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NON-DEGENERACY OF THE DISCRIMINANT

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To Professor Arkadiusz Płoski on his 65th birthday

Abstract. Let $(\ell, f) : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^2, 0)$ be the germ of a holomorphic mapping such that $\ell = 0$ is a smooth curve and $f = 0$ has an isolated singularity at $0 \in \mathbf{C}^2$. We assume that $\ell = 0$ is not a branch of $f = 0$. The direct image of the critical locus of this mapping is called the discriminant curve. The role of Puiseux exponents of the branches of the discriminant is mysterious, and it is therefore of interest to determine when there is non-degeneracy. In this paper we describe the weighted initial forms of the discriminant curve with respect to its Newton diagram.Then we study the pairs (ℓ, f) for which the discriminant curve is non-degenerate in the Kouchnirenko sense.

1. Introduction

Let $(\ell, f) : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}^2, 0)$ be a holomorphic mapping given by $u = \ell(x, y)$, $v = f(x, y)$, where $\ell = 0$ is a smooth curve and $f = 0$ has an isolated singularity at $0 \in \mathbf{C}^2$. We assume that $\ell = 0$ is not a branch of $f = 0$. To any such morphism we can associate two analytic curves: the *polar curve*

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$\frac{\partial \ell}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial \ell}{\partial y}\frac{\partial f}{\partial x} = 0$ and its direct image $\mathcal{D}(u,v) = 0$ which is called the *discriminant curve* of the morphism (ℓ, f) (see [20], [1]). A series $\mathcal{D}(u,v)$, defined up to multiplication by an invertible power series, is called the *discriminant.* In [20] and [22] Teissier introduced the *Jacobian Newton diagram,* which is the Newton diagram of $\mathcal{D}(u,v)$. The Jacobian Newton diagram depends only on the topological type of (ℓ, f) (see [20] for the case where ℓ is generic, Merle [18] and Ephraim [3] for one branch and [6], [17] and [19] for general case). Decompositions of the polar curve can be found in the literature (see [18], [3], [2], [4]). In the spirit of Eggers [2] we propose a factorization of the discriminant $\mathcal{D}(u,v)$. The Newton diagram of every factor has only one compact edge. We specify formulas for the weighted initial forms of these factors. Using this description we study the pairs (ℓ, f) for which the discriminant is non-degenerate, in the Kouchnirenko sense [12], answering a question of Patrick Popescu-Pampu.

For the irreducible case we prove in Section 4:

THEOREM 1.1. *Let $f = 0$ be a branch. Then the discriminant of (ℓ, f) is non-degenerate if and only if there are no lattice points inside the compact edges of its Newton diagram.*

COROLLARY 1.2. *Let $f = 0$ be a branch. Then the non-degeneracy of the discriminant of (ℓ, f) depends only on the topological type of (ℓ, f).*

In the multi-branched case the topological type of (ℓ, f) does not determine whether the discriminant is non-degenerate. The non-degeneracy depends also on the analytical type of (ℓ, f) as shown in Examples 2.8 and 2.9. We shed light on that case in Proposition 5.6 and Theorem 5.7.

The structure of the paper is as follows: in Section 2 we start by recalling the notion of non-degeneracy. Then, after a change of coordinates, we may assume that the morphism that we consider has the form (x, f). We describe the discriminant by using Newton–Puiseux roots of the y-partial derivative of $f(x,y)$. For that the Lemma of Kuo-Lu plays an important role. Using the results of this section we construct examples of curves with many smooth branches, which determine non-degenerate discriminants.

In Section 3 we propose an analytical factorization of $\mathcal{D}(u,v)$. In Proposition 3.8 we compute the initial Newton polynomial of every factor and express it as a product of rational powers of quasi-homogeneous polynomials. Then in Section 4 we apply this formula to irreducible power series $f(x,y)$ and we characterize in Corollary 4.4 the equisingularity classes of branches for which the discriminant of (x, f) is non-degenerate.

In Section 5 we return to the general case. Taking up again Proposition 3.8 we give, in Proposition 5.6, a polynomial factorization of the initial Newton polynomials of the factors of $\mathcal{D}(u,v)$. As a consequence, in Theorem 5.7, we obtain a criterion for non-degeneracy of the factors of the discriminant.

We finish this section with another example of curves with as many singular branches as we wish, which determine non-degenerate discriminants.

In the last section we analyze what impact on the discriminant has a modification of ℓ or f in the morphism (ℓ, f). Theorem 6.2 shows that non-degeneracy of the discriminant of the morphism (ℓ, f) is independent of the choice of the representative of the curve $f = 0$. Theorem 6.6 shows that if $f = 0$ is unitangent and transverse to $\ell = 0$, then the non-degeneracy of the discriminant of the morphism (ℓ, f) depends only on the curve $f = 0$. The assumption that $f = 0$ has only one tangent cannot be omitted as it is shown in Example 6.7.

2. Preliminaries

We start this section recalling the notion of non-degeneracy. Then we reduce our study to the morphisms of the form (x, f). We describe the discriminant by using Newton-Puiseux roots of $\frac{\partial f}{\partial y}(x, y)$. The Lemma of Kuo-Lu plays an important role.

2.1. Non-degeneracy after Kouchnirenko. Set $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geqq 0\}$. Let $f(x,y) = \sum_{ij} a_{ij} x^i y^j \in \mathbf{C}\{x,y\} \backslash \{0\}$. The *Newton diagram* of f is

$$\Delta_f := \text{Convex Hull}\left(\left\{ (i,j) : \, a_{ij} \neq 0 \right\} + \mathbf{R}_+^2\right).$$

The Newton diagram of a product is the Minkowski sum of the Newton diagrams of the factors. That is $\Delta_{fg} = \Delta_f + \Delta_g$, where

$$\Delta_f + \Delta_g = \{a + b : \, a \in \Delta_f, \; b \in \Delta_g\}.$$

In particular if f and g differ by an invertible factor $u \in \mathbf{C}\{x,y\}$, $u(0,0) \neq 0$ then $\Delta_f = \Delta_g$.

The *initial Newton polynomial* of $f(x,y) = \sum_{i,j} a_{ij} x^i y^j$, denoted by $\mathrm{in}_{\mathcal{N}} f$, is the sum of all terms $a_{ij} x^i y^j$ such that (i,j) belongs to a compact edge of Δ_f.

Following Teissier [21] we introduce *elementary Newton diagrams.* For $m, n > 0$ we put $\left\{\frac{n}{m}\right\} = \Delta_{x^n + y^m}$. We put also $\left\{\frac{n}{\infty}\right\} = \Delta_{x^n}$ and $\left\{\frac{\infty}{m}\right\} = \Delta_{y^m}$. By definition the *inclination* of $\left\{\frac{L}{M}\right\}$ is L/M with the conventions that $L/\infty = 0$ and $\infty/M = +\infty$. Any Newton diagram can be written as a Minkowski sum of elementary Newton diagrams, where inclinations of successive elementary diagrams form an increasing sequence.

Let S be a compact edge of Δ_f of inclination p/q, where p and q are coprime integers. The *initial part of $f(x,y)$ with respect to S* is the quasi-homogeneous polynomial $f_S(x,y) = \sum a_{ij} x^i y^j$ where the sum runs over all lattice points $(i,j)\in S$. Observe that if Δ_f is an elementary Newton diagram then the initial part of $f(x,y)$ with respect to the only compact edge of Δ_f coincides with the initial Newton polynomial of $f(x,y)$.

Decomposing $f_S(x,y)$ into irreducible factors in $\mathbf{C}[x,y]$ we get

$$f_S(x,y)=cx^k y^l\prod_{i=1}^{r}(y^q-a_i x^p)^{s_i}, \tag{1}$$

where k and l are non-negative integers, c and a_i are nonzero complex numbers and $a_i\neq a_j$ for $i\neq j$.

The series $f(x,y)$ is *non-degenerate on the compact edge* S of Δ_f if in (1) $s_i=1$ for all $i\in\{1,\dots,r\}$. In particular f is non-degenerate on the compact edge S if there are no lattice points inside S. The converse is not true as $(y-x)(y-2x)$ shows. The series $f(x,y)$ is *non-degenerate* if it is non-degenerate on every compact edge of its Newton diagram (see [12]).

2.2. Newton–Puiseux roots. Let $\mathbf{C}\{x\}^*$ be the ring of Puiseux series in x, that is the set of series of the form

$$\alpha(x)=a_1x^{N_1/D}+a_2x^{N_2/D}+\cdots,\quad a_i\in\mathbf{C},$$

where $N_1<N_2<\dots$ are non-negative integers, D is a positive integer and $a_1t^{N_1}+a_2t^{N_2}+\cdots$ has a positive radius of convergence. In this paper $+\cdots$ means *plus higher order terms.* If $a_1\neq 0$ then the *order* of $\alpha(x)$ is $\operatorname{ord}\alpha(x)=N_1/D$ and the *initial part* of $\alpha(x)$ equals $\operatorname{in}\alpha(x)=a_1x^{N_1/D}$. By convention the order of the zero series is $+\infty$. For any Puiseux series $\alpha(x)$, $\gamma(x)$ we denote by $O(\alpha,\gamma)=\operatorname{ord}\big(\alpha(x)-\gamma(x)\big)$ and call this number the *contact order* between $\alpha(x)$ and $\gamma(x)$. If $Z\subset\mathbf{C}\{x\}^*$ is a finite set then the *contact* between $\alpha\in\mathbf{C}\{x\}^*$ and Z is $\operatorname{cont}(\alpha,Z)=\max_{\gamma\in Z}O(\alpha,\gamma)$.

By a *fractional power series* we mean a Puiseux series of positive order.

Let $g(x,y)\in\mathbf{C}\{x,y\}$ be a convergent power series. A fractional power series $\gamma(x)$ is called a *Newton–Puiseux root* of $g(x,y)$ if $g\big(x,\gamma(x)\big)=0$ in $\mathbf{C}\{x\}^*$. We denote by $\operatorname{Zer}g$ the set of all Newton–Puiseux roots of $g(x,y)$.

If $g=g_1^{a_1}\cdots g_r^{a_r}$ where the g_i are irreducible and pairwise coprime elements of $\mathbf{C}\{x,y\}$, then the curves $g_i=0$ are called the *branches* of $g=0$. We say that $g=0$ is *reduced* if $a_1=\cdots=a_r=1$. Notice that g has an isolated singularity at $0\in\mathbf{C}^2$ if and only if it is singular and reduced.

2.3. The lemma of Kuo-Lu. Consider the morphism (ℓ,f) as in Introduction, where f is a reduced curve. An analytic change of coordinates does not affect the discriminant curve (see for example [1], Section 3). Hence in what follows we assume that $\ell(x,y)=x$. Then $\frac{\partial f}{\partial y}=0$ is the polar curve of (x,f).

The Newton–Puiseux factorizations of $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are of the form

$$f(x,y) = u(x,y)\prod_{i=1}^{p}\left[y-\alpha_i(x)\right], \tag{2}$$

$$\frac{\partial f}{\partial y}(x,y) = \tilde{u}(x,y)\prod_{j=1}^{p-1}\left[y-\gamma_j(x)\right], \tag{3}$$

where $u(x,y)$, $\tilde{u}(x,y)$ are units in $\mathbf{C}\{x,y\}$ and $\alpha_i(x)$, $\gamma_j(x)$ are fractional power series. Since f is reduced, $\alpha_i(x) \neq \alpha_j(x)$ for $i \neq j$.

The following lemma, which is a part of Lemma 3.3 in [13] (for the transverse case; see [7], Corollary 3.5 and [10], Proposition 2.2 for the general case), describes the contacts between Newton–Puiseux roots of $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$.

Lemma 2.1. *For every $\gamma_j \in \operatorname{Zer}\frac{\partial f}{\partial y}$ there exist $\alpha_k, \alpha_l \in \operatorname{Zer} f$, $k \neq l$ such that*

$$O(\alpha_k,\gamma_j) = O(\alpha_l,\gamma_j) = O(\alpha_k,\alpha_l) = \max_{i=1}^{p} O(\alpha_i,\gamma_j).$$

In what follows we recall the *tree model* introduced in [13] which encodes the contact orders between Newton–Puiseux roots of $f(x,y)$.

Definition 2.2. Let $\alpha \in \mathbf{C}\{x\}^*$ and let h be a positive rational number. The pseudo-ball $B(\alpha,h)$ is the set $B(\alpha,h) = \left\{\gamma \in \mathbf{C}\{x\}^* : O(\gamma,\alpha) \geqq h\right\}$. We call $h(B) := h$ the height of $B := B(\alpha,h)$.

Note that $h(B)$ is well-defined since $h(B) = \inf\left\{O(\gamma,\beta) : \gamma,\beta \in B\right\}$.

Consider the following set of pseudo-balls

$$T(f) := \left\{B\big(\alpha, O(\alpha,\alpha')\big) : \alpha,\alpha' \in \operatorname{Zer} f,\ \alpha \neq \alpha'\right\}.$$

The elements of $T(f)$ can be identified with bars of the *tree model* of f defined in [13] (for a short presentation see also Section 8 of [11]). It follows from Lemma 2.1 that for every $\gamma \in \operatorname{Zer}\frac{\partial f}{\partial y}$ there exists exactly one $B \in T(f)$ such that $\gamma \in B$ and $h(B) = \operatorname{cont}(\gamma, \operatorname{Zer} f)$. Following [14] we say that γ *leaves* $T(f)$ at B.

Take a pseudo-ball $B \in T(f)$. Every $\gamma \in B$ has the form

$$\gamma(x) = \lambda_B(x) + c_\gamma x^{h(B)} + \cdots, \tag{4}$$

where $\lambda_B(x)$ is obtained from an arbitrary $\alpha(x) \in B$ by omitting all the terms of order bigger than or equal to $h(B)$.

We call the complex number c_γ the *leading coefficient* of γ with respect to B and we denote it by $\mathrm{lc}_B(\gamma)$. Remark that c_γ can be zero.

We need next two Lemmas from [7] (see also [15] and [16], Corollary 3.7 and Proposition 3.6).

LEMMA 2.3 ([7], Lemma 3.3). *Let $B \in T(f)$. There exist a polynomial $F_B(z) \in \mathbf{C}[z]$, depending on f, and a rational number $q(B)$ such that for every $\gamma(x) = \lambda_B(x) + c_\gamma x^{h(B)} + \cdots$*

$$f\big(x, \gamma(x)\big) = F_B(c_\gamma)x^{q(B)} + \cdots . \tag{5}$$

Moreover

$$F_B(z) = C \prod_{i\colon \alpha_i \in B} \big(z - \mathrm{lc}_B(\alpha_i)\big), \tag{6}$$

where C is a nonzero constant.

REMARK 2.4. It follows from the proof of Lemma 3.3 in [7] that if f is a Weierstrass polynomial and $\alpha_j(x) \in B$, then the constant C in (6) is expressed by the formula

$$Cx^{q(B)} = \prod_{i\colon \alpha_i \notin B} \mathrm{in}\big(\alpha_j(x) - \alpha_i(x)\big) \prod_{i\colon \alpha_i \in B} x^{h(B)}.$$

LEMMA 2.5 ([7], Lemma 3.4). *Let $B \in T(f)$. Then*

$$\frac{d}{dz}F_B(z) = C' \prod_{j\colon \gamma_j \in B} \big(z - \mathrm{lc}_B(\gamma_j)\big),$$

where C' is a nonzero constant.

Using the above lemmas we characterize the Newton–Puiseux roots of $\frac{\partial f}{\partial y}(x,y)$ leaving $T(f)$ at a fixed B.

LEMMA 2.6. *Let $B \in T(f)$ and $\gamma \in B$. Then γ leaves $T(f)$ at B if and only if $F_B\big(\mathrm{lc}_B(\gamma)\big) \neq 0$.*

PROOF. For $\gamma \in B$ the inequality $F_B\big(\mathrm{lc}_B(\gamma)\big) \neq 0$ is equivalent to $\mathrm{lc}_B(\gamma) \neq \mathrm{lc}_B(\alpha_i)$ for all $\alpha_i \in B$, and this is equivalent to $\mathrm{cont}(\gamma, \mathrm{Zer}\, f) = h(B)$. $\square$

Given $B, B' \in T(f)$, we say that B' is a *direct successor* of B in $T(f)$ if $B \supset B'$ and there is no $B'' \in T(f)$ (different from B and B') such that $B \supset B'' \supset B'$. The next lemma follows from Theorem C in [13]. For convenience of the reader we present a proof:

LEMMA 2.7. *Let $B, B' \in T(f)$. Suppose that B' is a direct successor of B in $T(f)$. Then $q(B') - q(B) = \sharp(B' \cap \operatorname{Zer} f)\left[h(B') - h(B)\right]$, where the symbol $\sharp$ stands for the number of the elements of a set. If $B \in T(f)$ is the pseudo-ball of the minimal height then $q(B) = \sharp(\operatorname{Zer} f)h(B)$.*

PROOF. Let $\delta(x) = \lambda_B(x) + cx^{h(B)}$ where $F_B(c) \neq 0$ and $\delta'(x) = \lambda_{B'}(x) + c'x^{h(B')}$ where $F_{B'}(c') \neq 0$. Then following (2) and Lemma 2.3

$$q(B) = \operatorname{ord} f\big(x, \delta(x)\big) = \sum_{\alpha \in \operatorname{Zer} f} O(\delta, \alpha) \tag{7}$$

and

$$q(B') = \operatorname{ord} f\big(x, \delta'(x)\big) = \sum_{\alpha \in \operatorname{Zer} f} O(\delta', \alpha). \tag{8}$$

We have $O(\delta, \alpha) = h(B)$, $O(\delta', \alpha) = h(B')$ for $\alpha \in \operatorname{Zer} f \cap B'$. Using the strong triangle inequality property of the contact order one checks that $O(\delta, \alpha) = O(\delta', \alpha)$ for $\alpha \in \operatorname{Zer} f \backslash B'$. Substracting (7) from (8) we get the first statement of the lemma. The second statement of the lemma is a consequence of (7). $\square$

Following Lemma 5.4 in [5] the discriminant of the morphism (x, f) can be written as

$$\mathcal{D}(u, v) = \prod_{j=1}^{p-1} \big(v - f\big(u, \gamma_j(u)\big)\big). \tag{9}$$

EXAMPLE 2.8. Let

$$h(x, y) = \left(y - x^2 - x^3\right)\left(y - x^2 + x^3\right)\left(y + x^2 - x^3\right)\left(y + x^2 + x^3\right)$$

and let

$$f_1(x, y) = x^{10} + \int_0^y h(x, t)\, dt.$$

Since $\frac{\partial f_1}{\partial y}(x, y) = h(x, y)$, we get by (9)

$$\operatorname{in}_{\mathcal{N}} \mathcal{D}(u, v) = \left(v - \frac{23}{15}u^{10}\right)^2 \left(v - \frac{7}{15}u^{10}\right)^2.$$

Thus the discriminant of (x, f_1) is degenerate. One can also show that it remains degenerate after any analytical change of coordinates.

EXAMPLE 2.9. Let $f_2(x,y) = y^5 + x^8y + x^{10}$. As $f_2(x,y)$ is a quasi-homogeneous polynomial, all its Newton–Puiseux roots are monomials of the same order. The same applies to $\frac{\partial f_2}{\partial y}$. The tree model $T(f_2)$ has only one pseudo-ball B of the height 2. We have $F_B(z) = f_2(1,z) = z^5 + z + 1$. All critical values $w_j = F_B(z_j)$, where $z_1, \dots, z_4$ are critical points of $F_B(z)$, are pairwise different. By (9) and Lemma 2.5 we get

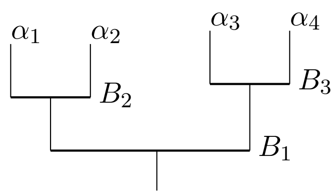
$$\mathcal{D}(u,v) = \prod_{j=1}^{4} \bigl(v - f_2\bigl(u, z_j u^2\bigr)\bigr) = \prod_{j=1}^{4} \bigl(v - w_j u^{10}\bigr).$$

Hence the discriminant of (x, f_2) is non-degenerate.

The curves $f_1(x,y) = 0$ and $f_2(x,y) = 0$ are equisingular. Nevertheless the discriminant of (x, f_1) is degenerate while the discriminant of (x, f_2) is non-degenerate.

EXAMPLE 2.10. Let $f(x,y) = \prod_{i=1}^{4} \bigl(y - \alpha_i(x)\bigr)$ where $\alpha_1(x) = x + x^3$, $\alpha_2(x) = x - x^3$, $\alpha_3(x) = -x + x^4$ and $\alpha_4(x) = -x - x^4$. The curve $f = 0$ has four smooth branches.

The tree model $T(f)$ is given in the picture below. Following [13] we draw pseudo-balls of finite height as horizontal bars. The tree $T(f)$ has three bars: B_1 of height 1, B_2 of height 3 and B_3 of height 4.



In order to compute the polynomial $F_B(z)$ for $B \in T(f)$ it is enough to find the lowest order term of $f\bigl(x, \lambda_B(x) + z x^{h(B)}\bigr)$.

Since $\lambda_{B_1}(x) = 0$ and $h(B_1) = 1$, we get $f\bigl(x, \lambda_{B_1}(x) + z x^{h(B_1)}\bigr) = f(x, zx)$ $= (z-1)^2(z+1)^2x^4 + \cdots$.

Similarly

$$f\bigl(x, \lambda_{B_2}(x) + z x^{h(B_2)}\bigr) = f\bigl(x, x + z x^3\bigr) = 4(z-1)(z+1)x^8 + \cdots$$

and

$$f\bigl(x, \lambda_{B_3}(x) + z x^{h(B_3)}\bigr) = f\bigl(x, -x + z x^4\bigr) = 4(z-1)(z+1)x^{10} + \cdots.$$

Hence

$$F_{B_1}(z) = (z-1)^2(z+1)^2, \quad q(B_1) = 4,$$

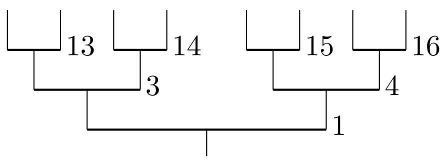
$$F_{B_2}(z) = 4(z-1)(z+1), \quad q(B_2) = 8,$$

$$F_{B_3}(z) = 4(z-1)(z+1), \quad q(B_3) = 10.$$

Each of the above polynomials has exactly two roots. Thus for every $i \in \{1,2,3\}$ there exists a unique critical point z_i, $F'_{B_i}(z_i) = 0$ such that the critical value $w_i = F_{B_i}(z_i)$ is nonzero. It follows from Lemmas 2.5 and 2.6 that $z_i = \mathrm{lc}_{B_i}\, \gamma_i$ for some $\gamma_i \in \mathrm{Zer}\, \frac{\partial f}{\partial y}$ which leaves $T(f)$ at B_i. By Lemma 2.3 we have $f\big(x, \gamma_i(x)\big) = w_i x^{q(B_i)} + \cdots$. In view of equality (9) the initial Newton polynomial of the discriminant $\mathcal{D}(u,v)$ is the initial Newton polynomial of $\prod_{i=1}^{3} \big(v - w_i u^{q(B_i)}\big)$. Since this polynomial does not have multiple factors, the discriminant $\mathcal{D}(u,v)$ is non-degenerate.

What matters in Example 2.10 is that different $B \in T(f)$ have different $q(B)$ and also that $T(f)$ is a binary tree, hence for every $B \in T(f)$ the polynomial $F_B(z)$ has exactly two roots and consequently there exists exactly one $\gamma \in \mathrm{Zer}\, \frac{\partial f}{\partial y}$ which leaves $T(f)$ at B. We use this idea in the next example.

Example 2.11. Let $g(x,y)$ be a power series which tree model $T(g)$ is presented in the figure below. The numbers attached to the bars are the heights of corresponding pseudo-balls. Applying Lemma 2.7 one can check that $\big\{ q(B) : B \in T(g) \big\} = \{8, 16, 20, 36, 38, 42, 44\}$. By the same argument as before the discriminant of the morphism (x, g) is non-degenerate.



The curve $g = 0$ from the above example decomposes into eight smooth branches. Following the idea of Example 2.11 one can construct new examples of multibranched curves, with more levels in their tree models, whose discriminants are non-degenerate.

3. Factorization of the discriminant

Assume that all the Newton–Puiseux roots of $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ belong to $\mathbf{C}\big\{ x^{1/D} \big\}$ for some positive integer D. We define the action of the multiplicative group $\mathbf{U}_D = \big\{ \theta \in \mathbf{C} : \theta^D = 1 \big\}$ of D-th complex roots of unity on $\mathbf{C}\big\{ x^{1/D} \big\}$.

Take $\theta \in \mathbf{U}_D$ and $\phi \in \mathbf{C}\{ x^{1/D} \}$ of the form

$$\phi(x) = a_1 x^{N_1/D} + a_2 x^{N_2/D} + \cdots ,$$

where $0 \leqq N_1 < N_2 < \cdots$. By definition

$$\theta * \phi(x) = a_1 \theta^{N_1} x^{N_1/D} + a_2 \theta^{N_2} x^{N_2/D} + \cdots .$$

Following [14] we call the series $\theta * \phi$ *conjugate* to ϕ.

It is well-known (see for example [23]) that if $g(x,y)$ is an irreducible power series such that $\operatorname{Zer} g \subset \mathbf{C}\{ x^{1/D} \}$ then the conjugate action of $\mathbf{U}_D$ permutes transitively the Newton–Puiseux roots of $g(x,y)$. The conjugate action of $\mathbf{U}_D$ preserves the contact order, i.e. $O(\phi,\psi) = O(\theta * \phi, \theta * \psi)$ for $\phi, \psi \in \mathbf{C}\{ x^{1/D} \}$ and $\theta \in \mathbf{U}_D$.

The *index* of a fractional power series $\beta(x)$ is the smallest positive integer N such that $\beta(x) \in \mathbf{C}\{ x^{1/N} \}$. Following [23] we get:

Property 3.1. *Let $\beta(x) \in \mathbf{C}\{ x^{1/D} \}$ be a fractional power series. Then the following conditions are equivalent*:

1. *The index of $\beta(x)$ equals N.*
2. *The set $\{ \theta * \beta(x) : \theta^D = 1 \}$ has N elements.*
3. *If $g(x,y)$ is an irreducible power series such that $g\big(x, \beta(x)\big) = 0$ then* $\operatorname{ord} g(0,y) = N$.

The action of $\mathbf{U}_D$ on $\operatorname{Zer} f$ induces an action of this group on $T(f)$ as follows. Let $B = B\big(\alpha_k, O(\alpha_k, \alpha_l)\big)$ and let $\theta \in \mathbf{U}_D$. Set

$$\theta * B = B\big(\theta * \alpha_k, O(\alpha_k, \alpha_l)\big) .$$

The properties of the conjugate action imply that $\theta * B$ is an element of $T(f)$ and $\theta * B = B\big(\theta * \lambda_B, h(B)\big)$. Hence the definition of $\theta * B$ does not depend on the choice of $\alpha_k \in B \cap \operatorname{Zer} f$.

Proposition 3.2. *Let $B \in T(f)$, $\theta \in \mathbf{U}_D$ and $B' = \theta * B$. Then $q(B) = q(B')$ and $\theta^{q(B)D} F_B(z) = F_{B'}\big(\theta^{h(B)D} z\big)$.*

Proof. Acting by θ on the equation

$$f\big(x, \lambda_B(x) + c x^{h(B)}\big) = F_B(c) x^{q(B)} + \cdots$$

we get

$$f\big(x, \lambda_{B'}(x) + c\theta^{h(B)D} x^{h(B)}\big) = F_B(c) \theta^{q(B)D} x^{q(B)} + \cdots .$$

By Lemma 2.3

$$f\big(x, \lambda_{B'}(x) + c\theta^{h(B)D} x^{h(B)}\big) = F_{B'}(c\theta^{h(B)D}) x^{q(B')} + \cdots .$$

Since c is arbitrary, equating the right hand sides of the formulas above gives the proof. $\square$

For every $B \in T(f)$ we denote by $\overline{B}$ the orbit $\mathbf{U}_D * B$ and by $E(f)$ the set of all orbits in $T(f)$.

Fix $B \in T(f)$. Let $\mathcal{D}_B(u,v) = \prod_j \big(v - f\big(u, \gamma_j(u)\big)\big)$ where the product runs over all j such that γ_j leaves $T(f)$ at B. Set $\mathcal{D}_{\overline{B}}(u,v) = \prod_{B' \in \overline{B}} \mathcal{D}_{B'}(u,v)$. Then $\mathcal{D}_{\overline{B}}(u,v)$ is a polynomial in v with coefficients in $\mathbf{C}\big\{ u^{1/D} \big\}$. Furthermore we have:

Lemma 3.3. $\mathcal{D}_{\overline{B}}(u,v) \in \mathbf{C}\{u\}[v]$.

Proof. It is enough to verify that for every complex number v_0 the index of $\mathcal{D}_{\overline{B}}(u,v_0) \in \mathbf{C}\big\{ u^{1/D} \big\}$ is 1, which is equivalent, by Property 3.1, that the action of $\mathbf{U}_D$ on this Puiseux series is trivial.

Take $\theta \in \mathbf{U}_D$ and $B' \in \overline{B}$. We have

$$\theta * \mathcal{D}_{B'}(u,v_0) = \prod_j \big(v_0 - f\big(u, \theta * \gamma_j(u)\big)\big),$$

where j runs over γ_j leaving $T(f)$ at B' and

$$\mathcal{D}_{\theta * B'}(u,v_0) = \prod_j \big(v_0 - f\big(u, \gamma_j(u)\big)\big),$$

where j runs over γ_j leaving $T(f)$ at $\theta * B'$.

Since $\gamma \in \operatorname{Zer} \frac{\partial f}{\partial y}$ leaves $T(f)$ at B' if and only if $\theta * \gamma$ leaves $T(f)$ at $\theta * B'$, we get $\theta * \mathcal{D}_{B'}(u,v_0) = \mathcal{D}_{\theta * B'}(u,v_0)$. As a consequence

$$\theta * \mathcal{D}_{\overline{B}}(u,v_0) = \theta * \prod_{B' \in \overline{B}} \mathcal{D}_{B'}(u,v_0) = \prod_{B' \in \overline{B}} \mathcal{D}_{\theta * B'}(u,v_0) = \mathcal{D}_{\overline{B}}(u,v_0). \quad \square$$

We conclude that $\prod_{\overline{B} \in E(f)} \mathcal{D}_{\overline{B}}(u,v)$ is an analytical factorization (not necessarily into irreducible factors) of the discriminant.

By Proposition 3.2 every factor $\mathcal{D}_{\overline{B}}(u,v)$ has an elementary Newton diagram of inclination $q(B)$. Observe that if $\mathcal{D}_{\overline{B}}(u,v)$ is degenerate then $\mathcal{D}(u,v)$ is also degenerate. The aim of this section is to compute the initial Newton polynomial of $\mathcal{D}_{\overline{B}}(u,v)$. For this we need the next auxiliary results:

Lemma 3.4. *Let A, B be positive integers. Then*

$$\prod_{\theta^A = 1} (z - \theta^B a) = \big(z^{A/\gcd(A,B)} - a^{A/\gcd(A,B)}\big)^{\gcd(A,B)}.$$

PROOF. Set $C = \gcd(A,B)$ and $A_1 = A/C$, $B_1 = B/C$. Then

$$\prod_{\theta^A=1} \left(z-\theta^B a\right) = \prod_{(\theta^C)^{A_1}=1} \left(z-\left(\theta^C\right)^{B_1} a\right) = \prod_{\substack{\omega^{A_1}=1\\ \theta^C=\omega}} \left(z-\omega^{B_1} a\right)$$

$$= \prod_{\omega^{A_1}=1} \left(z-\omega^{B_1} a\right)^C = \left(z^{A_1}-a^{A_1}\right)^C,$$

where the last equality holds since the numbers $\omega^{B_1}a$ for $\omega^{A_1}=1$ are all A_1-th complex roots of a^{A_1}. $\square$

LEMMA 3.5. *Let G be a finite group and A be a finite set. Assume that G acts on A transitively, that is $A = Ga_0$ for some $a_0 \in A$. Let P be a complex valued function on A. Set $G_0 := \{g \in G : ga_0 = a_0\}$. Then*

(i) $\sharp A \cdot \sharp G_0 = \sharp G$.

(ii) $\prod_{g\in G} P(ga_0) = \prod_{a\in A} \left(P(a)\right)^{\sharp G_0}$.

PROOF. The first statement is the orbit-stabilizer theorem.

To prove the second statement consider the function $h : G \to A$ given by $h(g) = ga_0$. Then

$$\prod_{g\in G} P(ga_0) = \prod_{a\in A} \prod_{g\in h^{-1}(a)} P\big(h(g)\big) = \prod_{a\in A} P(a)^{\sharp G_0}.$$

The last equality holds since the fibers of the function h are the left-cosets of G_0 in G. $\square$

Now, our aim is to give a formula for $F_B(z)$ from Lemma 2.3.

Fix a pseudo-ball B of $T(f)$. Let $f = f_1 \cdots f_r$ be the decomposition of f into irreducible factors. Assume that $\operatorname{Zer} f_j \cap B \neq \emptyset$ for $j \in \{1,\dots,s\}$ and $\operatorname{Zer} f_j \cap B = \emptyset$ for $j \in \{s+1,\dots,r\}$. Note that $s \geqq 1$ and perhaps $s = r$. For every $j \in \{1,\dots,s\}$ choose a Newton–Puiseux root of $f_j(x,y)$ of the form

$$\lambda_B(x) + c_j x^{h(B)} + \cdots . \tag{10}$$

Let N be the index of λ_B and write $h(B) = \frac{m}{nN}$ with m, n coprime.

FORMULA 3.6. *Keeping the above notations we have*

$$F_B(z) = C \prod_{j=1}^{s} \left(z^n - c_j^n\right)^{\frac{\operatorname{ord} f_j(0,y)}{nN}}$$

where C is a nonzero constant.

Proof. Fix $j \in \{1, \dots, s\}$ and a Newton–Puiseux root $\alpha(x)$ of $f_j(x,y)$ of the form (10). Since $f_j(x,y)$ is irreducible, the orbit $\mathbf{U}_D * \alpha$ is the set $\operatorname{Zer} f_j$. By Lemma 3.5 the stabilizer G_0 of $\alpha(x)$ has $D/(\sharp \operatorname{Zer} f_j) = D/\operatorname{ord} f_j(0,y)$ elements. Since every subgroup of a finite cyclic group is determined by the number of its elements, $G_0 = \mathbf{U}_{D/\operatorname{ord} f_j(0,y)}$.

Let us observe that $\theta * \alpha$ belongs to B if and only if $\theta * \lambda_B = \lambda_B$. By a similar argument as before, the stabilizer G_1 of λ_B is the subgroup $\mathbf{U}_{D/N}$ of $\mathbf{U}_D$. Hence $\operatorname{Zer} f_j \cap B = G_1 * \alpha$. By (ii) of Lemma 3.5 we get

$$\prod_{\theta \in G_1} \big(z - \operatorname{lc}_B(\theta * \alpha)\big) = \prod_{\alpha_i \in \operatorname{Zer} f_j \cap B} \big(z - \operatorname{lc}_B(\alpha_i)\big)^{\frac{D}{\operatorname{ord} f_j(0,y)}}. \tag{11}$$

On the other hand, following Lemma 3.4 we have

$$\prod_{\theta \in G_1} \big(z - \operatorname{lc}_B(\theta * \alpha)\big) = \prod_{\theta^{D/N}=1} (z - c_j\theta^{h(B)D}) = (z^n - c_j^n)^{D/nN}. \tag{12}$$

Comparing (11) and (12) we get

$$\prod_{\alpha_i \in \operatorname{Zer} f_j \cap B} \big(z - \operatorname{lc}_B(\alpha_i)\big) = (z^n - c_j^n)^{\operatorname{ord} f_j(0,y)/nN}.$$

Finally

$$F_B(z) = C \prod_{j=1}^{s} \prod_{\alpha_i \in \operatorname{Zer} f_j \cap B} \big(z - \operatorname{lc}_B(\alpha_i)\big) = C \prod_{j=1}^{s} (z^n - c_j^n)^{\operatorname{ord} f_j(0,y)/nN}. \quad \square$$

From now on up to the end of this section we fix $B \in T(f)$ and put $q(B) = \frac{L}{M}$ with L, M coprime.

Let $\frac{d}{dz} F_B(z) = C'(z - z_1) \cdots (z - z_l)$. Set $w_i = F(z_i)$ for $1 \leqq i \leqq l$ and let $I := \big\{ i \in \{1, \dots, l\} : w_i \neq 0 \big\}$. Keeping this notation we have:

Lemma 3.7. *The initial Newton polynomial of $\mathcal{D}_B(u,v)$ is*

$$\operatorname{in}_{\mathcal{N}} \mathcal{D}_B(u,v) = \prod_{i \in I} \big(v - w_i u^{q(B)}\big).$$

Proof. By Lemma 2.3 the initial Newton polynomial of $\mathcal{D}_B(u,v)$ is equal to $\prod_j \big(v - F_B(\operatorname{lc}_B \gamma_j) u^{q(B)}\big)$ where the product runs over j such that γ_j leaves $T(f)$ at B. It follows from Lemmas 2.5 and 2.6 that the above product equals $\prod_{i \in I} \big(v - w_i u^{q(B)}\big)$. $\square$

Proposition 3.8. *Let $f(x,y)=0$ be a reduced complex plane curve. Take a pseudo-ball B of $T(f)$ such that $q(B)=\frac{L}{M}$ with L, M coprime. Let N be the index of λ_B. Then*

$$\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\overline{B}}(u,v)=\prod_{i\in I}\left(v^M-w_i^M u^L\right)^{N/M}. \tag{13}$$

Proof. Recall that $\overline{B}$ is the orbit of B under the $*$ action of the group $\mathbf{U}_D$. Since $\theta * B=B$ if and only if $\theta*\lambda_B=\lambda_B$, the stabilizer of B is the subgroup $\mathbf{U}_{D/N}$ (see the proof of Formula 3.6).

We claim that under the assumptions of Lemma 3.7 one has

$$\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\theta*B}(u,v)=\prod_{i\in I}\left(v-w_i\theta^{q(B)D}u^{q(B)}\right).$$

Indeed, by Proposition 3.2 the critical values of $F_{\theta*B}$ are the critical values of F_B times $\theta^{q(B)D}$, which proves the claim.

By (ii) of Lemma 3.5 we have

$$\prod_{\theta\in\mathbf{U}_D}\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\theta*B}(u,v)=\prod_{B'\in\overline{B}}\operatorname{in}_{\mathcal{N}} \mathcal{D}_{B'}(u,v)^{D/N}=\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\overline{B}}(u,v)^{D/N}.$$

On the other hand, by the claim and Lemma 3.4 we have

$$\prod_{\theta\in\mathbf{U}_D}\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\theta*B}(u,v)=\prod_{\theta^D=1}\prod_{i\in I}\left(v-w_i\theta^{q(B)D}u^{q(B)}\right)=\prod_{i\in I}\left(v^M-w_i^M u^L\right)^{D/M}.$$

Comparing the above equalities we get the proposition. □

4. The irreducible case

We assume in this section that $f(x,y)\in\mathbf{C}\{x,y\}$ is irreducible. Let $p:=\operatorname{ord}_y f(0,y)>1$ and $\operatorname{Zer} f=\left\{\alpha_i(x)\right\}_{i=1}^{p}$. The contacts $\left\{O(\alpha_i,\alpha_j)\right\}_{i\neq j}$, called *the characteristic exponents of* $f(x,y)$, form a finite set of rational numbers $\left\{\frac{b_k}{p}\right\}_{k=1}^{h}$, where $b_1<\cdots<b_h$. Set $b_0=p$. The sequence $(b_0,b_1,\dots,b_h)$ is named *Puiseux characteristic.* Since $f(x,y)$ is irreducible, its Newton–Puiseux roots conjugate and all the pseudo-balls with the same height belong to the same conjugate class in $E(f)$. Write $E(f)=\left\{\overline{B}_1,\dots,\overline{B}_h\right\}$, where $h(B_k)=\frac{b_k}{p}$ for $k\in\{1,\dots,h\}$. By Lemma 2.7 $q(B_1)<q(B_2)<\cdots<q(B_h)$. The discriminant $\mathcal{D}(u,v)$ is the product $\prod_{k=1}^{h}\mathcal{D}_{\overline{B}_k}(u,v)$.

We now characterize the factors appearing in this product. Let $B\in T(f)$. By Formula 3.6, we have $F_B(z)=C(z^n-c^n)^{\frac{p}{nN}}$. This polynomial has only one nonzero critical value $w = F_B(0)$ of multiplicity $n-1$. By Proposition 3.8, we have

$$\mathrm{in}_{\mathcal{N}}\, \mathcal{D}_{\overline{B}}(u,v) = \left(v^M - w^M u^L\right)^{(n-1)N/M},$$

where $q(B) = \frac{L}{M}$, $\gcd(L,M) = 1$ and N is the index of λ_B. We stress that in the next corollary we only use the fact that $\mathrm{in}_{\mathcal{N}}\, \mathcal{D}_{\overline{B}}(u,v)$ is a power of a quasi-homogeneous irreducible polynomial.

Corollary 4.1. *The power series $\mathcal{D}_{\overline{B}_i}(u,v)$ is non-degenerate if and only if there are no lattice points inside the only compact edge of its Newton diagram.*

Theorem 1.1 is a consequence of Corollary 4.1 since the Newton diagram of $\mathcal{D}(u,v)$ is the sum of the elementary Newton diagrams of $\mathcal{D}_{\overline{B}_i}(u,v)$.

According to Merle [18] and Ephraim [3] the *semigroup* Γ (see for example [24] in the transversal case and [8] in the general case) of $f(x,y) = 0$ admits the minimal sequence of generators $\overline{b_0} := \operatorname{ord} f(0,y), \overline{b_1} < \cdots < \overline{b_h}$ and the Newton diagram of the discriminant $\mathcal{D}(u,v)$ is

$$\sum_{k=1}^{h} \left\{ \frac{(n_k-1)\overline{b_k}}{n_1 \cdots n_{k-1}(n_k-1)} \right\}, \tag{14}$$

where

$$n_k := \frac{\gcd(\overline{b_0}, \overline{b_1}, \dots, \overline{b_{k-1}})}{\gcd(\overline{b_0}, \overline{b_1}, \dots, \overline{b_k})} = \frac{\gcd(b_0, \dots, b_{k-1})}{\gcd(b_0, \dots, b_k)}$$

and by convention $n_0 = 1$. The inclinations of the edges of the Newton diagram (14) are $q(B_1), \dots, q(B_h)$. They are called *polar invariants* of the pair (x,f).

Since the Newton diagram of a product is the sum of the Newton diagrams of its factors and the sequence $\left(q(B_k)\right)$ is increasing, the Newton diagram of $\mathcal{D}_{\overline{B}_k}(u,v)$ is the k-th term of (14).

Corollary 4.2. *The power series $\mathcal{D}_{\overline{B}_k}(u,v)$ is non-degenerate if and only if* $(n_k - 1)\gcd(\overline{b_k}, n_1 \cdots n_{k-1}) = 1$.

Proof. Since the Newton diagram of $\mathcal{D}_{\overline{B}_k}(u,v)$ is

$$\left\{ \frac{(n_k-1)\overline{b_k}}{n_1 \cdots n_{k-1}(n_k-1)} \right\}$$

the statement follows from Corollary 4.1. $\square$

REMARK 4.3. Note that if for $k > 1$ the polar invariant $q(B_k)$ is an integer then $\left\{\frac{(n_k-1)\overline{b_k}}{n_1 \cdots n_{k-1}(n_k-1)}\right\}$ has lattice points inside its compact edge and $\mathcal{D}(u, v)$ is degenerate.

Observe that a necessary condition for $\mathcal{D}(u, v)$ to be non-degenerate is $n_1 = n_2 = \cdots = n_h = 2$, where h is the number of characteristic exponents of $f = 0$.

COROLLARY 4.4. *Let $f(x, y) = 0$ be a branch with h characteristic exponents. We have*

1. *If $h = 1$ then the discriminant $\mathcal{D}(u, v)$ is non-degenerate if and only if* $\operatorname{ord} f(0, y) = 2$.
2. *If $h = 2$ then the discriminant $\mathcal{D}(u, v)$ is non-degenerate if and only if* $\operatorname{ord} f(0, y) = 4$.
3. *If $h > 2$ then $\mathcal{D}(u, v)$ is degenerate.*

5. The general case

In this section we specify the polynomial factorization of $\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\overline{B}}(u, v)$. We start with four technical lemmas. Their sole purpose is to show that the factors of (16) and (17) in Proposition 5.6 are polynomials.

LEMMA 5.1. *Let $0 \leqq a \leqq b$ and let $f : [a, b] \to \mathbf{R}$ be a continuous function such that $f(x) \geqq 0$ for $a \leqq x \leqq b$. Let c be a positive integer. Then $\max_{x\in[a,b]} \frac{(m-x)^c}{m^c} f(x) \to \max_{x\in[a,b]} f(x)$ as $m \to \infty$.*

PROOF. Let x_0 be the point of the interval $[a, b]$ such that $f(x_0) = \max_{x\in[a,b]} f(x)$. We have

$$\frac{(m - x_0)^c}{m^c} f(x_0) \leqq \max_{x\in[a,b]} \frac{(m - x)^c}{m^c} f(x) \leqq \max_{x\in[a,b]} f(x)$$

for large m. Passing to the limits we get the lemma. $\square$

LEMMA 5.2. *Let $a_1, \ldots, a_n$ be positive integers. Then there exist pairwise different nonzero complex numbers $d_1, \ldots, d_n$ such that the polynomial*

$$H(t) = \prod_{j=1}^{n} (t - d_j)^{a_j}$$

has $n - 1$ pairwise different nonzero critical values, and all of them differ from $H(0)$.

Proof. It suffices to construct step-by-step a sequence $0 < d_1 < d_2 < \cdots < d_n$ such that the polynomials

$$W_k(t) = \prod_{j=1}^{k} (t - d_j)^{2a_j} \quad \text{for} \quad k \in \{1, \dots, n\}$$

satisfy conditions $W_k(0) < \max_{t\in[d_1,d_2]} W_k(t) < \cdots < \max_{t\in[d_{k-1},d_k]} W_k(t)$.

Assume that the numbers $0 < d_1 < \cdots < d_k$ and the polynomial $W_k(t)$ are already constructed. Applying Lemma 5.1 to every interval $[d_{j-1}, d_j]$ and to the interval $[0, 0]$ we conclude that for sufficiently large $m =: d_{k+1}$ the maximal values of the polynomial

$$\frac{1}{m^{2a_{k+1}}} W_{k+1}(t) = \frac{(m-t)^{2a_{k+1}}}{m^{2a_{k+1}}} W_k(t)$$

in the intervals $[0, 0]$, $[d_1, d_2]$, ..., $[d_{k-1}, d_k]$ form an increasing sequence and are bigger than $W_{k+1}(0)/m^{2a_{k+1}}$.

To assure that

$$\max_{t\in[0,d_k]} W_{k+1}(t) < \max_{t\in[d_k,d_{k+1}]} W_{k+1}(t)$$

it is enough to observe that in the sequence of inequalities

$$\max_{t\in[0,d_k]} W_{k+1}(t) \leqq m^{2a_{k+1}} \max_{t\in[0,d_k]} W_k(t) < \left(\frac{m-d_k}{2}\right)^{\deg W_{k+1}(t)}$$

$$\leqq W_{k+1}\left(\frac{m+d_k}{2}\right) \leqq \max_{t\in[d_k,d_{k+1}]} W_{k+1}(t),$$

the second inequality holds for all m big enough. Finally taking $H(t) := \prod_{j=1}^{n} (t-d_j)^{a_j}$ we see that the nonzero critical values of $W_n(t)$ are the squares of the nonzero critical values of $H(t)$ and we prove the lemma. $\square$

Corollary 5.3. *Let $H(t)$ be a complex polynomial of the form*

$$H(t) = t^{a_0} \prod_{j=1}^{n} (t - d_j)^{a_j}, \tag{15}$$

where a_j are positive integers for $j \in \{0, 1, \dots, n\}$. Then for some $d_1, \dots, d_n$ the polynomial $H(t)$ has n pairwise different nonzero critical values.

Proof. By Lemma 5.2 we can choose a sequence $e_0, e_1, \dots, e_n$ such that the polynomial $H_1(t) = \prod_{j=0}^{n} (t-e_j)^{a_j}$ has n pairwise different nonzero critical values. We finish by putting $H(t) = H_1(t + e_0)$ and $d_j = e_j - e_0$ for $j \in \{1, \dots, n\}$. $\square$

In the next lemma we change the notation slightly. Notice that the polynomial $F_B(z)$ and the exponent $q(B)$ in Lemma 2.3 depend not only on B but also on the power series $f(x,y)$. We write $F_{B,f}(z)$ for the polynomial and $q(B,f)$ for the exponent to stress this dependence.

LEMMA 5.4. *Let $f(x,y)$ be a reduced power series such that $f(0,y)\neq 0$. Fix $B\in T(f)$. Let N be the index of λ_B and write $h(B)=\frac{m}{nN}$ with m, n coprime. Assume that $F_{B,f}(z)=Cz^{a_0}\prod_{j=1}^{s}(z^n-d_j)^{a_j}$, where d_j are pairwise different nonzero complex numbers, a_0 is a nonnegative integer and a_j are positive integers for $j\in\{1,\dots,s\}$.*

Then for every sequence of pairwise different nonzero complex numbers $\tilde d_1,\dots,\tilde d_s$ there exists a reduced power series $\tilde f(x,y)$ such that $B\in T(\tilde f)$, $q(B,\tilde f)=q(B,f)$ and $F_{B,\tilde f}(z)=Cz^{a_0}\prod_{j=1}^{s}(z^n-\tilde d_j)^{a_j}$.

PROOF. Let $f=f_1\cdots f_r$ be the decomposition of $f(x,y)$ into irreducible factors. Without loss of generality we may assume that $\operatorname{Zer} f_i\cap B\neq\emptyset$ for $i\in\{1,\dots,k\}$ and $\operatorname{Zer} f_i\cap B=\emptyset$ for $i\in\{k+1,\dots,r\}$. For every $i\in\{1,\dots,k\}$ choose a Newton–Puiseux root of f_i of the form $\alpha_i(x)=\lambda_B(x)+c_ix^{h(B)}+\cdots$. Let $\mathcal{C}=\left\{c_i^n : i\in\{1,\dots,k\}\right\}$. Then it follows from Formula 3.6 that $\mathcal{C}\setminus\{0\}=\{d_1,\dots,d_s\}$, $a_0=\frac{1}{N}\sum_{i:\,c_i=0}\operatorname{ord} f_i(0,y)$ and $a_j=\frac{1}{nN}\sum_{i:\,c_i^n=d_j}\operatorname{ord} f_i(0,y)$ for $j=1,\dots,s$.

For every $i\in\{1,\dots,k\}$ take the fractional power series

$$\tilde\alpha_i(x)=\alpha_i(x)+(\tilde c_i-c_i)x^{h(B)}=\lambda_B(x)+\tilde c_ix^{h(B)}+\cdots$$

where $\tilde c_i=0$ if $c_i=0$ and $\tilde c_i^n=\tilde d_j$ if $c_i^n=d_j$. Set $\tilde f=a\tilde f_1\cdots\tilde f_kf_{k+1}\cdots f_r$, where $\tilde f_i(x,y)$ are irreducible power series such that $\tilde\alpha_i\in\operatorname{Zer}\tilde f_i$ for $i\in\{1,\dots,k\}$ and a is a constant which will be specified later. Clearly B is an element of $T(\tilde f)$.

Now let us compute $F_{B,\tilde f}$. One has $\operatorname{ord} f_i(0,y)=\operatorname{ord}\tilde f_i(0,y)$ for $i=1,\dots,k$ since $\alpha_i(x)$ and $\tilde\alpha_i(x)$ have the same index. By the first part of the proof it is clear that $F_{B,\tilde f}(z)=\tilde Cz^{a_0}\prod_{j=1}^{s}(z^n-\tilde d_j)^{a_j}$. By a suitable choice of the complex number a we get $\tilde C=C$.

It remains to prove that $q(B,f)=q(B,\tilde f)$. Let $\gamma(x)=\lambda_B(x)+cx^{h(B)}$ where c is a generic constant. Then

$$q(B,f)=\operatorname{ord} f\big(x,\gamma(x)\big)=\sum_{i=1}^{r}\operatorname{ord} f_i\big(x,\gamma(x)\big)$$

and an analogous formula holds for $q(B,\tilde f)$.

Fix $i\in\{1,\dots,k\}$. For generic c we have

$$\operatorname{cont}(\gamma,\operatorname{Zer} f_i)=\operatorname{cont}(\gamma,\operatorname{Zer}\tilde f_i)=h(B).$$

Since the Puiseux characteristics of both irreducible power series are the same, we get $\operatorname{ord} f_i\big(x,\gamma(x)\big) = \operatorname{ord} \tilde{f}_i\big(x,\gamma(x)\big)$ (see for example [18], Proposition 2.4 for the transverse case and [9], Proposition 3.3 for the general case). $\square$

REMARK. One can show that the power series $\tilde{f}(x,y)$ constructed in the proof of Lemma 5.4 has the same equisingularity type as $f(x,y)$.

We introduce a new polynomial $H_B(t)$ associated with $B \in T(f)$ whose critical values provide a polynomial factorization of $\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\overline{B}}(u,v)$.

LEMMA 5.5. *Fix $B \in T(f)$. Let N be the index of λ_B. Write $h(B) = \frac{m}{nN}$ and $q(B) = \frac{L}{M}$ where* $\gcd(m,n) = \gcd(L,M) = 1$. *Then there exists a unique polynomial $H_B(t)$ such that $H_B(z^n) = F_B(z)^M$.*

PROOF. Assume as earlier that all Newton–Puiseux roots of $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ belong to $\mathbf{C}\big\{x^{1/D}\big\}$ for some positive integer D. We use the properties of the conjugate action introduced in Section 3. One easily checks that $\theta * B = B$ for $\theta \in \mathbf{U}_{D/N}$ (see the proof of Proposition 3.8). Set $D = D_0 nN$ and take $\theta \in \mathbf{U}_{D/N}$ such that $\omega := \theta^{D_0}$ is an n-th primitive root of unity. By Proposition 3.2 we get $\theta^{q(B)D} F_B(z) = F_B(\theta^{h(B)D} z)$. Hence $F_B(z)^M = F_B(\omega^m z)^M$. Comparing the terms of both sides we see that all monomials appearing in the polynomial $F_B(z)^M$ are powers of z^n. $\square$

PROPOSITION 5.6. *Let $f(x,y) = 0$ be a reduced curve. Fix $B \in T(f)$. Let N be the index of λ_B. Write $h(B) = \frac{m}{nN}$ and $q(B) = \frac{L}{M}$ where* $\gcd(m,n) = \gcd(L,M) = 1$. *Let $H'_B(t) = C(t-t_1)\cdots(t-t_r)$. Set $\mathsf{w}_0 = H_B(0)$, $\mathsf{w}_j = H_B(t_j)$ and $J = \big\{ j \in \{1,\dots,r\} : \mathsf{w}_j \neq 0 \big\}$. Then*

$$\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\overline{B}}(u,v) = \big(v^M - \mathsf{w}_0 u^L\big)^{(n-1)N/M} \prod_{j\in J} \big(v^M - \mathsf{w}_j u^L\big)^{nN/M} \quad \text{if } \mathsf{w}_0 \neq 0, \tag{16}$$

$$\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\overline{B}}(u,v) = \prod_{j\in J} \big(v^M - \mathsf{w}_j u^L\big)^{nN/M} \quad \text{if } \mathsf{w}_0 = 0. \tag{17}$$

Moreover (16) *and* (17) *give a polynomial factorization of* $\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\overline{B}}(u,v)$.

PROOF. The above formulas follow from Proposition 3.8 and the equality $M F_B(z)^{M-1} F'_B(z) = n z^{n-1} H'_B(z^n)$ which allows to express critical values of F_B in terms of critical values of H_B.

Using Lemma 5.4 we can replace $f(x,y)$ by such a power series $\tilde{f}(x,y)$ that conclusions of Lemma 5.2 or Corollary 5.3, for $H(t) = H_B(t)$, are satisfied. Then $\{\mathsf{w}_j\}_{j\in J\cup\{0\}}$ is a sequence of pairwise different complex numbers.

The polynomials $v^M - \mathsf{w}_j u^L$ are irreducible and pairwise coprime. Hence the exponents $(n-1)N/M$, nN/M in (16) or nN/M in (17) are integers. $\square$

THEOREM 5.7. *Let $f(x,y)=0$ be a reduced curve and let $B \in T(f)$. Let N be the index of λ_B. Write $h(B)=\frac{m}{nN}$ and $q(B)=\frac{L}{M}$ where* $\gcd(m,n)=\gcd(L,M)=1$.

1. *If $H_B(t)$ has only one root (possibly multiple), then $\mathcal{D}_{\overline{B}}(u,v)$ is non-degenerate if and only if $(n-1)N=M$.*
2. *Otherwise $\mathcal{D}_{\overline{B}}(u,v)$ is non-degenerate if and only if $nN=M$ and all nonzero critical values of $H_B(t)$ are simple.*

PROOF. Assume that $H_B(t)$ has only one root. By Proposition 5.6 $\mathrm{in}_{\mathcal{N}}\, \mathcal{D}_{\overline{B}}(u,v) = \left(v^M - \mathsf{w}_0 u^L\right)^{(n-1)N/M}$. This polynomial is non-degenerate if and only if $(n-1)N=M$.

Suppose that $H_B(t)$ has at least two different roots. Assume that $\mathsf{w}_0 = 0$. Then (17) is a reduced polynomial if and only if $nN=M$ and all nonzero critical values of $H_B(t)$ are simple. Assume now that $\mathsf{w}_0 \neq 0$. Then the polynomial (16) is reduced if and only if $nN/M=1$ and $(\mathsf{w}_j)_{j\in J}$ is a sequence of pairwise different complex numbers. In this case the only difficulty arrives from the term $\left(v^M - \mathsf{w}_0 u^L\right)^{(n-1)N/M}$ but the exponents nN/M and $(n-1)N/M$ are integers, so the condition $nN/M=1$ implies $(n-1)N/M=0$. $\square$

We finish this section with another example of a multibranched curve $f=0$ such that the discriminant of the morphism (x,f) is non-degenerate. For the construction we use the *Eggers tree* whose construction we now recall. We assume that $x=0$ and $f=0$ are transverse. Recall that $E(f)$ is the set of all conjugate classes of B for $B \in T(f)$. An element of $E(f)$ is uniquely determined by its height $h(\overline{B}) := h(B)$ and the set of irreducible factors f_i of f such that $\operatorname{Zer} f_i \cap B \neq \emptyset$ (see [14], Section 6). The tree structure on $T(f)$ induces a tree structure on $E(f) \cup \{f_0,\dots,f_k\}$. This newly constructed tree is called the Eggers tree of f ([2], see also [4]). In Eggers' terminology the vertices from $E(f)$ are called *black points* and those from $\{f_0,\dots,f_k\}$ are called *white points.* The Eggers tree is an oriented tree where the root is the black point of the minimal height and the leaves are the white points. The *outdegree* of a vertex Q is the number of edges joining Q with its successors.

REMARK 5.8. The first part in Theorem 5.7 corresponds to simple points (i.e. vertices of outdegree 1) in the Eggers tree. The second part corresponds to bifurcation points (vertices of outdegree greater than 1) in the Eggers tree. The number of irreducible factors of $H_B(t)$ is equal to the outdegree of the vertex $\overline{B}$.

EXAMPLE 5.9. Set $n_0 = 1$ and let $n_1, \dots, n_k$ be pairwise coprime integers bigger than 1. We construct a singular power series $f = f_0 f_1 \cdots f_k$, where f_i are irreducible power series, $\operatorname{ord} f_i(0, y) = n_0 \cdots n_i$ for $i \in \{0, \dots, k\}$, and such that the discriminant of the morphism (x, f) is non-degenerate.

Let $h_i = 1 + \frac{1}{n_1} + \cdots + \frac{1}{n_i}$ for $i \in \{1, \dots, k\}$. We claim that h_i can be written as $\frac{b_i}{n_1 \cdots n_i}$, with b_i and $n_1 \cdots n_i$ coprime. The proof runs by induction on i. For $i = 1$ we have $h_1 = \frac{n_1 + 1}{n_1}$. Assume that $\gcd(b_i, n_1 \cdots n_i) = 1$. By the equality $\frac{b_{i+1}}{n_1 \cdots n_{i+1}} = \frac{b_i}{n_1 \cdots n_i} + \frac{1}{n_{i+1}}$ we get $b_{i+1} = b_i n_{i+1} + n_1 \cdots n_i$. Thus

$$\gcd(b_{i+1}, n_{i+1}) = \gcd(n_1 \cdots n_i, n_{i+1}) = 1,$$

$$\gcd(b_{i+1}, n_1 \cdots n_i) = \gcd(b_i n_{i+1}, n_1 \cdots n_i) = 1$$

and consequently we get $\gcd(b_{i+1}, n_1 \cdots n_{i+1}) = 1$.

Let

$$\alpha_0(x) = 0,$$

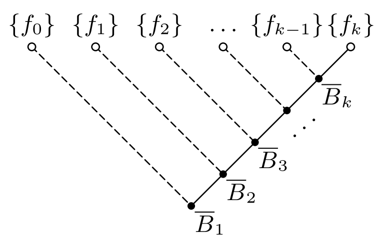
$$\alpha_1(x) = x^{h_1},$$

$$\alpha_2(x) = x^{h_1} + x^{h_2},$$

$$\vdots$$

$$\alpha_k(x) = x^{h_1} + x^{h_2} + \cdots + x^{h_k}.$$

We consider $f = f_0 f_1 \cdots f_k$ where f_i are irreducible power series such that $\alpha_i \in \operatorname{Zer} f_i$. By Property 3.1 the order of $f_i(0, y)$ is $n_0 \cdots n_i$ for $i \in \{0, \dots, k\}$. Let $B_i = B(\alpha_{i-1}, h_i)$ for $i \in \{1, \dots, k\}$. Then $E(f) = \{\overline{B}_1, \dots, \overline{B}_k\}$. The Eggers tree of f is drawn below.



Since $\lambda_{B_i}(x) = \alpha_{i-1}(x)$ we have, with the notations of Formula 3.6, $N = n_0 \cdots n_{i-1}$ and $n = n_i$. Hence

$$F_{B_i}(z) = C(z^n - 0)^{\frac{\operatorname{ord} f_{i-1}(0,y)}{nN}} \prod_{j=i}^{k} (z^n - 1)^{\frac{\operatorname{ord} f_j(0,y)}{nN}} = Cz(z^{n_i} - 1)^{A_i}, \tag{18}$$

where A_i is a positive integer.

Now we show that $q(B_i)$ could be written as $\frac{L_i}{M_i} = \frac{L_i}{nN}$ with L_i and nN coprime. Since $h(B_i) = \frac{b_i}{nN}$ with b_i and nN coprime, it is enough to prove by induction on i that for $i \in \{1, \dots, k\}$ the difference $q(B_i) - h(B_i)$ is an integer. By Lemma 2.7 and (18) we get

$$q(B_1) = \sharp(\operatorname{Zer} f) h(B_1) = \deg F_{B_1}(z) h(B_1) = (1 + n_1 A_1) h(B_1).$$

Hence $q(B_1) - h(B_1) = b_1 A_1$. Now, again by (18) and Lemma 2.7 we get

$$q(B_{i+1}) - q(B_i) = (1 + n_{i+1} A_{i+1}) \frac{1}{n_{i+1}} = \frac{1}{n_{i+1}} + A_{i+1}.$$

Thus by the inductive hypothesis

$$q(B_{i+1}) - h(B_{i+1}) = q(B_i) + \frac{1}{n_{i+1}} + A_{i+1} - h(B_{i+1}) = q(B_i) - h(B_i) + A_{i+1}$$

is an integer.

The only roots of $H_{B_i}(t)$ are 0 and 1. Therefore this polynomial has a unique nonzero critical value w_i. By equality $nN = M_i$ and Proposition 5.6 we get $\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\overline{B}_i}(u, v) = v^{L_i} - \mathsf{w}_i u^{M_i}$.

The polynomials $\operatorname{in}_{\mathcal{N}} \mathcal{D}_{\overline{B}_i}(u, v)$, for $1 \leqq i \leqq k$, are irreducible and pairwise coprime. Hence the discriminant $\mathcal{D}(u, v) = \mathcal{D}_{\overline{B}_1}(u, v) \cdots \mathcal{D}_{\overline{B}_k}(u, v)$ of the morphism (x, f) is non-degenerate.

6. Stability of the discriminant's initial Newton polynomial

To simplify subsequent statements we say that the power series $H_1(u, v)$, $H_2(u, v)$ are *equal up to rescaling variables* if there exist nonzero constants A, B, C such that $H_1(u, v) = C H_2(Au, Bv)$. The Kouchnirenko non-degeneracy of a power series in two variables does not depend on rescaling variables.

LEMMA 6.1. *Let $\mathcal{D}(u, v)$ be the discriminant of the morphism (f, g). Then for any nonzero constants A, B the discriminant curve of the morphism (Af, Bg) admits the equation $\mathcal{D}(u/A, v/B) = 0$.*

PROOF. Let $u = Au'$, $v = Bv'$. As $(u, v) = \big(Af(x, y), Bg(x, y)\big)$ then $(u', v') = \big(f(x, y), g(x, y)\big)$. Hence, the discriminant curve of the morphism (Af, Bg) admits the equation $\mathcal{D}(u', v') = 0$ which gives the lemma. □

THEOREM 6.2. *Let $f = 0$ be a reduced singular curve and let $\ell = 0$ be a smooth curve which is not a branch of $f = 0$. Then for every invertible power series $u_1(x, y) \in \mathbf{C}\{x, y\}$ the initial Newton polynomials of the discriminants of (ℓ, f) and $(\ell, u_1 f)$ are equal up to rescaling variables.*

PROOF. An analytic change of coordinates does not affect the equation of the discriminant. Hence, we may assume that $\ell(x,y)=x$. By Lemma 6.1 we may also assume that $u_1(0,0)=1$. Since f and $u_1 f$ have the same Newton–Puiseux roots, their tree models coincide. Let $B\in T(f)$. Applying Lemma 2.3 to f and $u_1 f$ we show that $F_{B,f}(z)=F_{B,u_1 f}(z)$ and $q(B,f)=q(B,u_1 f)$. By Lemma 3.7 the initial Newton polynomial of the discriminant depends only on $F_B(z)$ and $q(B)$ for pseudo-balls B from the tree model. This proves Theorem 6.2. $\square$

In what follows we need a few auxiliary results about fractional power series.

Consider the fractional power series $\phi(x)=x+\cdots=x\bigl(1+\phi_1(x)\bigr)$. We define the formal root

$$\phi(x)^{1/n}:=x^{1/n}\sqrt[n]{1+\phi_1(x)},\quad \text{where}\quad \sqrt[n]{1+z}:=1+\frac{1}{n}z+\cdots$$

is an analytic branch of the n-th complex root of $1+z$. Then, having a power series $\psi(x)=\overline{\psi}(x^{1/n})$, where $\overline{\psi}(t)$ is a convergent power series, we define the formal substitution $\psi\bigl(\phi(x)\bigr)$ as the fractional power series $\overline{\psi}\bigl(\phi(x)^{1/n}\bigr)$.

LEMMA 6.3. *Let*

$$\alpha_i(x)=x+\sum_{k=n+1}^{N-1}a_k x^{k/n}+c_i x^{N/n}+\cdots$$

$$\beta_i(y)=y+\sum_{k=n+1}^{N-1}b_k y^{k/n}+d_i y^{N/n}+\cdots$$

for $i=1,2$. *If* $\beta_1\bigl(\alpha_1(x)\bigr)=\beta_2\bigl(\alpha_2(x)\bigr)$ *then* $c_1-c_2=d_2-d_1$.

PROOF. Write $\lambda(y)=\sum_{k=n+1}^{N-1}b_k y^{k/n}$. Then

$$0=\beta_1\bigl(\alpha_1(x)\bigr)-\beta_2\bigl(\alpha_2(x)\bigr)$$

$$=\bigl[\alpha_1(x)-\alpha_2(x)\bigr]+\bigl[\lambda\bigl(\alpha_1(x)\bigr)-\lambda\bigl(\alpha_2(x)\bigr)\bigr]$$

$$+\Bigl[d_1\bigl(\alpha_1(x)\bigr)^{N/n}-d_2\bigl(\alpha_2(x)\bigr)^{N/n}\Bigr]+\cdots$$

$$=\bigl[(c_1-c_2)x^{N/n}+\cdots\bigr]+\bigl[(d_1-d_2)x^{N/n}+\cdots\bigr]+\bigl[\lambda\bigl(\alpha_1(x)\bigr)-\lambda\bigl(\alpha_2(x)\bigr)\bigr]$$

$$=\bigl[(c_1-c_2+d_1-d_2)x^{N/n}+\cdots\bigr]+\bigl[\lambda\bigl(\alpha_1(x)\bigr)-\lambda\bigl(\alpha_2(x)\bigr)\bigr].$$

To finish the proof it suffices to show that the fractional power series $\lambda\big(\alpha_1(x)\big) - \lambda\big(\alpha_2(x)\big)$ does not contain the term of order N/n. This task reduces to

Claim. *For every $k > n$ the order of $\big(\alpha_1(x)\big)^{k/n} - \big(\alpha_2(x)\big)^{k/n}$ is bigger than N/n.*

Proof. For every convergent power series $g(z) \in \mathbf{C}\{z\}$ there exists $G(z,w) \in \mathbf{C}\{z,w\}$ such that $g(z) - g(w) = (z-w)G(z,w)$.

Let $\alpha_i(x) = x\big(1 + \tilde{\alpha}_i(x)\big)$ for $i = 1, 2$. Using the above equality for $g(z) = \sqrt[n]{1+z}$ we get

$$\big(\alpha_1(x)\big)^{k/n} - \big(\alpha_2(x)\big)^{k/n} = x^{k/n}\Big(\big(\sqrt[n]{1+\tilde{\alpha}_1(x)}\big)^k - \big(\sqrt[n]{1+\tilde{\alpha}_2(x)}\big)^k\Big)$$

$$= x^{k/n}\big(\tilde{\alpha}_1(x) - \tilde{\alpha}_2(x)\big)\, G\big(\tilde{\alpha}_1(x), \tilde{\alpha}_2(x)\big)$$

$$= x^{(k-n)/n}\big(\alpha_1(x) - \alpha_2(x)\big)\, G\big(\tilde{\alpha}_1(x), \tilde{\alpha}_2(x)\big)$$

which proves the Claim. $\square$

Lemma 6.4. *Let $f(x,y) = (y-x)^n + \cdots$ be an irreducible complex power series. Then for every Newton–Puiseux root $y = \alpha(x)$ of $f(x,y)$ there exists a Newton–Puiseux root $x = \beta(y)$ of $f(x,y)$ such that $\beta\big(\alpha(x)\big) = x$.*

Proof. Fix a Newton–Puiseux root $y = \alpha(x)$ of $f(x,y)$. Let $\beta_1(y), \dots, \beta_n(y)$ be the solutions of $f(x,y) = 0$ in $\mathbf{C}\{y\}^*$. Then there exists a unit $v(x,y) \in \mathbf{C}\{x,y\}$ such that $f(x,y) = v(x,y)\prod_{j=1}^n \big(x - \beta_j(y)\big)$. By Property 3.1 the index of every $\beta_j(y)$ is n and we can write $\beta_j(y) = \bar{\beta}_j(y^{1/n})$. Substituting $y := s^n$ we get $f(x,s^n) = v(x,s^n)\prod_{j=1}^n \big(x - \bar{\beta}_j(s)\big)$. By putting $s := \alpha(x)^{1/n}$ we obtain

$$0 = f\big(x, \alpha(x)\big) = v\big(x, \alpha(x)\big) \prod_{j=1}^{n} \big(x - \bar{\beta}_j\big(\alpha(x)^{1/n}\big)\big)$$

and the lemma follows. $\square$

Remark 6.5. By Lemma 6.4 for every fractional power series $y = \alpha(x) = x + \cdots$ there exists a fractional power series $x = \beta(y)$ such that $\beta\big(\alpha(x)\big) = x$. We call $x = \beta(y)$ a *formal inverse* of $y = \alpha(x)$. By Lemma 6.3 a formal inverse is unique. One can also show that if $x = \beta(y)$ is the formal inverse of $y = \alpha(x)$ then $y = \alpha(x)$ is the formal inverse of $x = \beta(y)$.

Theorem 6.6. *Let $f = 0$ be a unitangent reduced singular curve and let $\ell_1 = 0$, $\ell_2 = 0$ be smooth curves transverse to $f = 0$. Then the initial Newton* *polynomials of the discriminants of morphisms* (ℓ_1, f), (ℓ_2, f) *are equal up to rescaling variables.*

PROOF. Assume that the curves $\ell_1 = 0$, $\ell_2 = 0$ are transverse. Then there exists a system of local analytic coordinates $(\tilde{x}, \tilde{y})$ such that $\ell_1 = \tilde{x}$ and $\ell_2 = \tilde{y}$. By assumption the curve $f = 0$ has only one tangent $\tilde{y} = c\tilde{x}$, where $c \neq 0$. In the new coordinates $(x, y) = (c\tilde{x}, \tilde{y})$ this tangent becomes $y = x$.

Let $g(x, y)$ be the Weierstrass polynomial of $f(x, y)$ and $g'(x, y)$ be the Weierstrass polynomial of $f(-y, x)$. Then by Lemma 6.1 and Theorem 6.2 the initial Newton polynomials of the discriminants of the morphisms (ℓ_1, f) and (x, g) are equal up to rescaling variables. The same applies to the morphisms (ℓ_2, f) and (x, g').

Write $\operatorname{Zer} g = \{\, \alpha_1(x), \dots, \alpha_p(x) \,\}$. Let $\beta_i(y)$ be the formal inverse of $\alpha_i(x)$ for $i = 1, \dots, p$. It follows from Lemma 6.4 that $\alpha'_i(x) = -\beta_i(x)$ are Newton-Puiseux roots of $g'(x, y)$ for $i = 1, \dots, p$. By Lemma 6.3 $\operatorname{in}\big(\alpha_i(x) - \alpha_j(x)\big) = \operatorname{in}\big(\alpha'_i(x) - \alpha'_j(x)\big)$ for $1 \leqq i < j \leqq p$. We get $\operatorname{Zer} g' = \{\, \alpha'_1(x), \dots, \alpha'_p(x) \,\}$.

The mapping $B\big(\alpha_i, O(\alpha_i, \alpha_j)\big) \mapsto B\big(\alpha'_i, O(\alpha'_i, \alpha'_j)\big)$ gives a one-to-one correspondence between pseudo-balls of the tree models $T(g)$ and $T(g')$. Moreover, for every $B \in T(g)$ and the corresponding $B' \in T(g')$ there exists a constant a_B such that $\operatorname{lc}_{B'}(\alpha'_i) = \operatorname{lc}_B(\alpha_i) + a_B$ for $\alpha_i \in B$, $\alpha'_i \in B'$.

By Remark 2.4, the leading coefficients of

$$F_{B,g}(z) = C \prod_{i\colon \alpha_i \in B} \big(z - \operatorname{lc}_B(\alpha_i)\big)$$

and

$$F_{B',g'}(z) = C' \prod_{i\colon \alpha'_i \in B'} \big(z - \operatorname{lc}_{B'}(\alpha'_i)\big)$$

are given respectively by

$$C x^{q(B,g)} = \prod_{i\colon \alpha_i \notin B} \operatorname{in}\big(\alpha_j(x) - \alpha_i(x)\big) \prod_{i\colon \alpha_i \in B} x^{h(B)}$$

and

$$C' x^{q(B',g')} = \prod_{i\colon \alpha'_i \notin B'} \operatorname{in}\big(\alpha'_j(x) - \alpha'_i(x)\big) \prod_{i\colon \alpha'_i \in B'} x^{h(B')},$$

where α_j is a fixed element of B. Hence $C = C'$, $q(B, g) = q(B', g')$ and $F_{B,g}(z) = F_{B',g'}(z + a_B)$. By Lemma 3.7 the initial Newton polynomial of the discriminant depends only on the critical values of $F_B(z)$ and on $q(B)$ for B from the tree model. This proves Theorem 6.6 in transverse case.

To prove Theorem 6.6 in the case when $\ell_1 = 0$ and $\ell_2 = 0$ are tangent it is enough to take a smooth curve $\ell_3 = 0$ which is transverse to $\ell_1\ell_2 f = 0$ and apply the previously proved part to pairs of morphisms (ℓ_1, f), (ℓ_3, f) and (ℓ_3, f), (ℓ_2, f). $\square$

EXAMPLE 6.7. Let $f = \left(y^2 - x^2\right)^2 + 2x^4$. The discriminant of (x, f) is degenerate while the discriminant of $(x + y, f)$ is non-degenerate. The second discriminant can be easily computed after the change of variables $x = x' - y'$, $y = y'$.

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