branch. Put $r(C) := \mu(C) - \tau(C)$, where $\mu(C)$ and $\tau(C)$ are the Milnor number and the Tjurina number of C, respectively. Observe that $r(C)$ is a nonnegative integer. In zero characteristic, from [15, Theorem 4] we get $r(C) = 0$ if and only if C is analytically equivalent to the curve $y^{s_0} - x^{s_1} = 0$ for two coprime integers s_0 and s_1 greater than 1. Later, in [1], the authors describe all plane branches defined over an algebraically closed field of characteristic zero, modulo analytic equivalence, having the property that the difference between their Milnor and Tjurina numbers is 1 or 2. In particular the authors determined the normal forms of the branches of this family, which show us that the Zariski invariant of these branches are determined by the two first generators of their semigroup. By [1, Corollary 5] we know that if $r(C) \neq 0$ then $r(C) \geq 2^{g-1}$. Hence, if $r(C) = 1$ then $g = 1$ and if $r(C) = 2$ then $g \leq 2$. In this section we will describe the topological type of the discriminant curve $D(u, v) = 0$ of branches C with $\mu(C) - \tau(C) \leq 2$.

Theorem 4.1 *Let $C : f(x, y) = 0$ be a plane branch with $r(C) := \mu(C) - \tau(C) \leq 2$. Then the discriminant curve $D(u, v) = 0$ is degenerate and its topological type is given by the semigroup $S(f)$.*

Proof Suppose first that $r(C) = 1$. By [1, Corollary 8] the branch C is analytically equivalent to the curve defined by the equation $f(x, y) = y^{s_0} - x^{s_1} + x^{s_1-2}y^{s_0-2}$, where $2 \leq s_0 < s_1$ are coprime integers. Hence $f_y(x, y) = s_0 y^{s_0-1} + (s_0 - 2)x^{s_1-2}y^{s_0-3} = y^{s_0-3}(s_0 y^2 + (s_0 - 2)x^{s_1-2})$, whose Newton–Puiseux roots are $\alpha_1 = 0$ (with multiplicity $s_0 - 3$), $\alpha_2 = \sqrt{\frac{2-s_0}{s_0}}x^{\frac{s_1-2}{2}}$ and $\alpha_3 = -\sqrt{\frac{2-s_0}{s_0}}x^{\frac{s_1-2}{2}}$. After (2.3) we obtain that the Newton–Puiseux roots of the discriminant curve are

(1) $\delta_1 = u^{s_1}$ with multiplicity $s_0 - 3$,

(2) $\delta_2 = -u^{s_1} + \left(\left(\sqrt{\frac{2-s_0}{s_0}} \right)^{s_0} + \left(\sqrt{\frac{2-s_0}{s_0}} \right)^{s_0-2} \right) u^{\frac{(s_1-2)s_0}{2}}$,

(3) $\delta_3 = -u^{s_1} - \left(\left(\sqrt{\frac{2-s_0}{s_0}} \right)^{s_0} + \left(\sqrt{\frac{2-s_0}{s_0}} \right)^{s_0-2} \right) u^{\frac{(s_1-2)s_0}{2}}$.

Hence, if s_1 and s_0 are odd then the discriminant curve is given by $D(u, v) = D_1(u, v)^{s_0-3} D_2(u, v)$, where $D_1(u, v) = (v - u^{s_1})$ and $D_2(u, v)$ is a branch of semigroup $\langle 2, (s_1 - 2)s_0 \rangle$ and the intersection multiplicity is $i_0(D_1, D_2) = \min\{s_1, (s_1 - 2)n\}$.

Otherwise $D(u, v)$ is the product of $D_1(u, v)^{s_0-3}$ and two smooth branches $D_2(u, v)$ and $D_3(u, v)$, where $i_0(D_1, D_k) = \min\left\{s_1, \frac{(s_1-2)s_0}{2}\right\}$ for $2 \leq k \leq 3$ and $i_0(D_2, D_3) = \frac{(s_1-2)s_0}{2}$.

Suppose now that $r(C) = 2$. In this case we get $g \leq 2$. If $g = 2$ then by [1, Corollary 13] the multiplicity of C is 4 and we study this case in Proposition 3.3.

For $g = 1$, after [1, Theorem 17, Corollary 18], we have two normal forms:

(A) $f(x, y) = y^{s_0} - x^{s_1} + x^{s_1-3}y^{s_0-2}$, with $2 < s_0 < s_1$.

(B) $f(x, y) = y^{s_0} - x^{s_1} + x^{s_1-2}y^{s_0-3} + \left(\sum_{k\geq 2}^{2+[\frac{s_1}{s_0}]} a_k x^{s_1-k}\right)y^{s_0-2}$, with $4 \leq s_0 < s_1$, $\frac{2s_0}{s_0-3} < s_1$ and $a_k \in \mathbb{C}$.

In the case (A) we get $f_y(x, y) = y^{s_0-3}(s_0 y^2 + (s_0 - 2)x^{s_1-3})$, whose Newton–Puiseux roots are $\alpha_1 = 0$ (with multiplicity $s_0 - 3$), $\alpha_2 = \sqrt{\frac{2-s_0}{s_0}} x^{\frac{s_1-3}{2}}$ and $\alpha_3 = -\sqrt{\frac{2-s_0}{s_0}} x^{\frac{s_1-3}{2}}$. After (2.3) we obtain that the Newton–Puiseux roots of the discriminant curve are

(1) $\delta_1 = u^{s_1}$ with multiplicity $s_0 - 3$,

(2) $\delta_2 = -u^{s_1} + \left(\left(\sqrt{\frac{2-s_0}{s_0}}\right)^{s_0} + \left(\sqrt{\frac{2-s_0}{s_0}}\right)^{s_0-2}\right)u^{\frac{(s_1-3)s_0}{2}}$,

(3) $\delta_3 = -u^{s_1} - \left(\left(\sqrt{\frac{2-s_0}{s_0}}\right)^{s_0} + \left(\sqrt{\frac{2-s_0}{s_0}}\right)^{s_0-2}\right)u^{\frac{(s_1-3)s_0}{2}}$.

Hence, if s_0 is odd and s_1 is even then the discriminant curve is given by $D(u, v) = D_1(u, v)^{s_0-3}D_2(u, v)$, where $D_1(u, v) = (v - u^{s_1})$ and $D_2(u, v)$ is a branch of the semigroup $\langle 2, (s_1 - 3)s_0 \rangle$ and the intersection multiplicity is $i_0(D_1, D_2) = \min\left\{s_1, (s_1 - 3)s_0\right\}$. Otherwise $D(u, v)$ is the product of $D_1(u, v)^{s_0-3}$ and two smooth branches $D_2(u, v)$ and $D_3(u, v)$, where $i_0(D_1, D_k) = \min\left\{s_1, \frac{(s_1-3)s_0}{2}\right\}$ for $2 \leq k \leq 3$ and $i_0(D_2, D_3) = \frac{(s_1-3)s_0}{2}$.

In the case (B) we have

$$f_y(x, y) = y^{s_0-4}\left(s_0 y^3 + (s_0 - 3)x^{s_1-2} + (s_0 - 2)\left(\sum_{k\geq 2}^{2+[\frac{s_1}{s_0}]} a_k x^{s_1-k}\right)y\right),$$

hence its Newton polygon coincides with the Newton polygon determined by $(0, 3)$, $(s_1-2, 0)$ and $(s_1 - 2 - [\frac{s_1}{s_0}], 1)$. But, after the inequality $\frac{2s_0}{s_0-3} < s_1$, we get that this Newton polygon has only two points which are its vertices: $(0, 3)$ and $(s_1 - 2, 0)$. The Newton–Puiseux roots of $f_y(x, y) = 0$ are $\alpha_1 = 0$ (with multiplicity $s_0 - 4$) and $\alpha_i = \xi_i \sqrt[3]{\frac{s_0-3}{s_0}} u^{\frac{s_0-2}{3}} + \cdots$, where ξ_i is a cubic root of unity, $1 \leq i \leq 3$. Then, after (2.3) the Newton–Puiseux roots of the discriminant curve are $\delta_1 = u^{s_1}$ with multiplicity $s_0 - 4$ and

$$\delta_i = -u^{s_1} + \left[\left(\xi_i\sqrt[3]{\frac{s_0-3}{s_0}}\right)^{s_0} + \left(\xi_i\sqrt[3]{\frac{s_0-3}{s_0}}\right)^{s_0-3}\right]u^{\frac{(s_1-2)s_0}{3}}$$

$$+ \left(\xi_i\sqrt[3]{\frac{s_0-3}{s_0}}\right)^{s_0-2} a_{2+[\frac{s_1}{s_0}]} u^{\frac{(s_1-2)(s_0-2)}{3}+(s_1-2)-[\frac{s_1}{s_0}]} + \cdots,$$

for $1 \leq i \leq 3$.

If $s_1 - 2$ (respectively s_0) and 3 are coprime then the discriminant curve is given by $D(u, v) = D_1(u, v)^{s_0-4} D_2(u, v)$, where $D_1(u, v) = (v - u^{s_1})$ and $D_2(u, v)$ is a branch of the semigroup $\langle 3, (s_1 - 2)s_0 \rangle$ and, after the Halphen–Zeuthen formula, the intersection multiplicity is $i_0(D_1, D_2) = 3s_1$. Otherwise $D(u, v)$ is the product of $D_1(u, v)^{s_0-4}$ and three smooth branches $D_k(u, v)$, $2 \le k \le 4$, where $i_0(D_1, D_k) = s_1$ for $2 \le k \le 4$ and $i_0(D_l, D_r) = \frac{(s_1-2)s_0}{3}$, for $2 \le l \ne r \le 4$. □

Remark 4.2 Observe that in Theorem 4.1 Case (B) with $s_0 = 4$ coincides with case σ_3 in (3.2) for $j = 2$. Hence this case was studied in Proposition 3.4.

5 Tables

In this section we collect, in some tables, the topological type of the discriminants for branches studied in this paper (See Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11).

Table 1 Discriminants of branches of semigroup $\langle 2, s_1 \rangle$

Multiplicity 2	
Normal form	Discriminant $D(u, v)$
$\lambda = 0$	$v + u^{s_1}$

Table 2 Discriminants of branches of semigroup $\langle 3, s_1 \rangle$

Multiplicity 3	
Normal form	Discriminant $D(u, v)$
$\lambda = 0$	$(v + u^{s_1})^2$
$\lambda \neq 0$	$D_1 D_2, \quad m(D_i) = 1, i_0(D_1, D_2) = \frac{s_1 + \lambda}{2}$ if $\gcd(2, s_1 + \lambda) = 2$
	$D_1, \qquad S(D_1) = \langle 2, s_1 + \lambda \rangle$ if $\gcd(2, s_1 + \lambda) = 1$

Table 3 Discriminants of branches of semigroup $\langle 4, 6, s_2 \rangle$

Multiplicity 4 and $g = 2$	
Normal form	Discriminant $D(u, v)$
$\lambda = s_2 - 6$	$D_1 D_2, \quad m(D_1) = 1, \quad S(D_2) = \langle 2, s_2 \rangle, \quad i_0(D_1, D_2) = 2s_1$

Table 4 Discriminants of branches of semigroup $\langle 4, s_1 \rangle$

Multiplicity 4 and $g = 1$	
Normal form	Discriminant $D(u, v)$
$\lambda = 0$	$(v + u^{s_1})^3$
$\lambda = 2s_1 - 4j \neq 0$ $2 \leq j \leq [\frac{s_1}{4}]$	$D_1 D_2 D_3, \quad m(D_i) = 1, \quad i_0(D_l, D_r) = \frac{2s_1 + \lambda_i}{3}$ if $\gcd(3, 2s_1 + \lambda_i) = 3,$
	$D_1, S(D_1) = \langle 3, 2s_1 + \lambda_i \rangle$ if $\gcd(3, 2s_1 + \lambda_i) = 1$
$\lambda = 3s_1 - 4j \neq 0$ $2 \leq j \leq [\frac{s_1}{2}]$	See Table 5

Table 5 Multiplicity 4, $g = 1, \lambda = 3s_1 - 4j \neq 0 \ \ 2 \leq j \leq [\frac{s_1}{2}]$

Multiplicity 4, $g = 1, \lambda = 3s_1 - 4j \neq 0 \ \ 2 \leq j \leq [\frac{s_1}{2}]$	
Depending on the analytical invariants	Discriminant $D(u, v)$
$\lambda \quad (a_i = 0$ for all $i)$	$D_1 D_2^2, \quad m(D_i) = 1, \quad i_0(D_1, D_2) = 2(s_1 - j)$
λ and $2s_1 - 4(j - [\frac{s_1}{4}] - k)$ where $k = \min\{j : a_j \neq 0\}$	see Table 6 (case B of Proposition 3.5)
$\lambda, \ 2s_1 - 4(j - [\frac{s_1}{4}] - k)$ and $2s_1 - 4(j - [\frac{s_1}{4}] - k - s)$ where $k = \min\{j : a_j \neq 0\}, s = \min\{j : a_{k+j} \neq 0, j > 0\}$	see Table 7 (case C of Proposition 3.5)

Table 6 Case B of Proposition 3.5

	Case B of Proposition 3.5 Discriminant $D(u,v)$
$\frac{2}{s_1-j} < \frac{1}{[\frac{s_1}{4}]+k}$	$D_1D_2D_3$, $\quad m(D_l)=1$, $\quad i_0(D_l,D_s)=\frac{4}{3}(s_1-j+[\frac{s_1}{4}]+k)$ $\quad$ if gcd$\{3,s_1-j+[\frac{s_1}{4}]+k\}=3$
	D_1, $\qquad S(D_1)=\langle 3,4(s_1-j+[\frac{s_1}{4}]+k)\rangle$ $\qquad\qquad\qquad\qquad$ if gcd$\{3,s_1-j+[\frac{s_1}{4}]+k\}=1$
$\frac{2}{s_1-j} > \frac{1}{[\frac{s_1}{4}]+k}$	$D_1D_2D_3$, $\quad m(D_l)=1$, $\quad i_0(D_1,D_s)=2(s_1-j)$, $\quad i_0(D_2,D_3)=\frac{3}{2}(s_1-j)+[\frac{s_1}{4}]+k$ $\quad$ if gcd$\{2,s_1-j\}=2$
	D_1D_2, $\qquad m(D_1)=1$ $\qquad S(D_2)=\langle 2,3(s_1-j)+2([\frac{s_1}{4}]+k)\rangle$ $\quad i_0(D_1,D_2)=4(s_1-j)$ $\quad$ if gcd$\{2,s_1-j\}=1$
$\frac{2}{s_1-j} = \frac{1}{[\frac{s_1}{4}]+k}$	$D_1D_2D_3$, $\quad m(D_l)=1$, $\quad i_0(D_l,D_s)=2(s_1-j)$ $\qquad\qquad\qquad\qquad\qquad$ if $a_k \neq \pm\frac{4\sqrt{6}}{9}$
	D_1D_2, $\qquad m(D_1)=1$, $\quad S(D_2)=\langle 2,5s_1-6j\rangle$, $\quad i_0(D_1,D_2)=4(s_1-j)$ $\quad$ if $a_k=\pm\frac{4\sqrt{6}}{9}$

Table 7 Case C of Proposition 3.5

$$ $$	Case C of Proposition 3.5 Discriminant $D(u,v)$
$\dfrac{2}{s_1-j} < \dfrac{1}{[\frac{s_1}{4}]+k}$	$D_1 D_2 D_3$, $m(D_1)=1$, $i_0(D_1, D_s) = \frac{4}{3}(s_1 - j + [\frac{s_1}{4}]+k)$ if gcd$\{3, s_1 - j + [\frac{s_1}{4}]+k\} = 3$ D_1, $S(D_1) = (3, 4(s_1 - j + [\frac{s_1}{4}]+k))$ if gcd$\{3, s_1 - j + [\frac{s_1}{4}]+k\} = 1$
$\dfrac{2}{s_1-j} > \dfrac{1}{[\frac{s_1}{4}]+k}$	$D_1 D_2 D_3$, $m(D_1)=1$, $i_0(D_1, D_s) = 2(s_1 - j)$, $i_0(D_2, D_3) = \frac{3}{2}(s_1 - j) + [\frac{s_1}{4}]+k$ if gcd$\{2, s_1 - j\} = 2$ $D_1 D_2$, $m(D_1)=1$ $S(D_2) = (2, 3(s_1 - j) + 2([\frac{s_1}{4}]+k))$ $i_0(D_1, D_2) = 4(s_1 - j)$ if gcd$\{2, s_1 - j\} = 1$
$\dfrac{2}{s_1-j} = \dfrac{1}{[\frac{s_1}{4}]+k}$	$D_1 D_2 D_3$, $m(D_1)=1$, $i_0(D_1, D_s) = 2(s_1 - j)$ if $a_k \neq \pm\frac{4\sqrt{6}}{9}$ see Table 8 if $a_k = \pm\frac{4\sqrt{6}}{9}$

Table 8 Case C of Proposition 3.5, when $\frac{2}{s_1-j} = \frac{1}{[\frac{s_1}{4}]+k}$ and $a_k = \pm\frac{4\sqrt{6}}{9}$

	Discriminant $D(u, v)$				
$s_1 - 2j > s$	$D_1 D_2$,	$m(D_1) = 1$,	$S(D_1) = \langle 2, 4(s_1 - j) + 3s\rangle$,	$i_0(D_1, D_2) = 4(s_1 - j)$	if s is odd
	$D_1 D_2 D_3$,	$m(D_1) = 1$,	$i_0(D_1, D_r) = 2(s_1 - j)$,	$i_0(D_2, D_3) = 2(s_1 - j) + \frac{3}{2}s$	if s is even
$s_1 - 2j < s$	$D_1 D_2$,	$m(D_1) = 1$,	$S(D_1) = \langle 2, 7s_1 - 10j\rangle$,	$i_0(D_1, D_2) = 4(s_1 - j)$	
$s_1 - 2j = s$ and $a_{k+s} \neq \pm\frac{4\sqrt{6}}{81}$	$D_1 D_2 D_3$,	$m(D_l) = 1$,	$i_0(D_l, D_r) = 2(s_1 - j)$		if $\gcd(2, s) = 2$
	$D_1 D_2$	$m(D_1) = 1$,	$S(D_2) = \langle 2, 7s_1 - 10j\rangle$,	$i_0(D_1, D_2) = 4(s_1 - j)$,	if $\gcd(2, s) = 1$
$s_1 - 2j = s$ and $a_{k+s} = \pm\frac{4\sqrt{6}}{81}$	$D_1 D_2 D_3$,	$m(D_l) = 1$,	$i_0(D_1, D_r) = 2(s_1 - j)$,	$i_0(D_2, D_3) = 4s_1 - 6j$.	

Table 9 Discriminants of branches with $r(C) = \mu(C) - \tau(C) \leq 2$

$r(C) = \mu(C) - \tau(C) \leq 2$	
Normal form	Discriminant $D(u, v)$
$r(C) = 1$ $S(f) = \langle s_0, s_1 \rangle$	$D_1^{s_0-3} D_2, \quad D_1 = (v - u^{s_1}), \quad S(D_2) = \langle 3, s_1 - 2 \rangle, \quad i_0(D_1, D_2) = \min\{s_1, (s_1 - 2)s_0\} \quad$ if s_1, s_0 coprime $D_1^{s_0-3} D_2 D_3 \quad D_1 = (v - u^{s_1}), \quad m(D_i) = 1, \quad i_0(D_1, D_k) = \min\left\{s_1, \frac{(s_1-2)s_0}{3}\right\}$ otherwise $i_0(D_2, D_3) = \frac{(s_1-2)s_0}{3}$
$r(C) = 2$ $S(f) = \langle 4, s_1, s_2 \rangle$	$D_1 D_2, \quad m(D_1) = 1, \quad S(D_2) = \langle 2, s_2 \rangle, \quad i_0(D_1, D_2) = 2s_1$
$r(C) = 2$ $S(f) = \langle s_0, s_1 \rangle$	see Table 10 and Table 11

Table 10 Case $r(C) = \mu(C) - \tau(C) = 2$, $S(f) = \langle s_0, s_1 \rangle$ and normal form (A)

Normal form: $f(x, y) = y^{s_0} - x^{s_1} + x^{s_1 - 3} y^{s_0 - 2}$, with $2 < s_0 < s_1$

Discriminant $D(u, v)$

s_0 odd, s_1 even	$D_1^{s_0 - 3} D_2$, $D_1 = (v - u^{s_1})$, $S(D_2) = \langle 2, (s_1 - 3)s_0 \rangle$, $i_0(D_1, D_2) = \min\{s_1, (s_1 - 3)s_0\}$
otherwise	$D_1^{s_0 - 3} D_2 D_3$, $m(D_i) = 1$, $i_0(D_1, D_i) = 1$, $i_0(D_1, D_k) = \min\left\{s_1, \frac{(s_1 - 3)s_0}{2}\right\}$, $i_0(D_2, D_3) = \frac{(s_1 - 3)s_0}{2}$

Table 11 Case $r(C) = \mu(C) - \tau(C) = 2$, $S(f) = \langle s_0, s_1 \rangle$ and normal form (B)

| | Normal form: $f(x, y) = y^{s_0} - x^{s_1} + x^{s_1-2}y^{s_0-3} + \left(\sum_{k \geq 2}^{2+[\frac{s_1}{s_0}]} a_k x^{s_1-k} \right) y^{s_0-2}$, with $4 \leq s_0 < s_1$, $\frac{2s_0}{s_0-3} < s_1$ and $a_k \in \mathbb{C}$ |
	Discriminant $D(u, v)$
$s_1 - 2$ (resp. s_0) and 3 coprime	$D_1^{s_0-4}D_2$, $\quad D_1 = (v - u^{s_1})$, $\quad S(D_2) = \langle 3, (s_1 - 2)s_0 \rangle$, $\quad i_0(D_1, D_2) = 3s_1$
Otherwise	$D_1^{s_0-4}D_2D_3D_4$, $\quad m(D_i) = 1$, $\quad i_0(D_1, D_k) = s_1$, $\quad i_0(D_l, D_r) = \frac{(s_1-2)s_0}{3}$, $\quad 2 \leq l \neq r \leq 4$

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