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An explicit deformation of a plane branch with constant δ-invariant

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A B S T R A C T

We construct an explicit deformation of a plane branch with constant δ-invariant, where δ denotes the *double point number* of the singularity. The generic fibers of the deformation are the union of two irreducible curves. Our construction works over algebraically closed fields of arbitrary characteristic.

1. Introduction

Let K be an algebraically closed field of arbitrary characteristic. Consider a non-zero power series $f(x,y) \in K[[x,y]]$ without constant term. Suppose that the curve $C \equiv \{f(x,y)=0\}$ is *singular* at the origin, that is the order of $f(x,y)$ is bigger than 1. A *deformation* of $f(x,y)$ is a uniparameter family of power series $\{f_c(x,y)\}_{c\in K}$ such that $f_c(0,0)=0$ and $f_0(x,y)=f(x,y)$. The Milnor number of $f_c(x,y)$ is by definition $\mu(f_c)=i_O\left(\frac{\partial f_c}{\partial x},\frac{\partial f_c}{\partial y}\right)$, where $i_O(g,h)$ denotes the *intersection multiplicity* at the origin $O=(0,0)$ of the curves $\{g(x,y)=0\}$ and $\{h(x,y)=0\}$ for any $g(x,y)$, $h(x,y)\in K[[x,y]]$ (see [8, page 230]). In zero-characteristic we have $\mu(f)=\mu(u\cdot f)$ for any unit $u(x,y)\in K[[x,y]]$. However, this is not true in positive characteristic (see [3, page 63]).

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It is well-known, for the complex numbers field, that in the case of analytical deformations, there exists an open neighborhood U of $0 \in \mathbb{C}$ such that $\mu(f_c)$ is constant and $\mu(f_0) \geq \mu(f_c)$ for any $c \in U \backslash \{0\}$. The *jump* of the deformation $\{f_c(x, y)\}_{c\in\mathbb{C}}$ is the constant difference $\mu(f_0) - \mu(f_c)$ with $c \in U\backslash\{0\}$. The *jump* of the Milnor number of $f(x, y)$ is the smallest nonzero value among the jumps of all deformations of $f(x, y)$. The jump of the Milnor number was studied by several authors (Arnold [1], Bodin [2], Gusein-Zade [11]). Gusein-Zade proves in [11] that, in the case where f is irreducible the jump equals one. His proof is not effective.

Inspired by [12] we present an explicit deformation of a plane branch with constant δ-invariant in the context of any algebraically closed field of arbitrary characteristic, where the δ-invariant is the *double point number* of the singularity (later we will give details about it). The reader can find more information about general theory on δ-constant deformations in [10] (for the complex field) and [5] (for arbitrary characteristic fields).

Consider an irreducible power series f in the ring $K[[x, y]]$. The semigroup associated with the branch $\{f = 0\}$ is

$$\Gamma(f) = \{i_O(f, g) \,:\, g \in K[[x, y]] \text{ such that } g \not\equiv 0 \pmod f\},$$

where $i_O(f, g)$ denotes the intersection multiplicity of f and g at the origin.

Suppose that $\{f = 0\}$ is singular, that is the order of f is bigger than 1. It is well-known (see for example [8, Lemma 3.1]) that $\Gamma(f)$ is a numerical semigroup, that is there exists $c \in \mathbb{N}$ such that any natural number greater than or equal to c belongs to $\Gamma(f)$ and $c - 1 \notin \Gamma(f)$. The number c is called the conductor of $\Gamma(f)$.

The semigroup $\Gamma(f)$ admits a minimal system of generators $b_0, b_1, \ldots, b_h$ such that $b_0 = i_O(f, x)$ and $b_i = \min \Gamma(f)\backslash(\mathbb{N}b_0 + \cdots + \mathbb{N}b_{i-1})$, for $1 \leq i \leq h$. We get $\max\{b_0, b_1\} < b_2 < \cdots < b_h$ and $\min\{b_0, b_1\} = \min(\Gamma(f)\backslash\{0\})$. Any element of $\Gamma(f)$ is of the form $\alpha_0 b_0 + \cdots + \alpha_h b_h$, where α_i are natural numbers. We will write $\Gamma(f) = \langle b_0, b_1, \ldots, b_h\rangle$.

Put $e_i = \gcd(b_0, \ldots, b_i)$ for $i \in \{0, \ldots, h\}$ and $n_i = \frac{e_{i-1}}{e_i}$ for $i \in \{1, \ldots, h\}$.

The minimal system of generators of $\Gamma(f)$ verifies

$$n_i b_i < b_{i+1} \text{ for } i \in \{1, \ldots, h-1\}. \tag{1}$$

See for example [8] for the proof of (1).

By [8, Theorem 3.2] or [15, Theorem 2.1], there exists a sequence of monic polynomials $f_1, \ldots, f_h \in K[[x]][y]$ such that

$$i_O(f_i, x) = e_0/e_{i-1}, \quad \text{and} \quad i_O(f, f_i) = b_i,$$

for $i \in \{1, ..., h\}$. Polynomials $f_0 = x, f_1, f_2, \ldots, f_h$ are called *key polynomials* of f.

Consider a reduced (without multiple factors) power series $f \in K[[x, y]]$. Denote by $\overline{\mathcal{O}}$ the normalization of the ring $\mathcal{O} = K[[x, y]]/(f)$. The *double point number* is by definition $\delta(f) = \dim_K \overline{\mathcal{O}}/\mathcal{O}$. Consider the conductor ideal $\mathcal{C}$ of $\overline{\mathcal{O}}$ in $\mathcal{O}$. The integer $c(f) = \dim_K \mathcal{O}/\mathcal{C}$ is called the degree of the conductor. Since $\mathcal{O}$ is Gorenstein we have $c(f) = 2\delta(f)$ (see also [15, Theorem 2.1]). In the case where f is irreducible the degree of the conductor equals the conductor of the semigroup $\Gamma(f)$.

If the characteristic of K is zero, then $\mu(f) = \delta(f) - r(f) + 1$, where $r(f)$ denotes the number of different irreducible factors of f. But if the characteristic is positive, then $\mu(f) \geq \delta(f) - r(f) + 1$, and equality means that f has no wild vanishing cycles (see [6] and [14]). Observe that in any characteristic $\delta(uf) = \delta(f)$ and $r(uf) = r(f)$ for any unit $u \in K[[x, y]]$. In what follows, $\bar{\mu}(f)$ will denote $\delta(f) - r(f) + 1$.

The main result of this note is

Theorem 1. *Let $f \in K[[x,y]]$ be an irreducible power series of order greater than one with semigroup $\Gamma(f) = \langle b_0, b_1, \dots, b_h \rangle$. Consider a sequence of key polynomials $f_1, f_2, \dots, f_h$ of f. Take $\alpha_0, \alpha_1, \dots, \alpha_h \in \mathbb{N}$ such that*

$$\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_h b_h = n_h b_h - 1$$

and let $g = x^{\alpha_0} f_1^{\alpha_1} \cdots f_h^{\alpha_h}$. Then the family $\mathcal{F}_c = f - cg$ has $c = 0$ as a special value and has at most one special value $c_0 \neq 0$. Moreover for $c \notin \{0, c_0\}$ we have

(i) *$\mathcal{F}_c$ is a product of two irreducible power series,*
(ii) *$\delta(\mathcal{F}_c) = \delta(f)$.*

We will prove Theorem 1 in Section 4. In this section we also determine the semigroups of two branches of $\mathcal{F}_c$, when c is a generic value. In order to prove Theorem 1 we need some arithmetical lemmas (see Section 2). In particular, we will prove, in Lemma 1, that equality $\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_h b_h = n_h b_h - 1$ holds for some $\alpha_0, \alpha_1, \dots, \alpha_h \in \mathbb{N}$. In Section 3 we present the notion of the dual resolution graph of a plane curve which is an important tool in the proof of Theorem 1.

2. Arithmetical lemmas

In this section we will present some lemmas, which are necessary for the proof of Theorem 1.

Lemma 1. *Let $\{f = 0\}$ be a branch with $\Gamma(f) = \langle b_0, b_1, \dots, b_h \rangle$. Then there exist $\alpha_0, \alpha_1, \dots, \alpha_h \in \mathbb{N}$ such that*

$$\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_h b_h = n_h b_h - 1. \tag{2}$$

Proof. By [8, Proposition 2.3] or [15, Proposition 1.17], the conductor of $\Gamma(f)$ is

$$c = 1 - b_0 + \sum_{k=1}^{h} (n_k - 1) b_k.$$

By the definition of the conductor it is enough to show that

$$n_h b_h - 1 \geq c. \tag{3}$$

If $h = 1$ then (3) reduces to $b_0 + b_1 \geq 2$.

Now we suppose that $h > 1$. By the formula of the conductor and inequality (1) we get

$$\begin{aligned} c - 1 &= \sum_{i=1}^{h} (n_i - 1) b_i - b_0 = \sum_{i=1}^{h} n_i b_i - \sum_{i=1}^{h} b_i - b_0 \\ &= (n_1 b_1 - b_2) + (n_2 b_2 - b_3) + \cdots + (n_{h-1} b_{h-1} - b_h) + n_h b_h - b_1 - b_0 \\ &\leq -(h-1) + n_h b_h - b_1 - b_0 \\ &= n_h b_h - b_0 - b_1 - h + 1 < n_h b_h - (b_0 + b_1) \leq n_h b_h - 2. \end{aligned}$$

This finishes the proof. ∎

Remark 1. Observe that $i_O(f, x^{\alpha_0} f_1^{\alpha_1} \cdots f_h^{\alpha_h}) = \alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_h b_h$ for any sequence $f_1, f_2, \ldots, f_h$ of key polynomials of f.

Lemma 2. *Keep assumptions of Lemma 1 and suppose that $b_h = n_{h-1}b_{h-1} + 1$. Then in (2) we have $\alpha_h = n_h - 1$. Moreover, if $\alpha_{h-1} \neq 0$ then $\alpha_{h-1} = n_{h-1}$ and $\alpha_i = 0$ for $i \in \{0, \ldots, n-2\}$.*

Proof. Clearly $\alpha_h < n_h$. Let $\lambda = n_h - \alpha_h$. Then

$$\lambda b_h - 1 = \alpha_0 b_0 + \cdots + \alpha_{h-1} b_{h-1}.$$

By the assumption

$$\lambda n_{h-1} b_{h-1} + \lambda - 1 = \alpha_0 b_0 + \cdots + \alpha_{h-1} b_{h-1}.$$

Reducing this equality modulo n_h we get $\lambda - 1 \equiv 0 \pmod{n_h}$. Hence $\lambda = 1$ and consequently $\alpha_h = n_h - 1$.

Suppose now that $\alpha_{h-1} \neq 0$. By the assumption and equalities $\alpha_h = n_h - 1$ and (2), we get $(n_{h-1} - \alpha_{h-1})b_{h-1} = \alpha_0 b_0 + \cdots + \alpha_{h-2} b_{h-2}$. Since n_{h-1} is the smallest positive integer m such that mb_{h-1} belongs to the semigroup generated by $b_0, \ldots, b_{h-2}$, we have $(n_{h-1} - \alpha_{h-1})b_{h-1} = 0$, that is, $\alpha_{h-1} = n_{h-1}$ and $\alpha_i = 0$ for $i \in \{0, \ldots, n-2\}$. ■

The next arithmetical lemma will be useful in Corollary 1.

Lemma 3. *Let p_i, q_i be positive integers for $i \in \{1, 2, 3\}$. If*

$$p_1 + 1 = q_1, \tag{4}$$

$$p_2 + p_3 = p_1, \tag{5}$$

$$q_2 + q_3 + 1 = q_1, \tag{6}$$

$$\frac{p_j}{q_j} > \frac{p_1}{q_1} \quad \textit{for } j = 2, 3, \tag{7}$$

then $\frac{p_2}{q_2} = \frac{p_3}{q_3} = 1$.

Proof. Substituting left hand sides of (5) and (6) to (4) we get

$$p_2 + p_3 = q_2 + q_3. \tag{8}$$

Hence by (7)

$$\frac{p_j}{q_j} > \frac{p_2 + p_3}{p_2 + p_3 + 1} \quad \text{for } j = 2, 3.$$

The above inequalities are equivalent with $(q_j - p_j)(p_2 + p_3) < p_j$ for $j = 2, 3$. Thus we get $q_j \leq p_j$ for $j = 2, 3$ and finally by (8) we obtain $q_j = p_j$ for $j = 2, 3$. ■

3. Dual graph

It is well-known, after Hironaka's results for arbitrary characteristic, that there exists a sequence of blowings-up such that their composition Π is a *minimal normal resolution* of f, that is, the strict transform of f is non-singular and the union of the strict transform of f and the curve $\Pi^{-1}(0)$ (called *exceptional* *divisor*) is a normal crossing divisor, that is, each irreducible component of $\Pi^{-1}(0)$ is non-singular and intersect each other transversally at only one point at most. The adjective *minimal* means minimal among the resolutions having this property. The geometric configuration of the resolution of f is represented by a weighted graph $T(f)$, called *dual graph*, defined as follows: every irreducible component of $\Pi^{-1}(0)$ is represented by a vertex and two vertices are joint if and only if the corresponding irreducible components of the exceptional divisor intersect. The weight of any vertex is the order number of appearance in the resolution process. Any irreducible component of the strict transform of f is represented by an arrow attached to the vertex corresponding to the irreducible component of the exceptional divisor that intersects it. A vertex of $T(f)$ is called a *rupture vertex* if the number of edges and arrows at the vertex is bigger than two. A *curvette* is a smooth curve intersecting, in a transversal way, an irreducible component of the exceptional divisor. We denote by $\mathcal{C}_i = \{g_i(x,y) = 0\}$ the corresponding irreducible curve passing through the origin which strict transform is a curvette intersecting the irreducible component E_i of the exceptional divisor.

Let $f, g \in K[[x,y]]$ be such that $O = (0,0)$ is an intersection point of $f = 0$ and $g = 0$. Max Noether theorem's states

$$i_O(f,g) = \sum_{P \in \sigma^{-1}(O)} i_P(\tilde{f}, \tilde{g}) + \mathrm{ord}\ f \cdot \mathrm{ord}\ g, \tag{9}$$

where $\tilde{f} = 0, \tilde{g} = 0$ are the proper preimages of $f = 0$ and $g = 0$ under the blowing-up σ.

If O_i is the sequence of points we blow-up in the resolution process of f we denote by $m_{O_i}(f)$ the multiplicity of the strict transform of f passing through O_i.

We can compute the delta invariant of any reduced curve $\{f(x,y) = 0\}$ using the sequence of multiplicities appearing in the resolution process of this curve:

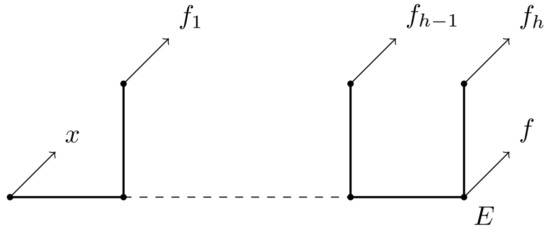
$$\delta(f) = \frac{1}{2} \sum_P m_P(f)(m_P(f) - 1), \tag{10}$$

where the sum runs over all points P we blow-up in the resolution process of desingularisation of f (see for example [9, Proposition 2.1 (iv)]).

If $f \in K[[x,y]]$ is an irreducible power series of order bigger than one with semigroup $\Gamma(f) = \langle b_0, b_1, \dots, b_h \rangle$ then the dual graph of f is determined by the continued fraction expansions of the rational numbers $\frac{b_1}{b_0}, \frac{b_2 - n_1 b_1}{e_1}, \dots \frac{b_h - n_{h-1} b_{h-1}}{e_{h-1}}$ (see [7] for zero-characteristic and $b_0 < b_1$, but in arbitrary characteristic the construction is the same). In particular,

- there are h rupture vertices in $T(f)$,
- if the k-th rupture vertex of $T(f)$ represents the irreducible component E_{i_0} of $\Pi^{-1}(0)$ then the intersection multiplicity $i_O(f, g_{i_0}) = n_k b_k$,
- the last two rupture vertices are jointed if and only if $b_h - n_{h-1} b_{h-1} = 1$.

Let $f_0 = x, f_1, f_2, \dots, f_h$ be key polynomials of the irreducible power series f. The dual resolution graph of $x \cdot f_1 \cdots f_h \cdot f$ has the following shape (when $\{x = 0\}$ and $\{f(x,y) = 0\}$ are transversal and we only draw the vertices which are endpoints), where E is the divisor obtained in the last blowing-up.



Let E_i be an irreducible component of the exceptional divisor of the resolution process of $x \cdot f_1 \cdots f_h \cdot f$. Denote by $\nu_f(E_i)$ the intersection multiplicity at the origin of $f = 0$ and the curve $\mathcal{C}_i$ associated with a curvette intersecting E_i. The *total transform* of f by Π is $\Pi^*(f) = \sum_{i=1}^{s} \nu_f(E_i)E_i + \tilde{f}$, where s is the number of irreducible components of $\Pi^{-1}(0)$ and $\tilde{f}$ denotes the strict transform of f. It follows from (2) and Remark 1 that

$$\nu_f(E) = \nu_g(E) + 1, \tag{11}$$

for $g = x^{\alpha_0} \cdot f_1^{\alpha_1} \cdots f_h^{\alpha_h}$, where the numbers α_i verify equation (2).

Denote $H_{E_i} := \frac{\nu_g(E_i)}{\nu_f(E_i)}$ the *Hironaka quotient* associated with E_i. By (11) we get $H_E < 1$.

Let us fix $g = f_0^{\alpha_0} f_1^{\alpha_1} \cdots f_h^{\alpha_h}$ satisfying equality (11), where $f_0 := x$. Consider the minimal connected component $\mathcal{A}$ of the dual graph $T(f.g)$ containing all the arrows corresponding to the strict transforms of the components of $f.g$. A *dead branch* of $T(f.g)$ is the closure of any connected component of $T(f.g)\backslash\{\mathcal{A}\}$. The *geodesic* of f_i is the minimal oriented connected path in $T(f.g)$ starting in the arrow of f_i and ending in the arrow of f. With this notation we have:

Lemma 4.

1. *The Hironaka quotients H_{E_i} are strictly decreasing in geodesic of f_j for any $j \in \{0, \dots, h\}$.*
2. *Let $\mathcal{B}$ be a dead branch in $T(f.g)$. Then $H_D = H_{D'}$ for any $D, D' \in \mathcal{B}$.*

Proof. Put $H_{j,E_i} := \frac{\nu_{f_j}(E_i)}{\nu_f(E_i)}$ for any irreducible factor $f_j \in K[[x, y]]$ of g. By [13, Théorème 1.2], which also holds in positive characteristic, the quotients H_{j,E_i} are strictly decreasing in the geodesic $\mathcal{G}_j$ of f_j. Moreover H_{j,E_i} is constant on the closure of any connected component of $T(f.g)\backslash\mathcal{G}_j$. The proof follows from the equality $H_{E_i} = \sum_{j=0}^{h} \alpha_j H_{j,E_i}$. ■

Consider $g = x^{\alpha_0} f_1^{\alpha_1} \cdots f_h^{\alpha_h}$ satisfying equality (11).

Denote by E' and E'' the divisors adjoint to E. Suppose that E' lives in the segment jointed E with the root of the dual graph.

Corollary 1. $H_{E'} = H_{E''} = 1$.

Proof. Let $E_{j_1} := E$, $E_{j_2} := E'$, $E_{j_3} := E''$ and $p_i := \nu_g(E_{j_i})$, $q_i := \nu_f(E_{j_i})$ for $i \in \{1, 2, 3\}$. Let Q be the point we blow-up in order to obtain the component E. Suppose that $\bar{f} = 0$ is the strict transform of $f = 0$ at Q. Consider $l \in K[[x, y]]$ such that $l = 0$ is smooth and transverse to $\bar{f}.E'.E'' = 0$. By properties of intersection multiplicities and Noether formula (9) we get

$$\nu_f(E') + \nu_f(E') + 1 = i_Q(\bar{f}.E'^{\,\nu_f(E')}.E''^{\,\nu_f(E'')}, l) = \nu_f(E).$$

In a similar way we obtain:

$$\nu_g(E') + \nu_g(E') = i_Q(\bar{f}.E'^{\,\nu_g(E')}.E''^{\,\nu_g(E'')}, l) = \nu_g(E).$$

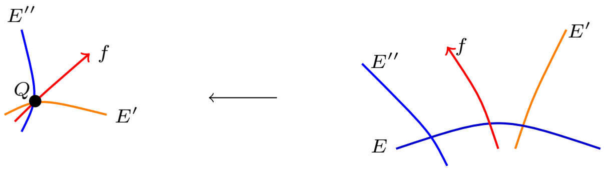


Fig. 1. Last blowing-up.

After the first part of Lemma 4 the numbers p_i, q_i verify the hypothesis of Lemma 3. Hence $H_{E'} = \frac{p_2}{q_2} = 1 = \frac{p_3}{q_3} = H_{E''}$. See Fig. 1 ■

Corollary 2. *If E' is not a rupture point then $H_{E_i} > 1$ for any $E_i \notin \{E, E', E''\}$.*

Proof. It is a consequence of Lemma 4 and Corollary 1. ■

Corollary 3. *Suppose that E' is a rupture point. Then $T(f)\backslash\{E'\}$ has 3 connected components, $\mathcal{C}_1$ containing the root E_1 of the dual resolution graph, $\mathcal{C}_2$ containing E, and $\mathcal{C}_3$. Moreover*

1. *If $\alpha_{h-1} \neq 0$ then $H_{E_i} = 1$ for any $E_i \in \mathcal{C}_1$ and $H_{E_i} > 1$ for any $E_i \in (\mathcal{C}_2 \cup \mathcal{C}_3)\backslash\{E, E''\}$.*
2. *If $\alpha_{h-1} = 0$ then $H_{E_i} = 1$ for any $E_i \in \mathcal{C}_3 \cup \{E', E''\}$. Otherwise $H_{E_i} > 1$ for $E_i \neq E$.*

Proof. It is clear that $T(f)\backslash\{E'\}$ has 3 connected components. Since E' is a rupture point then $b_h = n_{h-1}b_{h-1} + 1$. We finish using Lemmas 2 and 4. ■

Remark 2. Figs. 2, 3 and 4 represent the situation of Corollaries 2 and 3 (when $\{x = 0\}$ and $\{f(x, y) = 0\}$ are transversal). We color in blue the part where Hironaka quotients are equal to 1.

4. Proof of Theorem 1

Let us prove Theorem 1.

Recall that we denote by $\Pi : X \longrightarrow (K^2, 0)$ the resolution process of the singularity of f. Consider the rational function $F := \frac{f\circ\Pi}{g\circ\Pi} : X \dashrightarrow \mathbb{P}^1_K$. We claim that F is well defined in X. Indeed, observe that F is well

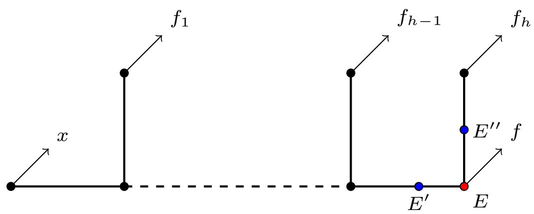


Fig. 2. E' is not a rupture point.

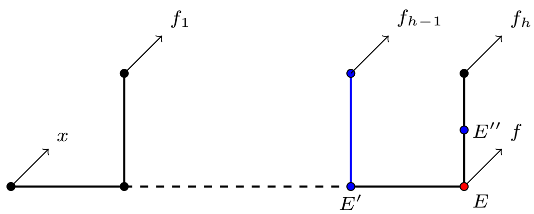


Fig. 3. E' is a rupture point and $\alpha_{h-1} = 0$.

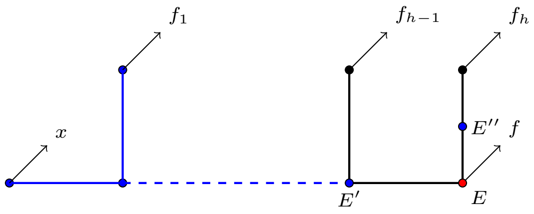


Fig. 4. E' is a rupture point and $\alpha_{h-1} \neq 0$.

defined in $X \backslash \Pi^{-1}(0)$. This function is also well-defined in any regular point P of $\Pi^{-1}(0)$ since in this case $F = unity \cdot t^m$ where $t = 0$ is the equation of the component of $\Pi^{-1}(0)$ containing P and $m \in \mathbb{Z}$. Finally if P is a singular point of $\Pi^{-1}(0)$ then P is the intersection point of two components, E_i and E_j, of $\Pi^{-1}(0)$. Suppose that $E_i \equiv t = 0$ and $E_j \equiv v = 0$. Then $F = unity \cdot t^{\nu_i(f)-\nu_i(g)} v^{\nu_j(f)-\nu_j(g)}$ and after Corollaries 1, 2 and 3, the exponents $\nu_i(f) - \nu_i(g)$ and $\nu_j(f) - \nu_j(g)$ do not have opposite signs.

Now, let us compute $F_{|E_i}$ for any component E_i of $\Pi^{-1}(0)$ different from E' and E''. If $E \equiv t = 0$ then, from equality (11) we get $F = \cancel{o}\, t$ and $F_{|E} = 0$, where $\cancel{o}$ denotes a unit in the local ring of holomorphic functions. If E' is not a rupture point then, from Corollary 2, we have $F_{|E_i} = \infty$ for any $E_i \notin \{E, E', E''\}$. If E' is a rupture point then, from Corollary 3, there is a connected component $\mathcal{C}$ of $T(f) \backslash \{E'\}$ such that F is a meromorphic function different to zero and ∞ at $\mathcal{C}$. Hence, $F_{|\mathcal{C}} = c_0$, where c_0 is a nonzero constant.

Let us prove that for any non special value c, the deformation $\mathcal{F}_c$ is a product of two irreducible power series. Observe that the strict transform of $\mathcal{F}_c = 0$ is the fiber $F^{-1}(c)$. Every $c \notin \{0, c_0\}$ is a regular value of F restricted to $\Pi^{-1}(0)$. On the other hand F restricted to $D \in \{E', E''\}$ is a one-to-one function to $\mathbb{P}^1$, since the only zero of F restricted to D is the intersection point of D with E and this zero has multiplicity one. Thus the fiber $F^{-1}(c)$ consists in two curvettes, one intersecting E' and the second intersecting E''. Hence $\mathcal{F}_c = 0$ is a plane curve with two irreducible components.

It rests to prove that $\delta(\mathcal{F}_c) = \delta(f)$. After Corollaries 1, 2 and 3 we have $\nu_f(E_i) = \nu_{\mathcal{F}_c}(E_i)$ for any E_i different from E. Denote by $P_0 = O, P_1, \ldots, P_k$ the sequence of points we blow-up in the resolution process of $\{f = 0\}$, that is, the component E_i of the divisor $\Pi^{-1}(0)$ arrives when we blow-up P_{i-1} for $i \in \{1, \ldots, k+1\}$ (where $E_{k+1} = E$). By [4, Proposition 17, page 530] we get $m_{P_i}(f) = m_{P_i}(\mathcal{F}_c)$ for any $i \in \{0, \ldots, k-1\}$. Moreover $m_Q(f), m_Q(\mathcal{F}_c) \in \{0, 1\}$ for $Q \in \{P_k, A_1, A_2\}$, where A_i are the intersection points of the strict transform of the two irreducible components of $\{\mathcal{F}_c = 0\}$ and $\Pi^{-1}(0)$. We finish the proof using equality (10). ■

Example 1. Consider the irreducible power series $f(x, y) = (y^2 + x^3)^2 + x^5 y \in K[[x, y]]$ which semigroup is $\Gamma(f) = \langle b_0, b_1, b_2 \rangle$, with $b_0 = 4$, $b_1 = 6$ and $b_2 = 13$. A sequence of key polynomials of f is $f_0 = x$, $f_1 = y$, $f_2 = y^2 + x^3$. We get $n_1 = n_2 = 2$. Hence $n_2 b_2 - 1 = 25 = 3 \cdot 4 + 0 \cdot 6 + 1 \cdot 13 = 0 \cdot 4 + 2 \cdot 6 + 1 \cdot 13$. The deformations, after Theorem 1, are $f - cx^3(y^2 + x^3)$, $f - cy^2(y^2 + x^3)$. The non zero special value (in both cases) is $c_0 = 1$. Observe that, in both cases, $\mathcal{F}_c = 0$ is a family of equisingular curves of two cusps for any $c \neq 0, 1$; $\mathcal{F}_0 = 0$ is the initial branch. In the first case $\mathcal{F}_1 = 0$ is the union of two smooth branches and one cusp, the smooth branches are different and tangent to the cusp. In the second case, $\mathcal{F}_1 = 0$ is the union of one cusp and the triple smooth branch $x^3 = 0$ which is transverse to the cusp. The reader familiar with Newton polygons can read this information from Fig. 5.

In general, we can obtain the equisingularity types of branches of the deformation using two different approaches:

Dual resolution graph: we draw the dual resolution graph of the branch $f(x, y) = 0$. Consider the vertices E, E', E'' as in the proof of Theorem 1. Then the strict transform of $\mathcal{F}_c$, for generic c, consists in two curvettes passing by E' and E''. Denote them by C' and C'' respectively. Now, in order to obtain the equisingularity type of the branch corresponding to C' we keep the arrow of C' and we remove all the other

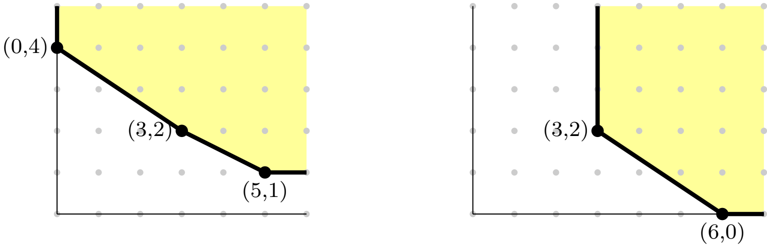


Fig. 5. The Newton polygon of $\mathcal{F}_1$ in both cases.

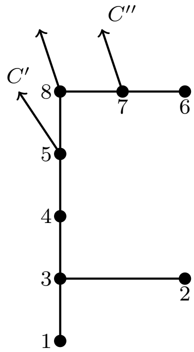


Fig. 6. The dual resolution graph of a generic fiber $\mathcal{F}_c = 0$ with $\Gamma(f) = \langle 6, 9, 25\rangle$.

arrows in the dual resolution graph. The dual graph of the minimal resolution of this branch is obtained by contraction of the initial dual graph (see [4, page 529]). We proceed in a similar way for C''. For example, the dual resolution graph of a generic fiber $\mathcal{F}_c = 0$, where $f(x, y) = (y^2 - x^3)^3 - x^{11}y$ with semigroup $\langle 6, 9, 25\rangle$ is in Fig. 6.

Fig. 7 shows the contraction procedure for C', obtaining a cusp of semigroup $\langle 2, 3\rangle$. Fig. 8 shows the contraction procedure for C'', obtaining a branch of semigroup $\langle 4, 6, 17\rangle$.

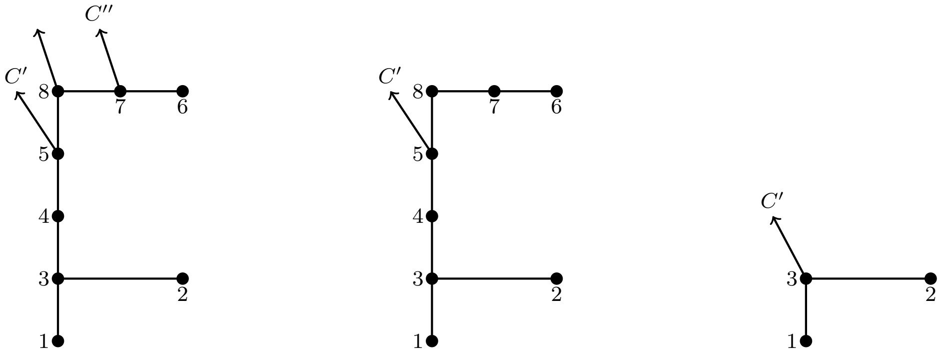


Fig. 7. Contraction procedure for C'.

Newton polygons: Suppose first that $\Gamma(f) = \langle b_0, b_1\rangle$, $f_0 = x$ and $f_1 = y$. After equality (2) we can write $\alpha_0 b_0 + \alpha_1 b_1 = b_0 b_1 - 1$, for some natural numbers α_0, α_1, and the deformation is $\mathcal{F}_c = f - cx^{\alpha_0} y^{\alpha_1}$. This kind of deformations was studied in [2]. We draw the Newton polygon of $\mathcal{F}_c$, for $c \neq 0$ on the left side of Fig. 9. The area of the triangle T of vertices $(0, b_0)$, (α_0, α_1) and $(b_1, 0)$ equals $1/2$. Observe that (α_0, α_1) is the only lattice point below the segment determined by $(0, b_0)$ and $(b_1, 0)$ having this property. By Pick's formula there are no lattice points in T, except the vertices. Hence $\mathcal{F}_c$ is Kouchnirenko nondegenerate and from its Newton polygon we conclude that $\mathcal{F}_c = w'w''$, where both factors are irreducible, $\Gamma(w') = \langle b_0 - \alpha_1, \alpha_0\rangle$ and

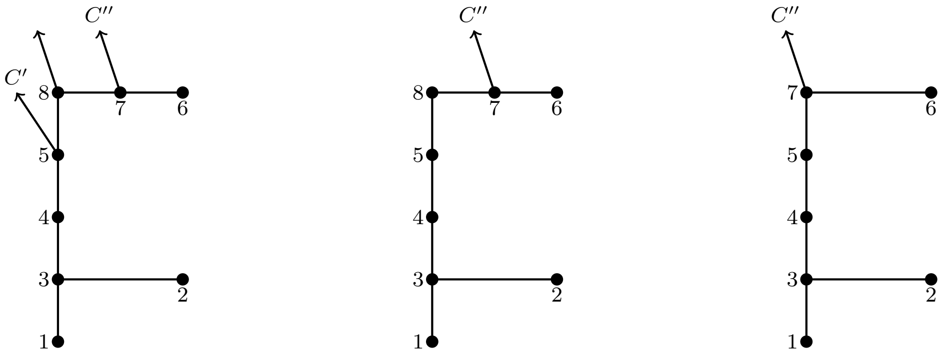


Fig. 8. Contraction procedure for C''.

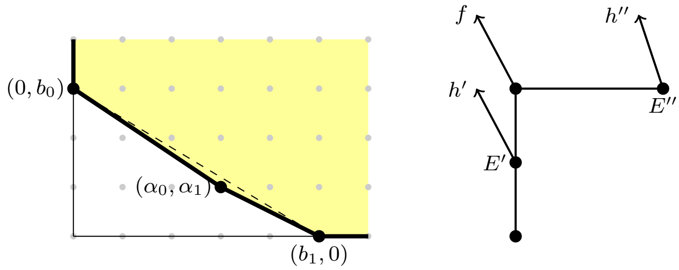


Fig. 9. The dual graph of $\mathcal{F}_c.f$ and the Newton polygon of $\mathcal{F}_c$.

$\Gamma(w'') = \langle \alpha_1, b_1 - \alpha_0\rangle$ (when $b_0 - \alpha_1 = 1$ or $\alpha_1 = 1$ the corresponding semigroup is $\mathbb{N}$). The intersection multiplicities are $i_O(w', w'') = \alpha_0\alpha_1$, $i_O(w', f) = \alpha_0 b_0$, and $i_O(w'', f) = \alpha_1 b_1$. We draw the dual graph in the case $\Gamma(f) = \langle 3, 5\rangle$ on the right side of Fig. 9.

Let us now study the general case $\Gamma(f) = \langle b_0, \dots, b_h\rangle$. Denote by R the component of the exceptional divisor corresponding to the $(h-1)$-th rupture point in the dual graph of $f(x, y) = 0$. Consider the partial modification where R is the last component of the exceptional divisor. In this moment the strict transforms $\bar{f} = 0$ (of the original branch $f = 0$), $\bar{w}' = 0$ and $\bar{w}'' = 0$ (of the two branches $w' = 0$ and $w'' = 0$ of the original deformation) intersect R at a smooth point P.

We get $\Gamma(\bar{f}) = \langle \bar{b}_0, \bar{b}_1\rangle$, where

$$\bar{b}_0 = e_{h-1} \quad \text{and} \quad \bar{b}_1 = b_h - n_{h-1}b_{h-1} + e_{h-1}. \tag{12}$$

Let $\bar{\alpha}_0, \bar{\alpha}_1$ be natural numbers such that $\bar{\alpha}_0\bar{b}_0 + \bar{\alpha}_1\bar{b}_1 = \bar{b}_0\bar{b}_1 - 1$. It follows from the previous approach for semigroups with two generators that $\Gamma(\bar{w}') = \langle \bar{b}_0 - \bar{\alpha}_1, \bar{\alpha}_0\rangle$ and $\Gamma(\bar{w}'') = \langle \bar{\alpha}_1, \bar{b}_1 - \bar{\alpha}_0\rangle$ (when $\bar{b}_0 - \bar{\alpha}_1 = 1$ or $\bar{\alpha}_1 = 1$ the corresponding semigroup is $\mathbb{N}$). Hence, $\Gamma(w') = \langle b'_0, \dots, b'_h\rangle$, where $b'_i = \frac{\bar{b}_0 - \bar{\alpha}_1}{e_{h-1}} b_i$ for $0 \leq i \leq h-1$ and by the second equality from (12) we get $b'_h = \bar{\alpha}_0 + (\bar{b}_0 - \bar{\alpha}_1)\left(\frac{n_{h-1}b_{h-1}}{e_{h-1}} - 1\right)$. Similarly, $\Gamma(w'') = \langle b''_0, \dots, b''_h\rangle$, where $b''_i = \frac{\bar{\alpha}_1}{e_{h-1}} b_i$ for $0 \leq i \leq h-1$ and by the second equality from (12) we get $b''_h = (\bar{b}_1 - \bar{\alpha}_0) + \bar{\alpha}_1\left(\frac{n_{h-1}b_{h-1}}{e_{h-1}} - 1\right)$.

Remember that the *Milnor number* of any curve $\{f(x, y) = 0\}$ is, by definition

$$\mu(f) = \dim_K K[[x, y]]/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

If $r(f)$ denotes the number of branches of $\{f(x, y) = 0\}$ then, after [6] and [14], we have

$$\mu(f) \geq 2\delta(f) - r(f) + 1 \tag{13}$$

in arbitrary characteristic with equality in zero-characteristic.

In [9] the authors denote by $\bar{\mu}(f)$ the value $2\delta(f)-r(f)+1$ of any reduced power series $f(x,y) \in K[[x,y]]$ for any algebraically closed field of arbitrary characteristic. They prove (see [9, Proposition 2.1 (i)]) that $\bar{\mu}(f) = \bar{\mu}(uf)$ for any unit $u \in K[[x,y]]$.

Corollary 4. *Let $f \in \mathbb{K}[[x,y]]$ be an irreducible power series of order bigger than 1 and consider a sequence of key polynomials $f_1, f_2, \ldots, f_h$ of f. Then there exist $\alpha_0, \alpha_1, \ldots, \alpha_h \in \mathbb{N}$ and a 1-parameter family $\mathcal{F}_c = f - cx^{\alpha_0} f_1^{\alpha_1} \cdots f_h^{\alpha_h}$ such that $\bar{\mu}(\mathcal{F}_c) = \bar{\mu}(f) - 1$, for all $c \in \mathbb{C}$ except at most two values.*

Proof. Apply Theorem 1. ■

If in Corollary 4 $K = \mathbb{C}$ then $\bar{\mu}(f) = \mu(f)$, hence $\mu(\mathcal{F}_c) = \mu(f) - 1$ which is the main result of [12].

Remark 3. We can construct δ-constant deformations of a branch with more than two branches: consider the complex power series $\mathcal{F}_c = y^3 + x^5 - cxy^2$ which Newton polygon is in Fig. 10. For any $c \neq 0$, $\mathcal{F}_c$ is a δ-constant deformation with three irreducible factors. The key point in this example is the non-existence of lattice points inside the red area. Using this idea we can construct new examples where the number of irreducible factors of $\mathcal{F}_c$ is greater.

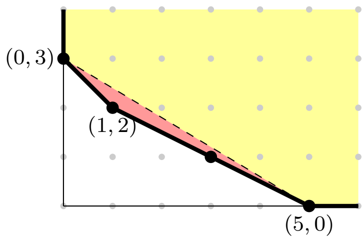


Fig. 10. The Newton polygon of $y^3 - x^5 - cxy^2$.

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CRediT authorship contribution statement

Evelia R. García Barroso: Writing – original draft, Investigation. **Janusz Gwoździewicz:** Investigation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] V. Arnold, Arnold's Problems, Springer-Verlag, Berlin, 2004.
[2] A. Bodin, Jump of Milnor numbers, Bull. Braz. Math. Soc. New Ser. 38 (3) (2007) 389–396.
[3] Y. Boubakri, G.M. Greuel, T. Markwig, Invariants of hypersurface singularities in positive characteristic, Rev. Mat. Complut. 25 (2012) 61–85.
[4] E. Brieskorn, H. Knörrer, Plane Algebraic Curves, Birkhäuser Verlag, Basel, 1986.
[5] A. Campillo, G.M. Greuel, C. Lossen, Equisingular deformations of plane curves in arbitrary characteristic, Compos. Math. 143 (4) (2007) 829–882.
[6] P. Deligne, La Formule de Milnor. Groupes de Monodromie en Géométrie Algébrique, Lecture Notes in Mathematics, vol. 340, Springer, Berlin, 1973, pp. 197–211.
[7] E.R. García Barroso, Invariants des Singularités de Courbes Planes et Courbure des Fibres de Milnor, Tesis, Univ. La Laguna, 1996. See also in Courbes polaires et courbure des fibres de Milnor des courbes planes. Thèse, Univ. Paris 7, 2000. Available at http://ergarcia.webs.ull.es/tesis.pdf.
[8] E.R. García Barroso, A. Płoski, An approach to plane algebroid branches, Rev. Mat. Complut. 28 (1) (2015) 227–252, https://doi.org/10.1007/s13163-014-0155-5.
[9] E.R. García Barroso, A. Płoski, On the Milnor formula in arbitrary characteristic, in: G.M. Greuel, L. Narváez, S. Xambó-Descamps (Eds.), Singularities, Algebraic Geometry, Commutative Algebra and Related Topics. Festschrift for Antonio Campillo on the Occasion of his 65th Birthday, Springer, 2018, pp. 119–133.
[10] G.M. Greuel, C. Lossen, E. Shustin, Introduction to Singularities and Deformations, Springer Monographs in Mathematics, Springer, Berlin, 2007.
[11] S.M. Gusein-Zade, On singularities from which an A_1 can be split off, Funct. Anal. Appl. 27 (1) (1993) 57–59.
[12] A. Lenarcik, M. Masternak, Effective proof of Gusein-Zade theorem that branches may be deformed with jump one. Analytic and Algebraic Geometry 4. T. Krasiński, S. Spodzieja, Łódź University Press, 2022, pp. 95–119.
[13] H. Maugendre, Discriminant d'un germe $(g, f) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ et quotients of contact dans la résolution de $f \cdot g$, Ann. Fac. Sci. Toulouse 6^e série 7 (3) (1998) 497–525.
[14] A. Melle-Hernández, C.T.C. Wall, Pencils of curves on smooth surfaces, Proc. Lond. Math. Soc., III Ser. 83 (2001) 257–278.
[15] G. Pfister, A. Płoski, Plane Algebroid Curves in Arbitrary Characteristic, IMPAN Lecture Notes, vol. 4, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 2022, p. 134.