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# CHARACTERIZATION OF NON-DEGENERATE PLANE CURVE SINGULARITIES

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**Abstract.** We characterize plane curve germs (nondegenerate in Kouchnirenko's sense) in terms of characteristics and intersection multiplicities of branches.

**1. Introduction.** In this paper we consider (reduced) plane curve germs $C$, $D, \ldots$ centered at a fixed point $O$ of a complex nonsingular surface. Two germs $C$ and $D$ are *equisingular* if there exists a bijection between their branches which preserves characteristic pairs and intersection numbers. Let $(x, y)$ be a chart centered at $O$. Then a plane curve germ has a local equation of the form $\sum c_{\alpha,\beta} x^\alpha y^\beta = 0$. Here $\sum c_{\alpha,\beta} x^\alpha y^\beta$ is a convergent power series without multiple factors. The *Newton diagram* $\Delta_{x,y}(C)$ is defined to be the convex hull of the union of quadrants $(\alpha, \beta) + (\mathbb{R}_+)^2$, $c_{\alpha,\beta} \neq 0$. Recall that the *Newton boundary* $\partial\Delta_{x,y}(C)$ is the union of the compact faces of $\Delta_{x,y}(C)$. A germ $C$ is called *non-degenerate* with respect to the chart $(x, y)$ if the coefficients $c_{\alpha,\beta}$, where $(\alpha, \beta)$ runs over integral points lying on the faces of $\Delta_{x,y}(C)$, are *generic* (see Preliminaries to this Note for the precise definition). It is a well-known fact that the equisingularity class of a germ $C$ *non-degenerate* with respect to $(x, y)$ depends exclusively on the Newton polygon formed by the faces of $\Delta_{x,y}(C)$: if $(r_1, s_1), (r_2, s_2), \ldots, (r_k, s_k)$ are subsequent vertices of $\partial\Delta_{x,y}(C)$, then the germs $C$ and $C'$ with local equation $x^{r_1}y^{s_1} + \cdots + x^{r_k}y^{s_k} = 0$ are equisingular. Our aim is to give an explicit description of the non-degenerate plane curve germs in terms of characteristic pairs and intersection numbers of branches. In particular we show that if two germs $C$ and $D$ are equisingular,

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then $C$ is non-degenerate if and only if $D$ is non-degenerate. The proof of our result is based on a refined version of Kouchnirenko's formula for the Milnor number and on the concept of contact exponent.

**2. Preliminaries.** Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. For any subsets $A, B$ of the quarter $\mathbb{R}_+^2$, we consider the arithmetic sum $A + B = \{a + b : a \in A \text{ and } b \in B\}$. If $S \subset \mathbb{N}^2$, then $\Delta(S)$ is the convex hull of the set $S + \mathbb{R}_+^2$. The subset $\Delta$ of $\mathbb{R}_+^2$ is a *Newton diagram* if $\Delta = \Delta(S)$ for a set $S \subset \mathbb{N}^2$ (see [**1**],[**5**]). Following Teissier we put $\{\frac{a}{b}\} = \Delta(S)$ if $S = \{(a, 0), (0, b)\}$, $\{\frac{a}{\infty}\} = (a, 0) + \mathbb{R}_+^2$ and $\{\frac{\infty}{b}\} = (0, b) + \mathbb{R}_+^2$ for any $a, b > 0$ and call such diagrams *elementary Newton diagrams*. The Newton diagrams form the semigroup $\mathcal{N}$ with respect to the arithmetic sum. The elementary Newton diagrams generate $\mathcal{N}$. If $\Delta = \sum_{i=1}^{r} \{\frac{a_i}{b_i}\}$, then $a_i/b_i$ are the inclinations of edges of the diagram $\Delta$ (by convention, $\frac{a}{\infty} = 0$ and $\frac{\infty}{b} = \infty$ for $a, b > 0$). We also put $a + \infty = \infty$, $a \cdot \infty = \infty$, $\inf\{a, \infty\} = a$ if $a > 0$ and $0 \cdot \infty = 0$.

*Minkowski's area* $[\Delta, \Delta'] \in \mathbb{N} \cup \{\infty\}$ of two Newton diagrams $\Delta, \Delta'$ is uniquely determined by the following conditions

$(m_1)$ $[\Delta_1 + \Delta_2, \Delta'] = [\Delta_1, \Delta'] + [\Delta_2, \Delta']$,
$(m_2)$ $[\Delta, \Delta'] = [\Delta', \Delta]$,
$(m_3)$ $[\{\frac{a}{b}\}, \{\frac{a'}{b'}\}] = \inf\{ab', a'b\}$.

We define the *Newton number* $\nu(\Delta) \in \mathbb{N} \cup \{\infty\}$ by the following properties:

$(\nu_1)$ $\nu(\sum_{i=1}^{k} \Delta_i) = \sum_{i=1}^{k} \nu(\Delta_i) + 2\ \sum_{1 \leq i < j \leq k} [\Delta_i, \Delta_j] - k + 1$,
$(\nu_2)$ $\nu(\{\frac{a}{b}\}) = (a - 1)(b - 1)$, $\nu(\{\frac{1}{\infty}\}) = \nu(\{\frac{\infty}{1}\}) = 0$.

A diagram $\Delta$ is *convenient* (resp., *nearly convenient*) if $\Delta$ intersects both axes (resp. if the distances of $\Delta$ to the axes are $\leq 1$). Note that $\Delta$ is nearly convenient if and only if $\nu(\Delta) \neq \infty$. Fix a complex nonsingular surface, i.e., a complex holomorphic variety of dimension 2. Throughout this paper, we consider *reduced* plane curve germs $C, D, \ldots$ centered at a fixed point $O$ of this surface. We denote by $(C, D)$ the *intersection multiplicity* of $C$ and $D$ and by $m(C)$ the *multiplicity* of $C$. There is $(C, D) \geq m(C)m(D)$; if $(C, D) = m(C)m(D)$, then we say that $C$ and $D$ *intersect transversally*. Let $(x, y)$ be a chart centered at $O$. Then a plane curve germ $C$ has a local equation $f(x, y) = \sum c_{\alpha\beta} x^\alpha y^\beta \in \mathbb{C}\{x, y\}$ without multiple factors. We put $\Delta_{x,y}(C) = \Delta(S)$, where $S = \{(\alpha, \beta) \in \mathbb{N}^2 : c_{\alpha\beta} \neq 0\}$. Clearly, $\Delta_{x,y}(C)$ depends on $C$ and $(x, y)$. We note two fundamental properties of Newton diagrams:

$(N_1)$ If $(C_i)$ is a finite family of plane curve germs such that $C_i$ and $C_j$ $(i \neq j)$ have no common irreducible component, then

$$\Delta_{x,y}\left(\bigcup_i C_i\right) = \sum_i \Delta_{x,y}(C_i)\ .$$

$(N_2)$ If $C$ is an irreducible germ (a branch) then

$$\Delta_{x,y}(C) = \left\{\frac{(C, y=0)}{(C, x=0)}\right\}\ .$$

For the proof, we refer the reader to [**1**], pp. 634–640.

The topological boundary of $\Delta_{x,y}(C)$ is the union of two half-lines and a finite number of compact segments (faces). For any face $S$ of $\Delta_{x,y}(C)$ we let $f_S(x,y) = \sum_{(\alpha,\beta)\in S} c_{\alpha,\beta}x^\alpha y^\beta$. Then $C$ is *non-degenerate* with respect to the chart $(x,y)$ if for all faces $S$ of $\Delta_{x,y}(C)$ the system

$$\frac{\partial f_S}{\partial x}(x,y) = \frac{\partial f_S}{\partial y}(x,y) = 0$$

has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$. We say that the germ $C$ is *non-degenerate* if there exists a chart $(x,y)$ such that $C$ is non-degenerate with respect to $(x,y)$.

For any reduced plane curve germs $C$ and $D$ with irreducible components $(C_i)$ and $(D_j)$, we put $d(C,D) = \inf_{i,j}\{(C_i,D_j)/(m(C_i)m(D_j))\}$ and call $d(C,D)$ the *order of contact* of germs $C$ and $D$. Then for any $C, D$ and $E$:

- $(d_1)$ $d(C,D) = \infty$ if and only if $C = D$ is a branch,
- $(d_2)$ $d(C,D) = d(D,C)$,
- $(d_3)$ $d(C,D) \geq \inf\{d(C,E), d(E,D)\}$.

The proof of $(d_3)$ is given in [**2**] for the case of irreducible $C, D, E$, which implies the general case. Condition $(d_3)$ is equivalent to the following: at least two of three numbers $d(C,D)$, $d(C,E)$, $d(E,D)$ are equal and the third is not smaller than the other two. For each germ $C$, we define

$$d(C) = \sup\{d(C,L) :\ L \text{ runs over all smooth branches}\}$$

and call $d(C)$ the *contact exponent* of $C$ (see [**4**], Definition 1.5 where the term "characteristic exponent" is used). Using $(d_3)$ we check that $d(C) \leq d(C,C)$.

$(d_4)$ For every finite family $(C^i)$ of plane curve germes we have

$$d(\bigcup_i C^i) = \inf\{\inf_i d(C^i), \inf_{i,j} d(C^i, C^j)\}\ .$$

The proof of $(d_4)$ is given in [**3**] (see Proposition 2.6). We say that a smooth germ $L$ has *maximal contact* with $C$ if $d(C,L) = d(C)$. Note that $d(C) = \infty$ if and only if $C$ is a smooth branch. If $C$ is singular then $d(C)$ is a rational number and there exists a smooth branch $L$ which has maximal contact with $C$ (see [**4**], [**1**]).

**3. Results.** Let $C$ be a plane curve germ. A finite family of germs $(C^{(i)})_i$ is called a *decomposition* of $C$ if $C = \cup_i C^{(i)}$ and $C^{(i)}, C^{(i_1)}$ $(i \neq i_1)$ have no common branch. The following definition will play a key role.

DEFINITION 3.1. A plane curve $C$ is *Newton's germ* (shortly an $N$-germ) if there exists a decomposition $(C^{(i)})_{1\le i\le s}$ of $C$ such that the following conditions hold

(1) $1 \le d(C^{(1)}) < \ldots < d(C^{(s)}) \le \infty$.

(2) Let $(C_j^{(i)})_j$ be branches of $C^{(i)}$. Then

- (a) if $d(C^{(i)}) \in \mathbb{N} \cup \{\infty\}$ then the branches $(C_j^{(i)})_j$ are smooth,
- (b) if $d(C^{(i)}) \notin \mathbb{N} \cup \{\infty\}$ then there exists a pair of coprime integers $(a_i, b_i)$ such that each branch $C_j^{(i)}$ has exactly one characteristic pair $(a_i, b_i)$. Moreover, $d(C_j^{(i)}) = d(C^{(i)})$ for all $j$.

(3) If $C_l^{(i)} \neq C_k^{(i_1)}$, then $d(C_l^{(i)}, C_k^{(i_1)}) = \inf\{d(C^{(i)}), d(C^{(i_1)})\}$.

A branch is Newton's germ if it is smooth or has exactly one characteristic pair. Let $C$ be Newton's germ. The decomposition $\{C^{(i)}\}$ satisfying (1), (2) and (3) is not unique. Take for example a germ $C$ that has all $r > 2$ branches smooth intersecting with multiplicity $d > 0$. Then for any branch $L$ of $C$, we may put $C^{(1)} = C \setminus \{L\}$ and $C^{(2)} = \{L\}$ (or simply $C^{(1)} = C$). If $C$ and $D$ are equisingular germs, then $C$ is an $N$-germ if and only if $D$ is an $N$-germ.

Our main result is

THEOREM 3.2. *Let $C$ be a plane curve germ. Then the following two conditions are equivalent*

1. *The germ $C$ is non-degenerate with respect to a chart $(x, y)$ such that $C$ and $\{x = 0\}$ intersect transversally,*
2. *$C$ is Newton's germ.*

We give a proof of Theorem 3.2 in Section 5 of this paper. Let us note here

COROLLARY 3.3. *If a germ $C$ is unitangent, then $C$ is non-degenerate if and only if $C$ is an $N$-germ.*

Every germ $C$ has the *tangential decomposition* $(\tilde{C}^i)_{i=1,\ldots,t}$ such that

1. $\tilde{C}^i$ are unitangent, that is for every two branches $\tilde{C}^i_j$, $\tilde{C}^i_k$ of $\tilde{C}^i$ there is $d(\tilde{C}^i_j, \tilde{C}^i_k) > 1$.
2. $d(\tilde{C}^i, \tilde{C}^{i_1}) = 1$ for $i \neq i_1$.

We call $(\tilde{C}^i)_i$ tangential components of $C$. Note that $t(C) = t$ (the number of tangential components) is an invariant of equisingularity.

THEOREM 3.4. *If* $(\tilde{C}^i)_{i=1,\dots,t}$ *is the tangential decomposition of the germ* $C$ *then the following two conditions are equivalent*

1. *The germ* $C$ *is non-degenerate.*
2. *All tangential components* $\tilde{C}^i$ *of* $C$ *are N-germs and at least* $t(C)-2$ *of them are smooth.*

Using Theorem 3.4, we get

COROLLARY 3.5. *Let* $C$ *and* $D$ *be equisingular plane curve germs. Then* $C$ *is non-degenerate if and only if* $D$ *is non-degenerate.*

## 4. Kouchnirenko's theorem for plane curve singularities.

Let $\mu(C)$ be the *Milnor number* of a reduced germ $C$. By definition, $\mu(C) = \dim \mathbb{C}\{x,y\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, where $f = 0$ is an equation without multiple factors of $C$. The following properties are well-known (see for example [**9**]).

($\mu_1$) $\mu(C) = 0$ if and only if $C$ is a smooth branch.

($\mu_2$) If $C$ is a branch with the first characteristic pair $(a,b)$ then $\mu(C) \geq (a-1)(b-1)$. $\mu(C) = (a-1)(b-1)$ if and only if $(a,b)$ is the unique characteristic pair of $C$.

($\mu_3$) If $(C^{(i)})_{i=1,\dots,k}$ is a decomposition of $C$, then

$$\mu(C) = \sum_{i=1}^{k} \mu(C^{(i)}) + 2 \sum_{1\leq i<j\leq k} (C^{(i)}, C^{(j)}) - k + 1.$$

Now we can give a refined version of Kouchnirenko's theorem in two dimensions.

THEOREM 4.1. *Let* $C$ *be a reduced plane curve germ. Fix a chart* $(x,y)$. *Then* $\mu(C) \geq \nu(\Delta_{x,y}(C))$ *with equality holding if and only if* $C$ *is non-degenerate with respect to* $(x,y)$.

PROOF. Let $f = 0$, $f \in \mathbb{C}\{x,y\}$ be the local equation without multiple factors of the germ $C$. To abbreviate the notation, we put $\mu(f) = \mu(C)$ and $\Delta(f) = \Delta_{x,y}(C)$. If $f = x^a y^b \varepsilon(x,y)$ in $\mathbb{C}\{x,y\}$ with $\varepsilon(0,0) \neq 0$ then the theorem is obvious. Then we can write $f = x^a y^b f_1$ in $\mathbb{C}\{x,y\}$, where $a,b \in \{0,1\}$ and $f_1 \in \mathbb{C}\{x,y\}$ is an appropriate power series. A simple calculation based on properties ($\mu_2$), ($\mu_3$) and ($\nu_1$), ($\nu_2$) shows that $\mu(f) - \nu(\Delta(f)) = \mu(f_1) - \nu(\Delta(f_1))$. Moreover $f$ is non-degenerate if and only if if $f_1$ is non-degenerate and the theorem reduces to the case of an appropriate power series which is proved in [**8**] (Theorem 1.1). □

REMARK 4.2. The implication "$\mu(C) = \nu(\Delta_{x,y}(C)) \Rightarrow C$ is non-degenerate" is not true for hypersurfaces with isolated singularity (see [**5**], Remarque 1.21).

COROLLARY 4.3. *For any reduced germ $C$, there is $\mu(C) \geq (m(C)-1)^2$. The equality holds if and only if $C$ is an ordinary singularity, i.e., such that $t(C) = m(C)$.*

PROOF. Use Theorem 4.1 in generic coordinates. $\square$

**5. Proof of Theorem 3.2.** We start with the implication (1)⇒(2). Let $C$ be a plane curve germ and let $(x, y)$ be a chart such that $\{x = 0\}$ and $C$ intersect transversally. The following result is well-known ([**7**], Proposition 4.7).

LEMMA 5.1. *There exists a decomposition $(C^{(i)})_{i=1,\dots,s}$ of $C$ such that*

1. $\Delta_{x,y}(C^{(i)}) = \left\{ \dfrac{(C^{(i)}, y=0)}{m(C^{(i)})} \right\}$ .
2. *Let $d_i = \frac{(C^{(i)},y=0)}{m(C^{(i)})}$. Then $1 \leq d_1 < \cdots < d_s \leq \infty$ and $d_s = \infty$ if and only if $C^{(s)} = \{y = 0\}$.*
3. *Let $n_i = m(C^{(i)})$ and $m_i = n_i d_i = (C^{(i)}, y = 0)$. Suppose that $C$ is non-degenerate with respect to the chart $(x, y)$. Then $C^{(i)}$ has $r_i =$ g.c.d.$(n_i, m_i)$ branches $C_j^{(i)} : y^{n_i/r_i} - a_{ij}x^{m_i/r_i} + \cdots = 0$ ($j = 1, \dots, r_i$ and $a_{ij} \neq a_{ij'}$, if $j \neq j'$).*

Using the above lemma, we prove that any germ $C$ which is non-degenerate with respect to $(x, y)$ is an $N$-germ. From $(d_4)$ we get $d(C^{(i)}) = d_i$. Clearly, each branch $C_j^{(i)}$ has exactly one characteristic pair $(\frac{n_i}{r_i}, \frac{m_i}{r_i})$ or is smooth. A simple calculation shows that

$$d(C_j^{(i)}, C_{j_1}^{(i_1)}) = \frac{(C_j^{(i)}, C_{j_1}^{(i_1)})}{m(C_j^{(i)})m(C_{j_1}^{(i_1)})} = \inf\{d_i, d_{i_1}\} \ .$$

To prove the implication (2)⇒(1), we need some auxiliary lemmas.

LEMMA 5.2. *Let $C$ be a plane curve germ whose all branches $C_i$ ($i = 1, \dots, s$) are smooth. Then there exists a smooth germ $L$ such that $(C_i, L) = d(C)$ for $i = 1, \dots, s$.*

PROOF. If $d(C) = \infty$, then $C$ is smooth and we take $L = C$. If $d(C) = 1$, then we take a smooth germ $L$ such that $C$ and $L$ are transversal. Let $k = d(C)$ and suppose that $1 < k < \infty$. By formula $(d_4)$, we get $\inf\{(C_i, C_j) : i, j = 1, \dots, s\} = k$. We may assume that $(C_1, C_2) = \ldots = (C_1, C_r) = k$ and $(C_1, C_j) > k$ for $j > r$ for an index $r$, $1 \leq r \leq s$. There is a system of coordinates $(x,y)$ such that $C_j$ $(j=1,\dots,r)$ have equations $y=c_jx^k+\dots$ It suffices to take $L:\ y-cx^k=0$, where $c\neq c_j$ for $j=1,\dots,r$. $\square$

LEMMA 5.3. *Suppose that $C$ is an $N$-germ and let $(C^{(i)})_{1\leq i\leq s}$ be a decomposition of $C$ as in Definition 3.1. Then there is a smooth germ $L$ such that $d(C_j^{(i)},L)=d(C^{(i)})$ for all $j$.*

PROOF. *Step 1.* There is a smooth germ $L$ such that $d(C_j^{(s)},L)=d(C^{(s)})$ for all $j$. If $d(C^{(s)})\in\mathbb{N}\cup\{\infty\}$, then the existence of $L$ follows from Lemma 5.2. If $d(C^{(s)})\notin\mathbb{N}\cup\{\infty\}$, then all components $C_j^{(s)}$ have the same characteristic pair $(a_s,b_s)$. Fix a component $C_{j_0}^{(s)}$ and let $L$ be a smooth germ such that $d(C_{j_0}^{(s)},L)=d(C_{j_0}^{(s)})=d(C^{(s)})$.
Let $j_1\neq j_0$. Then $d(C_{j_1}^{(s)},L)\geq\inf\{d(C_{j_1}^{(s)},C_{j_0}^{(s)}),d(C_{j_0}^{(s)},L)\}=d(C^{(s)})$. On the other hand, $d(C_{j_1}^{(s)},L)\leq d(C_{j_1}^{(s)})=d(C^{(s)})$ and we get $d(C_{j_1}^{(s)},L)=d(C^{(s)})$.
*Step 2.* Let $L$ be a smooth germ such that $d(C_j^{(s)},L)=d(C^{(s)})$ for all $j$. We will check that $d(C_j^{(i)},L)=d(C^{(i)})$ for each $i$ and $j$. To this purpose, fix $i<s$. Let $C_{j_0}^{(s)}$ be a component of $C^{(s)}$. Then $d(C_j^{(i)},C_{j_0}^{(s)})=\inf\{d(C^{(i)}),d(C^{(s)})\}=d(C^{(i)})$. By $(d_3)$ we get $d(C_j^{(i)},L)\geq\inf\{d(C_j^{(i)},C_{j_0}^{(s)}),d(C_{j_0}^{(s)},L)\}=\inf\{d(C^{(i)}),d(C^{(s)})\}=d(C^{(i)})$. On the other hand $d(C_j^{(i)},L)\leq d(C_j^{(i)})=d(C^{(i)})$, which completes the proof. $\square$

REMARK 5.4. In the notation of the above lemma we have $(C^{(i)},L)=m(C^{(i)})d(C^{(i)})$ for $i=1,\dots,s$.

Indeed, if $C_j^{(i)}$ are branches of $C^{(i)}$, then

$$\begin{aligned}(C^{(i)},L)&=\sum_j(C_j^{(i)},L)=\sum_j m(C_j^{(i)})d(C_j^{(i)},L)\\&=\sum_j m(C_j^{(i)})d(C^{(i)})=m(C^{(i)})d(C^{(i)})\ .\end{aligned}$$

LEMMA 5.5. *Let $C$ be an $N$-germ and let $(C^{(i)})_{1\leq i\leq s}$ be a decomposition of $C$ as in Definition 3.1. Then*

$$\begin{aligned}\mu(C)=&\sum_i(m(C^{(i)})-1)(m(C^{(i)})d(C^{(i)})-1)\\&+2\sum_{i<j}m(C^{(i)})m(C^{(j)})\inf\{d(C^{(i)}),d(C^{(j)}\}-s+1\ .\end{aligned}$$

PROOF. Use properties $(\mu_1)$,$(\mu_2)$ and $(\mu_3)$ of the Milnor number. $\square$

To prove implication (2)⇒(1) of Theorem 3.2, suppose that $C$ is an $N$-germ and let $(C^{(i)})_{i=1,\dots,s}$ be a decomposition of $C$ such as in Definition 3.1. Let $L$ be a smooth branch such that $(C^{(i)}, L) = m(C^{(i)})d(C^{(i)})$ for $i = 1, \dots, s$ (such a branch exists by Lemma 5.3 and Remark 5.4). Take a system of coordinates such that $\{x = 0\}$ and $C$ are transversal and $L = \{y = 0\}$. Then we get

$$\Delta_{x,y}(C) = \sum_{i=1}^{s} \Delta_{x,y}(C^{(i)}) = \sum_{i=1}^{s} \left\{ \frac{(C^{(i)}, \{y=0\})}{m(C^{(i)})} \right\} = \sum_{i=1}^{s} \left\{ \frac{m(C^{(i)})d(C^{(i)})}{m(C^{(i)})} \right\}$$

and consequently

$$\begin{aligned}\nu(\Delta_{x,y}(C)) =& \sum_{i=1}^{s}(m(C^{(i)}) - 1)(m(C^{(i)})d(C^{(i)}) - 1) \\ &+ 2 \sum_{1 \le i < j \le s} m(C^{(i)})m(C^{(j)}) \inf\{d(C^{(i)}), d(C^{(j)})\} - s + 1 \\ =& \mu(C)\end{aligned}$$

by Lemma 5.5. Therefore, $\mu(C) = \nu(\Delta_{x,y}(C))$ and $C$ is non-degenerate with respect to $(x, y)$ by Theorem 4.1.

**6. Proof of Theorem 3.4.** The Newton number $\nu(C)$ of the plane curve germ $C$ is defined to be $\nu(C) = \sup\{\nu(\Delta_{x,y}(C)) \ : \ (x, y)$ runs over all charts centered at $O\}$.

Using Theorem 4.1, we get

LEMMA 6.1. *A plane curve germ $C$ is non-degenerate if and only if $\nu(C) = \mu(C)$.*

The proposition below shows that we can reduce the computation of the Newton number to the case of unitangent germs.

PROPOSITION 6.2. *If $C = \bigcup_{k=1}^{t} \tilde{C}^k$ $(t > 1)$ where $\{\tilde{C}^k\}_k$ are unitangent germs such that $(\tilde{C}^k, \tilde{C}^l) = m(\tilde{C}^k)m(\tilde{C}^l)$ for $k \neq l$, then*

$$\nu(C) - (m(C)-1)^2 = \max_{1\le k<l\le t}\{(\nu(\tilde{C}^k) - (m(\tilde{C}^k)-1)^2) + (\nu(\tilde{C}^l) - (m(\tilde{C}^l)-1)^2)\}.$$

PROOF. Let $\tilde{n}_k = m(\tilde{C}^k)$. Suppose that $\{x = 0\}$ and $\{y = 0\}$ are tangent to $C$. Then there are two tangential components $\tilde{C}^{k_1}$ and $\tilde{C}^{k_2}$ such that $\{x = 0\}$ is tangent to $\tilde{C}^{k_1}$ and $\{y = 0\}$ is tangent to $\tilde{C}^{k_2}$. Now, there is

$$
\begin{aligned}
\nu(\Delta_{x,y}(C)) &= \nu(\sum_{k=1}^{t} \Delta_{x,y}(\tilde{C}^k)) = \nu(\Delta_{x,y}(\tilde{C}^{k_1})) + \nu(\Delta_{x,y}(\tilde{C}^{k_2})) \\
&\quad + \sum_{k \neq k_1, k_2} \nu(\Delta_{x,y}(\tilde{C}^k)) + 2 \sum_{1 \le k < l \le t} \left[\Delta_{x,y}(\tilde{C}^k), \Delta_{x,y}(\tilde{C}^l)\right] - t + 1 \\
&= \nu(\Delta_{x,y}(\tilde{C}^{k_1})) + \nu(\Delta_{x,y}(\tilde{C}^{k_2})) + \sum_{k \neq k_1, k_2} (\tilde{n}_k - 1)^2 + 2 \sum_{1 \le k < l \le t} \tilde{n}_k \tilde{n}_l - t + 1 \\
&= \nu(\Delta_{x,y}(\tilde{C}^{k_1})) - (\tilde{n}_{k_1} - 1)^2 \\
&\quad + \nu(\Delta_{x,y}(\tilde{C}^{k_2})) - (\tilde{n}_{k_2} - 1)^2 + (m(C) - 1))^2.
\end{aligned}
$$

The germs $\tilde{C}^{k_1}$ and $\tilde{C}^{k_2}$ are unitangent and transversal. Thus it is easy to see that there exists a chart $(x_1, y_1)$ such that $\nu(\Delta_{x_1,y_1}(\tilde{C}^k)) = \nu(\tilde{C}^k)$ for $k = k_1, k_2$.
If $\{x = 0\}$ (or $\{y = 0\}$) and $C$ are transversal, then there exists a $k \in \{1, \dots, t\}$ such that $\nu(\Delta_{x,y}(C)) = \nu(\Delta_{x,y}(\tilde{C}^k)) - (\tilde{n}_k - 1)^2 + (m(C) - 1))^2$ and the proposition follows from the previous considerations. □

Now we can pass to the proof of Theorem 3.4. If $t(C) = 1$ then $C$ is non-degenerate with respect to a chart $(x, y)$ such that $C$ and $\{x = 0\}$ intersect transversally and Theorem 3.4 follows from Theorem 3.2. If $t(C) > 1$, then by Proposition 6.2 there are indices $k_1 < k_2$ such that

($\alpha$) $\nu(C) - (m(C) - 1)^2 = \nu(\tilde{C}^{k_1}) - (m(\tilde{C}^{k_1}) - 1)^2 + \nu(\tilde{C}^{k_2}) - (m(\tilde{C}^{k_2}) - 1)^2$ .

On the other hand, from basic properties of the Milnor number we get

($\beta$) $\mu(C) - (m(C) - 1)^2 = \sum_k (\mu(\tilde{C}^k) - (m(\tilde{C}^k) - 1)^2)$ .

Using ($\alpha$), ($\beta$) and Lemma 6.1, we check that $C$ is non-degenerate if and only if $\mu(\tilde{C}^{k_1}) = \nu(\tilde{C}^{k_1})$, $\mu(\tilde{C}^{k_2}) = \nu(\tilde{C}^{k_2})$ and $\mu(\tilde{C}^k) = (m(\tilde{C}^k) - 1)^2$ for $k \neq k_1, k_2$. Now Theorem 3.4 follows from Lemma 6.1 and Corollary 4.3.

**7. Concluding remark.** M. Oka in [**6**] proved that the Newton number like the Milnor number is an invariant of equisingularity. Therefore, the invariance of non-degeneracy (Corollary 3.5) follows from the equality $\nu(C) = \mu(C)$ characterizing non-degenerate germs (Lemma 6.1).

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