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Quasi-Ordinary Singularities: Tree Model, Discriminant, and Irreducibility

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Let $f(Y) \in \mathbf{K}[[X_1, \ldots, X_d]][Y]$ be a quasi-ordinary Weierstrass polynomial with coefficients in the ring of formal power series over an algebraically closed field of characteristic zero. In this paper, we study the discriminant D_f of $f(Y) - V$, where V is a new variable. We show that the Newton polytope of D_f depends only on contacts between the roots of $f(Y)$. Then, we prove that $f(Y)$ is irreducible if and only if the Newton polytope of D_f satisfies some arithmetic conditions. Finally, we generalize these results to quasi-ordinary power series.

1 Introduction

Classically, the irreducibility of singular plane curves was studied by resolving the singularity or using approximate roots (Abhyankar criterion). More recently, in [10, 11] we use discriminants and the so-called *Jacobian Newton polygon* introduced by Teissier in [27]. In [2], Assi gives an irreducibility criterion for *quasi-ordinary polynomials* that generalizes the approach of Abhyankar for plane curves. In [14], González Villa characterizes

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the irreducible quasi-ordinary polynomials in terms of its *Newton process* (a way to encode the resolution). Previously, in [9, Theorem 3] the authors proved that if a power series is irreducible and has a *polygonal Newton polytope* (the maximal dimension of its compact faces equals one), then it has only one compact edge, which generalizes the case of plane curve germs.

In this note, we study the irreducibility of a quasi-ordinary Weierstrass polynomial $f(Y) \in \mathbf{K}[[\underline{X}]][Y]$ from the point of view [10, 11]. We consider the Newton polytope $\Delta(D_f)$ of the discriminant $D_f(\underline{X}, V) = \mathrm{Discr}_Y(f(Y) - V)$, where V is a new variable.

The main result of the article is Theorem 7.1 which states that if $p(Y)$, $f(Y)$ are quasi-ordinary Weierstrass polynomials such that $\Delta(D_p) = \Delta(D_f)$ and f is irreducible then p is also irreducible.

Our tool is the *tree model* associated with a quasi-ordinary polynomial, also called *Kuo-Lu* tree. This combinatorial object is a natural generalization of a tree introduced in [18]. The tree model $T(f)$ of a polynomial $f(Y)$ depends only on contacts between the roots of $f(Y)$.

In Theorem 4.1, we give an explicit formula expressing the Newton diagram $\Delta(D_f)$ by $T(f)$. Then, after some preparatory work, we characterize in Theorems 6.2 and 6.3 the tree models of irreducible quasi-ordinary Weierstrass polynomials. These are tree models with the highest possible level of symmetry.

The proof of Theorem 7.1 is based on above results and its idea is to show that if $\Delta(D_p) = \Delta(D_f)$ and the tree model $T(f)$ has a high level of symmetry, then $T(p)$ has the same structure as $T(f)$.

A consequence of the main result is Theorem 8.1 which presents an arithmetical test of irreducibility for quasi-ordinary Weierstrass polynomials. As an illustration we apply this test to three examples of quasi-ordinary polynomials from [2].

Finally in Section 9, we generalize the notion of the discriminant $D_f(\underline{X}, V)$, which was previously defined for quasi-ordinary Weierstrass polynomials, to *Y-regular quasi-ordinary power series* and we generalize the criterion of irreducibility to such power series.

2 Quasi-Ordinary Weierstrass Polynomials

While the term *quasi-ordinary* appears in the 60s with Zariski paper [28] and Lipman thesis [22], the study of these objects goes back at least to the paper [16] of Jung. In this section, we recall the notion of *quasi-ordinary Weierstrass polynomials* and some results that will be useful in the development of this note.

Let $\mathbf{K}$ be an algebraically closed field of characteristic zero and let

$$f(Y) = Y^n + a_1(X_1, \ldots, X_d)Y^{n-1} + \cdots + a_n(X_1, \ldots, X_d) \in \mathbf{K}[[\underline{X}]][Y] \tag{1}$$

be a unitary polynomial with coefficients in the ring of formal power series in $\underline{X} = (X_1, \ldots, X_d)$. Such a polynomial is called *quasi-ordinary* if its Y-discriminant equals $X_1^{\alpha_1} \cdots X_d^{\alpha_d} u(\underline{X})$, where $\alpha_i \in \mathbf{N}$ and $u(\underline{X})$ is a unity in $\mathbf{K}[[\underline{X}]]$, that is $u(0) \neq 0$. We call $f(Y)$ a *Weierstrass polynomial* if $a_i(0) = 0$ for all $i = 1, \ldots, n$.

Theorem 2.1 (Abhyankar–Jung Theorem [1, 16, 24]). Let $f(Y) \in \mathbf{K}[[\underline{X}]][Y]$ be a quasi-ordinary Weierstrass polynomial. Then there is $k \in \mathbf{N} \setminus \{0\}$ such that $f(Y)$ has its roots in $\mathbf{K}[[X_1^{\frac{1}{k}}, \ldots, X_d^{\frac{1}{k}}]]$. □

For every d-tuple $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{Q}^d_{\geq 0}$ denote $\underline{X}^\alpha = X_1^{\alpha_1} \cdots X_d^{\alpha_d}$. Let $\operatorname{Zer} f = \{Y_1(\underline{X}), \ldots, Y_n(\underline{X})\}$ be the set of roots of $f(Y)$ in $\mathbf{K}[[X_1^{\frac{1}{k}}, \ldots, X_d^{\frac{1}{k}}]]$. As the differences of roots divide the discriminant, we have for $i \neq j$

$$Y_{ij}(\underline{X}) := Y_i(\underline{X}) - Y_j(\underline{X}) = \underline{X}^{\lambda_{ij}} u_{ij}(\underline{X}), \quad \text{for some } \lambda_{ij} \in (1/k)\mathbf{N}^d,\ u_{ij}(0) \neq 0.$$

In the next we will write Y_j instead of $Y_j(\underline{X})$ and Y_{ij} instead of $Y_{ij}(\underline{X})$. We call $O(Y_i, Y_j) := \lambda_{ij}$ *the contact between* Y_i *and* Y_j. We put $O(Y_i, Y_i) = +\infty$.

Let us introduce a partial order in $\mathbf{Q}^d$: $(\alpha_1, \ldots, \alpha_d) \leq (\beta_1, \ldots, \beta_d)$ if and only if $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, d$. Let us put by convention $\alpha < +\infty$ for $\alpha \in \mathbf{Q}^d$.

Lemma 2.2 ([4, Lemma 4.7]). Let $\alpha, \beta, \gamma \in \mathbf{N}^d$ and let $a(\underline{X}), b(\underline{X}), c(\underline{X})$ be invertible elements of $\mathbf{K}[[\underline{X}]]$. If

$$a(\underline{X})\underline{X}^\alpha - b(\underline{X})\underline{X}^\beta = c(\underline{X})\underline{X}^\gamma,$$

then either $\alpha \leq \beta$ or $\beta \leq \alpha$. □

Applying Lemma 2.2 to Y_{ik}, Y_{jk}, and Y_{ij} we see that for every $Y_i, Y_j, Y_k \in \operatorname{Zer} f$ one has $O(Y_i, Y_k) \leq O(Y_j, Y_k)$ or $O(Y_i, Y_k) \geq O(Y_j, Y_k)$. Moreover, we have the *strong triangular inequality:* $O(Y_i, Y_j) \geq \min\{O(Y_i, Y_k), O(Y_j, Y_k)\}$. Consequently for every subset $A \subset \operatorname{Zer} f$ the set of contacts $\{O(Y_j, Y_k) : Y_j, Y_k \in A\}$ has the smallest element.

3 The Tree Model $T(f)$

In this section, we construct the tree model $T(f)$ which encodes the contacts between the roots of $f(Y)$. Given $h \in \mathbf{Q}^d_{\geq 0}$ we write $Y_i \equiv Y_j \bmod h^+$ if $O(Y_i, Y_j) > h$.

Let $B = \operatorname{Zer} f$ and let $h(B)$ be the minimal contact between the elements of B. We represent B as a horizontal bar and call $h(B)$ the *height* of B. The equivalence relation $\equiv \bmod h(B)^+$ divides B into equivalence classes $B_1, \dots, B_k$. From B we draw k vertical segments and at the end of the ith segment we place a horizontal bar representing B_i. The bar B_i is called a *postbar* of B and we write $B \perp B_i$. For each B_i we repeat this construction recursively. We do not draw the bars of infinite height.

Remark that for every bar $\bar{B} \in T(f)$ there exists a unique sequence $B \perp B' \perp B'' \perp \cdots \perp \bar{B}$ starting from the bar B of the minimal height.

Let $\#A$ denotes the number of elements of the set A.

Definition 3.1. To every bar $B \in T(f)$ we associate a d-tuple $q(B) \in \mathbf{Q}^d_{\geq 0}$ in the next way:

(i) If B is the bar of the lowest height then $q(B) = \#B \cdot h(B)$.

(ii) If $B \perp B'$ then $q(B') = q(B) + \#B'(h(B') - h(B))$.

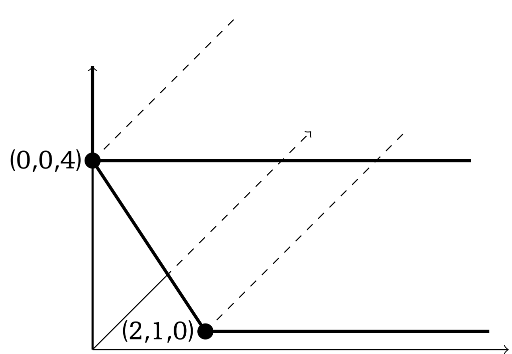
(iii) If $h(B)$ is infinite then $q(B)$ is also infinite. □

Remark 3.2. For $d = 1$, $q(B)$ becomes a rational number. In [10, 12] this number was defined by using the order of certain substitutions. However Lemma 2.7 in [12] states that $q(B)$ satisfies the recursive formula of Definition 3.1. Hence, that definition coincides with the present one. □

Let $q = (q_1, \dots, q_d) \in \mathbf{Q}^d_{\geq 0}$ and let k be a positive integer. We define the *elementary Newton polytope*

$$\left\{\frac{q}{k}\right\} := \text{Convex Hull } (\{(q_1, \dots, q_d, 0), (0, \dots, 0, k)\} + \mathbf{R}^{d+1}_{\geq 0}).$$

Example 3.3. The elementary Newton polytope $\left\{\frac{(2,1)}{4}\right\}$ is



□

With each tree model T we associate the Newton polytope

$$\Delta_T = \sum_{B \in \tilde{T}} \left\{ \frac{(t(B)-1)q(B)}{t(B)-1} \right\}, \tag{2}$$

where $\tilde{T}$ is the set of bars $B \in T$ of finite height, $t(B)$ is the number of postbars of B and the sum denotes the Minkowski sum (see [8, Chapter 4, Definition 1.1]).

4 Newton Polytope of the Discriminant

Let $h(\underline{X}) = \sum_{\underline{i}} a_{\underline{i}} \underline{X}^{\underline{i}}$ be a power series in s variables and coefficients in $\mathbf{K}$. The *Newton polytope* $\Delta(h)$ of h is the convex hull of the set $\bigcup_{a_{\underline{i}} \neq 0} \{\underline{i} + \mathbf{R}^s_{\geq 0}\}$. In two variables case the Newton polytope is called the *Newton diagram*.

If $g(Y) \in \mathbf{C}\{X_1\}[Y]$ is a Weierstrass polynomial then the Newton diagram of the Y-discriminant of $g(Y) - V$, where V is a new variable, is determined by the tree model of g (see Lemma 4.4). In this section, we generalize this result to quasi-ordinary Weierstrass polynomials in $\mathbf{K}[[X_1, \ldots, X_d]][Y]$.

Theorem 4.1. Let $f(Y) \in \mathbf{K}[[\underline{X}]][Y]$ be a quasi-ordinary Weierstrass polynomial and let $D_f(\underline{X}, V)$ be the Y-discriminant of the polynomial $f(Y) - V$, where V is a new variable. Then $\Delta(D_f) = \Delta_{T(f)}$. □

Set $f(Y) = Y^n + a_1(X_1, \ldots, X_d)Y^{n-1} + \cdots + a_n(X_1, \ldots, X_d) \in \mathbf{K}[[\underline{X}]][Y]$. Let $X_1 = T^{c_1}$, $\ldots$, $X_d = T^{c_d}$ be monomial substitutions, where T is a new variable and c_i are positive integers. Set $\underline{c} = (c_1, \ldots, c_d)$ and let

$$g(Y) = Y^n + a_1(T^{c_1}, \ldots, T^{c_d})Y^{n-1} + \cdots + a_n(T^{c_1}, \ldots, T^{c_d}) \in \mathbf{K}[[T]][Y]. \tag{3}$$

Remark that if $c_i \geq n$ for $i = 1, \ldots, d$ then the order of $a_i(T^{c_1}, \ldots, T^{c_d})$ is bigger than or equal to n for $i = 1, \ldots, n$. In particular the initial form of g, treated as a power series in variables T and Y, is not divisible by T since Y^n is one of its terms.

Lemma 4.2. There is a bijective correspondence between the bars of $T(f)$ and the bars of $T(g)$. Moreover, if B and $\bar{B}$ are corresponding bars of $T(f)$ and $T(g)$, respectively, then $h(\bar{B}) = \langle \underline{c}, h(B) \rangle$ and $q(\bar{B}) = \langle \underline{c}, q(B) \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product. □

Proof. Set $T^{\underline{c}} = (T^{c_1}, \ldots, T^{c_d})$. Clearly $\mathrm{Zer}\, g = \{Y_1(T^{\underline{c}}), \ldots, Y_n(T^{\underline{c}})\}$ and $O(Y_i(T^{\underline{c}}), Y_j(T^{\underline{c}})) = \langle \underline{c}, O(Y_i(\underline{X}), Y_j(\underline{X})) \rangle$ for $i \neq j$. Hence every bar $B = \{Y_{i_1}(\underline{X}), \ldots, Y_{i_k}(\underline{X})\}$ of $T(f)$ yields the bar $\bar{B} = \{Y_{i_1}(T^{\underline{c}}), \ldots, Y_{i_k}(T^{\underline{c}})\}$ of $T(g)$ of height $\langle \underline{c}, h(B)\rangle$. Taking the scalar product by $\underline{c}$ of the equations appearing in Definition 3.1 we get the second part of the lemma. ■

Further, in this section, we write $(\underline{x}, y)$ for $(x_1, \ldots, x_d, y) \in \mathbf{R}^{d+1}$.

Corollary 4.3. Let $\pi : \mathbf{R}^{d+1} \to \mathbf{R}^2$ be the linear mapping given by $(\underline{x}, x_{d+1}) \mapsto (\langle \underline{c}, \underline{x}\rangle, x_{d+1})$. Then $\pi(\Delta_{T(f)}) = \Delta_{T(g)}$. □

Proof. Corollary 4.3 follows from Lemma 4.2 and two simple observations: $\pi\left(\left\{\frac{q}{k}\right\}\right) = \left\{\frac{\langle \underline{c}, q\rangle}{k}\right\}$ for every elementary Newton polytope $\left\{\frac{q}{k}\right\} \subset \mathbf{R}^{d+1}_{\geq 0}$, and $\pi(\Delta_1 + \Delta_2) = \pi(\Delta_1) + \pi(\Delta_2)$ for all Newton polytopes $\Delta_1, \Delta_2 \subset \mathbf{R}^{d+1}_{\geq 0}$. ■

Lemma 4.4. Let $D_g(T, V)$ be the Y-discriminant of the polynomial $g(Y) - V$, where V is a new variable. Assume that T does not divide the initial form of g treated as a power series in two variables. Then $\Delta(D_g) = \Delta_{T(g)}$. □

Lemma 4.4 was proved in [10, p. 691] for Weierstrass polynomials in $\mathbf{C}\{T\}[Y]$. However, its proof can be generalized without any problems to Weierstrass polynomials with coefficients in the ring $\mathbf{K}[[T]]$.

Proof of Theorem 4.1. For every Newton polytope $\Delta \subset \mathbf{R}^k_{\geq 0}$ and every $v \in \mathbf{R}^k_{\geq 0}$ we define the *support function* $l(v, \Delta) = \min\{\langle v, \alpha\rangle : \alpha \in \Delta\}$. To prove the theorem it is enough to show that the support functions $l(\cdot, \Delta(D_f))$ and $l(\cdot, \Delta_{T(f)})$ are equal. As these functions are continuous it suffices to show the equality on a dense subset of $\mathbf{R}^{d+1}_{\geq 0}$.

Let $c = (c_1, \ldots, c_{d+1}) = (\underline{c}, c_{d+1}) \in \mathbf{R}^{d+1}_{\geq 0}$, where $\underline{c} = (c_1, \ldots, c_d)$.

Perturbing c a little we may assume that the hyperplane $\{\alpha \in \mathbf{R}^{d+1} : \langle c, \alpha\rangle = l(c, \Delta(D_f))\}$ supports $\Delta(D_f)$ at exactly one point $\check{\alpha} = (\underline{\check{\alpha}}, \check{\alpha}_{d+1})$. Since after a small change of c the support point remains the same, we can assume, perturbing c again if necessary, that all c_i are positive rational numbers.

We will show that

$$l(c, \Delta_{T(f)}) = l(c, \Delta(D_f)). \tag{4}$$

Multiplying c by the common denominator of $c_1, \ldots, c_{d+1}$ we may assume that all c_i are integers bigger than or equal to $\deg f$. At this point of the proof we fixed c. Let $g(Y)$ be the Weierstrass polynomial given by (3). We claim that $l(c, \Delta_{T(f)}) = l((1, c_{d+1}), \Delta_{T(g)})$ and $l(c, \Delta(D_f)) = l((1, c_{d+1}), \Delta(D_g))$.

First equality follows from Corollary 4.3 and the identity $\langle c, \alpha\rangle = \langle (1, c_{d+1}), \pi(\alpha)\rangle$ for $\alpha \in \mathbf{R}^{d+1}$.

Let $D_f(\underline{X}, V) = \sum_\alpha d_\alpha \underline{X}^{\underline{\alpha}} V^{\alpha_{d+1}}$, where $\alpha = (\underline{\alpha}, \alpha_{d+1})$. As the discriminant commutes with base change we get by (3) $D_g(T, V) = \sum_\alpha d_\alpha T^{\langle \underline{c}, \underline{\alpha}\rangle} V^{\alpha_{d+1}}$. Since the hyperplane $\{\alpha \in \mathbf{R}^{d+1} : \langle c, \alpha\rangle = l(c, \Delta(D_f)\}$ supports $\Delta(D_f)$ at $\check{\alpha}$, the monomial $d_{\check{\alpha}} T^{\langle \underline{c}, \underline{\check{\alpha}}\rangle} V^{\check{\alpha}_{d+1}}$ satisfies the equality $\langle \underline{c}, \underline{\check{\alpha}}\rangle + c_{d+1}\check{\alpha}_{d+1} = l(c, \Delta(D_f))$, while for all other monomials $d_\alpha T^{\langle \underline{c}, \underline{\alpha}\rangle} V^{\alpha_{d+1}}$ with $d_\alpha \neq 0$ appearing in the sum $\sum_\alpha d_\alpha T^{\langle \underline{c}, \underline{\alpha}\rangle} V^{\alpha_{d+1}}$ we have $\langle \underline{c}, \underline{\alpha}\rangle + c_{d+1}\alpha_{d+1} > l(c, \Delta(D_f))$. Hence $l((1, c_{d+1}), \Delta(D_g)) = \langle \underline{c}, \underline{\check{\alpha}}\rangle + c_{d+1}\check{\alpha}_{d+1} = l(c, \Delta(D_f))$.

By Lemma 4.4 $\Delta_{T(g)} = \Delta(D_g)$ which together with the just proved claim gives (4). This completes the proof because c is sufficiently general. ■

From Theorem 4.1, Corollary 4.3 and Lemma 4.4 we get $\pi(\Delta(D_f)) = \pi(\Delta_{T(f)}) = \Delta_{T(g)} = \Delta(D_g)$, which gives us

Corollary 4.5. Let $\pi : \mathbf{R}^{d+1} \to \mathbf{R}^2$ be the linear mapping given by $(\underline{x}, x_{d+1}) \mapsto (\langle \underline{c}, \underline{x}\rangle, x_{d+1})$. Then $\pi(\Delta(D_f)) = \Delta(D_g)$, where f and g are quasi-ordinary Weierstrass polynomials given by the equations (1) and (3), respectively. □

5 Symmetry of the Tree Model

In this section, we describe symmetries of the tree model associated with a quasi-ordinary Weierstrass polynomial $f(Y)$.

Let $U = \{\omega \in \mathbf{K} : \omega^k = 1\}$ be the multiplicative group of kth roots of unity. With every d-tuple $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_d) \in U^d$ we associate the $\mathbf{K}$-algebra homomorphism $\phi_{\underline{\epsilon}} : \mathbf{K}[[X_1^{\frac{1}{k}}, \ldots, X_d^{\frac{1}{k}}]] \to \mathbf{K}[[X_1^{\frac{1}{k}}, \ldots, X_d^{\frac{1}{k}}]]$, such that $\phi_{\underline{\epsilon}}(X_i^{\frac{1}{k}}) = \epsilon_i X_i^{\frac{1}{k}}$ for $i = 1, \ldots, d$. Since $\phi_{\underline{\epsilon}}(X_i) = \epsilon_i^k X_i = X_i$, the homomorphism $\phi_{\underline{\epsilon}}$ is the identity on $\mathbf{K}[[\underline{X}]]$. For every $\underline{\epsilon}, \underline{\omega} \in U^d$ we have $\phi_{\underline{\epsilon}} \circ \phi_{\underline{\omega}} = \phi_{\underline{\epsilon}\cdot\underline{\omega}}$, where the product $\underline{\epsilon}\cdot\underline{\omega}$ is componentwise. Hence the *star operation* $\underline{\epsilon} * \psi(\underline{X}) := \phi_{\underline{\epsilon}}(\psi(\underline{X}))$ is an action of the group U^d on $\mathbf{K}[[X_1^{\frac{1}{k}}, \ldots, X_d^{\frac{1}{k}}]]$.

If $\psi(\underline{X}) = \sum_{\underline{\alpha} \in (1/k)\mathbf{N}^d} c_{\underline{\alpha}} \underline{X}^{\underline{\alpha}}$ then $\underline{\epsilon} * \psi(\underline{X}) = \sum_{\underline{\alpha} \in (1/k)\mathbf{N}^d} c_{\underline{\alpha}} \underline{\epsilon}^{k\underline{\alpha}} \underline{X}^{\underline{\alpha}}$.

We will show that the star operation permutes the set $\mathrm{Zer}\, f$ and is transitive on $\mathrm{Zer}\, f$ providing $f(Y)$ is irreducible in $\mathbf{K}[[\underline{X}]][Y]$. Moreover, it preserves the contact.

To be more precise, we have

Property 5.1.

(i) $\underline{\epsilon} * \mathrm{Zer}\, f = \mathrm{Zer}\, f$ for every $\underline{\epsilon} \in U^d$.

(ii) If $f(Y)$ is irreducible in $\mathbf{K}[[\underline{X}]][Y]$ then $\mathrm{Zer}\, f = U^d * Y_i$ for every $Y_i \in \mathrm{Zer}\, f$.

(iii) $O(Y_i, Y_j) = O(\underline{\epsilon} * Y_i, \underline{\epsilon} * Y_j)$ for every $\underline{\epsilon} \in U^d$ and $i \neq j$. □

Proof. Fix $\underline{\epsilon} \in U^d$. The homomorphism $\phi_{\underline{\epsilon}}$ naturally extends to the homomorphism $\Phi_{\underline{\epsilon}}$: $\mathbf{K}[[X_1^{\frac{1}{k}}, \ldots, X_d^{\frac{1}{k}}]][Y] \to \mathbf{K}[[X_1^{\frac{1}{k}}, \ldots, X_d^{\frac{1}{k}}]][Y]$. Acting by $\Phi_{\underline{\epsilon}}$ on $f(Y) = \prod_{i=1}^n [Y - Y_i]$ we get $f(Y) = \Phi_{\underline{\epsilon}}(f(Y)) = \prod_{i=1}^n [Y - \phi_{\underline{\epsilon}}(Y_i)]$ which proves (i).

Fix $Y_i \in \mathrm{Zer}\, f$ and let $f_1(Y) = \prod_{Y(\underline{X}) \in U^d * Y_i} [Y - Y(\underline{X})]$. For every $\underline{\epsilon} \in U^d$ we have $\Phi_{\underline{\epsilon}}(f_1(Y)) = f_1(Y)$. Since the action of U^d on $f_1(Y)$ is trivial the polynomial $f_1(Y)$ has coefficients in the ring $\mathbf{K}[[\underline{X}]]$. Because $U^d * Y_i \subseteq \mathrm{Zer}\, f$ all roots of $f_1(Y)$ are the roots of $f(Y)$. Assuming that $f(Y)$ is irreducible in $\mathbf{K}[[\underline{X}]][Y]$, we get $f_1(Y) = f(Y)$ which proves (ii).

Statement (iii) follows directly from the definition of the *star action*. ■

For every $\underline{\epsilon} \in U^d$ the mapping $\mathrm{Zer}\, f \ni Y_i \to \underline{\epsilon} * Y_i \in \mathrm{Zer}\, f$ preserves contacts. Let $B = \{Y_{i_1}, \ldots, Y_{i_s}\}$ be a bar of $T(f)$. Then $\underline{\epsilon} * B = \{\underline{\epsilon} * Y_{i_1}, \ldots, \underline{\epsilon} * Y_{i_s}\}$ is also a bar of $T(f)$ of the same height. Thus $U^d \times T(f) \ni (\underline{\epsilon}, B) \to \underline{\epsilon} * B \in T(f)$ is an action of the group U^d on $T(f)$ which for each fixed $\underline{\epsilon}$ yields a symmetry of $T(f)$ preserving heights.

Every bar $\underline{\epsilon} * B$ will be called *conjugate* to B. Further in this section we count the number of conjugates of $B \in T(f)$. To this aim we employ the theory of *dual groups*.

Let C be a cyclic group of order k and let G be a finite commutative group such that $kg = 0$ for every $g \in G$. Recall that the *dual* of G, denoted G^*, is the group of homomorphisms from G to C. The main theorem of dual groups states that G^* is isomorphic to G.

Let A, A' be commutative groups. The mapping $A \times A' \to C$, $(x, x') \to \langle x, x' \rangle$ is called a *pairing* if for every $x' \in A'$ the mapping $\phi_{x'} = \langle \cdot, x' \rangle$ is a homomorphism of A to C and for every $x \in A$ the mapping $\psi_x = \langle x, \cdot \rangle$ is a homomorphism of A' to C.

For every $a \in A$ and $a' \in A'$ we introduce the orthogonal relation $a \perp a'$ if and only if $\langle a, a' \rangle$ is the identity element of C. For every set $B \subset A$ we denote by $B^{\perp}$ the set $\{x' \in A' : b \perp x' \text{ for all } b \in B\}$. We make a similar definition of $(B')^{\perp}$ for $B' \subset A'$.

Theorem 5.2 ([20, Chapter 1, Theorem 9.2, p. 49])**.** Let $A \times A' \to C$ be a pairing of two abelian groups into a finite cyclic group C. Assume that A' is finite. Then $A'/A^{\perp}$ is isomorphic to the dual group of $A/(A')^{\perp}$. □

Corollary 5.3. Let $A \times A' \to C$ be a pairing of two abelian groups into a finite cyclic group C. Assume that A' is finite. If M, N are subgroups of A such that $A'^{\perp} \subset N \subset M$ then $[M : N] = [N^{\perp} : M^{\perp}]$. □

Proof. First, we will show that $(N^{\perp})^{\perp} = N$.

Let $a \in A \setminus N$. Then there exists $a' \in N^{\perp}$ such that $a \not\perp a'$. Indeed, if this is not the case then $N^{\perp} = N_1^{\perp}$, where N_1 is the group generated by $N \cup \{a\}$. By Theorem 5.2 the group $A'/N^{\perp} = A'/N_1^{\perp}$ would be dual of $N/(A')^{\perp}$ and of $N_1/(A')^{\perp}$ which is impossible because these groups have different number of elements since the coset of a belongs to $N_1/(A')^{\perp}$ but not to $N/(A')^{\perp}$. This shows that $a \notin (N^{\perp})^{\perp}$. Since a is an arbitrary element of $A \setminus N$, we have $(N^{\perp})^{\perp} \subset N$.

Let $a \in N$. Then for every $a' \in N^{\perp}$ we have $a \perp a'$. Consequently $a \in (N^{\perp})^{\perp}$ which gives $N \subset (N^{\perp})^{\perp}$. The first part of the proof is finished.

It follows from Theorem 5.2 applied to the pairing $M \times N^{\perp} \to C$ that $N^{\perp}/M^{\perp}$ is the dual of $M/(N^{\perp})^{\perp} = M/N$. Since a finite abelian group is isomorphic to its dual, we get $[M : N] = [N^{\perp} : M^{\perp}]$. ■

Let B' be a postbar of $B \in T(f)$. Since all $Y_i \in B'$ belong to the same equivalence class mod $h(B)^+$, they have the same term of exponent $h(B)$. Let c be the coefficient of such a term. Following [19] we write $B \perp_c B'$ and say that B' *is supported at* c on B. It is obvious that different postbars of B are supported at different points.

Definition 5.4. Let $B_0 \perp_{c_0} B_1 \perp_{c_1} \cdots \perp_{c_{r-2}} B_{r-1} \perp_{c_{r-1}} B_r = B$ be a sequence of bars of $T(f)$, where B_0 is the bar of the lowest height in $T(f)$. Let $H(B) = \{h(B_i) : c_i \neq 0,\ 0 \le i \le r-1\} = \{h_1, \ldots, h_s\}$. Then we call the lattice $N(B) = \mathbf{Z}^d + \mathbf{Z}h_1 + \cdots + \mathbf{Z}h_s$ the characteristic lattice of B. □

Note that if $Y(\underline{X})$ is any element of B then $H(B)$ consist of such heights $h(B_i)$, $0 \le i \le r-1$, that $\underline{X}^{h(B_i)}$ appears in $Y(\underline{X})$ with nonzero coefficient.

Consider the pairing $(1/k)\mathbf{Z}^d \times U^d \ni (\lambda, \underline{\epsilon}) \to \underline{\epsilon}^{k\lambda} \in U$. Directly from the definition it follows that for $\underline{\epsilon} \in U^d$ and $\lambda \in (1/k)\mathbf{N}^d$ holds $\underline{\epsilon} * \underline{X}^{\lambda} = \underline{X}^{\lambda}$ if and only if $\lambda \perp \underline{\epsilon}$. It is easy to check that $(U^d)^{\perp} = \mathbf{Z}^d$.

Theorem 5.5. Every $B \in T(f)$ has $[N(B) : \mathbf{Z}^d]$ conjugates.

Let $B \perp_c B'$.

1. If $c \neq 0$ then there are $n(B) = [N(B) + \mathbf{Z}h(B) : N(B)]$ postbars of B conjugate with B'.
2. If $c = 0$ then there are no postbars of B conjugate with B', expect B' itself. □

Proof. Given $Y_i \in B$ and $\underline{\epsilon} \in U^d$ the contact between Y_i and $\underline{\epsilon} * Y_i$ is bigger than or equal to $h(B)$ if and only if $h \perp \underline{\epsilon}$ for every $h \in H(B)$, since otherwise the monomial $\underline{X}^h$ would appear in the difference $\underline{\epsilon} * Y_i - Y_i$ with nonzero coefficient. It follows that $\underline{\epsilon} * B = B$ if and only if $\underline{\epsilon} \in N(B)^{\perp}$. Thus, the stabilizer of B under the action of U^d is the group $N(B)^{\perp}$. By the orbit stabilizer theorem and Corollary 5.3 the set $U^d * B$ has $[U^d : N(B)^{\perp}] = [N(B) : \mathbf{Z}^d]$ elements which proves the first part of the theorem.

Let B' be a postbar of B. Then $\underline{\epsilon} * B'$ is a postbar of B if and only if $\underline{\epsilon} * B = B$. Thus the set of postbars of B which are conjugate to B' is equal to $N(B)^{\perp} * B'$. By the just proven part of the theorem $N(B')^{\perp}$ is the stabilizer of B' under the action of U^d. By the orbit stabilizer theorem and Corollary 5.3, the number of elements of $N(B)^{\perp} * B'$ equals $[N(B)^{\perp} : N(B')^{\perp}] = [N(B') : N(B)]$.

Assume that $c \neq 0$. Then $[N(B') : N(B)] = [N(B) + \mathbf{Z}h(B) : N(B)]$.

Now, suppose that $c = 0$. Since $N(B') = N(B)$, we get $[N(B') : N(B)] = 1$, hence the set of postbars of B conjugate to B' has one element. ■

Corollary 5.6. If $B \in T(f)$ has $n(B)$ postbars then all of them are conjugate and they are supported at nonzero numbers. □

6 The Tree Model of an Irreducible Polynomial

Let $f(Y) \in \mathbf{K}[[\underline{X}]][Y]$ be an irreducible quasi-ordinary Weierstrass polynomial. By Property 5.1 the action of U^d on $\operatorname{Zer} f$ is transitive. This implies that for fixed Y_i, the set of contacts $\{O(Y_j, Y_i) : j \neq i\}$ does not depend on the choice of $Y_i \in \operatorname{Zer} f$. If $\{O(Y_j, Y_i) : j \neq i\} = \{h_1, \ldots, h_g\}$, where $h_1 < h_2 < \cdots < h_g$ then $h_1, h_2, \ldots, h_g$ is called the *sequence of characteristic exponents of* $f(Y)$. The next lemma is in [23, Remarks 5.8, p. 57] but we give the proof for convenience of the reader.

Lemma 6.1. A finite sequence $h_1, h_2, \ldots, h_g$ of elements from $\mathbf{Q}^d_{\geq 0}$ is a sequence of characteristic exponents of an irreducible quasi-ordinary Weierstrass polynomial $f(Y) \in \mathbf{K}[[X_1, \ldots, X_d]][Y]$ if and only if

(C1) $h_1 < h_2 < \cdots < h_g$ and

(C2) $h_i \notin N_{i-1} := \mathbf{Z}^d + \mathbf{Z}h_1 + \cdots + \mathbf{Z}h_{i-1}$ for $i = 1, \ldots, g$,

where $N_0 = \mathbf{Z}^d$. □

Proof. Let $f(Y)$ be an irreducible quasi-ordinary Weierstrass polynomial. Without loss of generality we may assume that all roots of $f(Y)$ belong to $\mathbf{K}[[X_1^{\frac{1}{k}}, \ldots, X_d^{\frac{1}{k}}]]$. Let Y_1 be a fixed root of $f(Y)$ and let $h_1, h_2, \dots, h_g$ be the sequence of its characteristic exponents. All roots of $f(Y)$ are conjugate by the action of U^d. Hence, by the definition of a sequence of characteristic exponents, for every $i \in \{1, \dots, g\}$ there exists $\underline{\epsilon}_i \in U^d$ such that $h_i = O(\underline{\epsilon}_i * Y_1, Y_1)$. This shows that all monomials $\underline{X}^{h_i}$ appear in Y_1 with nonzero coefficients. Moreover $\underline{\epsilon}_i * \underline{X}^{h_j} = \underline{X}^{h_j}$ for $1 \le j < i$ and $\underline{\epsilon}_i * \underline{X}^{h_i} \neq \underline{X}^{h_i}$. We get $\underline{\epsilon}_i \in (N_{i-1})^{\perp}$ and $\underline{\epsilon}_i^{kh_i} \neq 1$, hence $h_i \notin N_{i-1}$ for $i = 1, \dots, g$.

Now, assume that a sequence $h_1, h_2, \dots, h_g$ satisfies conditions (C1) and (C2). Let $Y_1 := \underline{X}^{h_1} + \cdots + \underline{X}^{h_g}$. Clearly $Y_1 \in \mathbf{K}[[X_1^{\frac{1}{k}}, \dots, X_d^{\frac{1}{k}}]]$ for some $k > 0$. Let $\{Y_1, \dots, Y_n\}$ be the set of conjugates of Y_1 by the action of U^d, where $U = \{\omega \in \mathbf{K} : \omega^k = 1\}$. Consider the polynomial $f(X) = \prod_{i=1}^n (Y - Y_i)$. As in the proof of Property 5.1 we show that $f(Y)$ is a polynomial with coefficients in the ring $\mathbf{K}[[\underline{X}]]$ and that is irreducible over this ring.

Condition (C1) implies that the difference of any two roots of $f(Y)$ has a form $w(\underline{X})\underline{X}^{h_l}$, where $w(0) \neq 0$ and $1 \le l \le g$. Thus the discriminant of $f(Y)$, being the product of differences of the roots, equals $X_1^{\alpha_1} \cdots X_d^{\alpha_d} u(\underline{X})$, where $\alpha_i \in \mathbf{N}$ and $u(\underline{X})$ is a unity in $\mathbf{K}[[\underline{X}]]$. This shows that $f(Y)$ is quasi-ordinary.

By Condition (C2) we get $N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_g$ and consequently $U^d = N_0^{\perp} \supsetneq N_1^{\perp} \supsetneq \cdots \supsetneq N_g^{\perp}$. Take $\underline{\epsilon} \in U^d$. If $\underline{\epsilon} \in N_{i-1}^{\perp} \setminus N_i^{\perp}$ then $O(\underline{\epsilon} * Y_1, Y_1) = h_i$ and if $\underline{\epsilon} \in N_g^{\perp}$ then $\underline{\epsilon} * Y_1 = Y_1$. Thus $h_1, h_2, \dots, h_g$ is the sequence of characteristic exponents of $f(Y)$. ■

Now we show that the tree model of an irreducible quasi-ordinary Weierstrass polynomial $f(Y)$ depends only on its sequence of characteristic exponents.

Theorem 6.2. Let $f(Y) \in \mathbf{K}[[\underline{X}]][Y]$ be an irreducible quasi-ordinary Weierstrass polynomial and let $h_1, h_2, \dots, h_g$ be the sequence of its characteristic exponents. Let $N_0 = \mathbf{Z}^d$ and $N_i = \mathbf{Z}^d + \mathbf{Z}h_1 + \cdots + \mathbf{Z}h_i$ for $i = 1, \dots, g$. Then the tree model $T(f)$ is characterized by two properties:

(i) the set of the heights of bars of $T(f)$ is $\{h_1, \dots, h_g, h_{g+1}\}$, where $h_{g+1} = \infty$,

(ii) every bar of height h_i has $[N_i : N_{i-1}]$ postbars and all of them have the height h_{i+1} for $i = 1, \dots, g$. □

Proof. Part (i) follows directly from the definition of the sequence of characteristic exponents. Moreover, since the action of U^d on $\mathrm{Zer} f$ is transitive, every bar of height h_i has only postbars of height h_{i+1}, for $i = 1, \dots, g$ and all bars of a fixed height are conjugate.

Let $B \in T(f)$. To prove part (ii) observe that if $h(B) = h_i$ then $N(B) = N_{i-1}$ since the monomials $\underline{X}^{h_i}$ for $1 \le i \le g$ appear with nonzero coefficients in every $Y(\underline{X}) \in \mathrm{Zer} f$.

Applying part (ii) of Theorem 5.5 to B we see that B has $[N_i : N_{i-1}]$ postbars conjugate with a given postbar B' of B. This completes the proof. ■

A tree model T satisfying conditions (i), (ii) of Theorem 6.2 will be called the *tree of type* $(h_1, h_2, \dots, h_g)$.

Theorem 6.3. If the tree model of a quasi-ordinary Weierstrass polynomial $f(Y) \in \mathbf{K}[[\underline{X}]][Y]$ is of type $(h_1, h_2, \dots, h_g)$ then $f(Y)$ is irreducible and $h_1, h_2, \dots, h_g$ is the sequence of its characteristic exponents. □

Proof. By conditions (i) and (ii) the tree $T(f)$ has $[N_g : N_{g-1}] \cdot [N_{g-1} : N_{g-2}] \cdots [N_1 : N_0] = [N_g : \mathbf{Z}^d]$ bars of infinite height.

The bar B of $T(f)$ of the lowest height $h(B) = h_1$ has at least two postbars. Let us choose one of them, B', which is supported at a nonzero number. Taking a similar choice of a postbar of B' and continuing this procedure $g-1$-times we arrive at a bar $\bar{B}$ of infinite height. It is clear that $N(\bar{B}) = N_g$. By Theorem 5.5 the number of conjugates of $\bar{B}$ equals $[N_g : \mathbf{Z}^d]$.

Thus all bars of infinite height are conjugate. It follows that all the roots of $f(Y)$ are conjugate by the action of U^d. Thus $f(Y)$ is irreducible in $\mathbf{K}[[\underline{X}]][Y]$. ■

7 Irreducibility Criterion

In this section, we consider two Weierstrass polynomials $p(Y)$ and $f(Y)$ such that $\Delta(D_p) = \Delta(D_f)$. We prove that $p(Y)$ is an irreducible quasi-ordinary polynomial if and only if $f(Y)$ is also.

Theorem 7.1. Let $f(Y), p(Y) \in \mathbf{K}[[\underline{X}]][Y]$ be quasi-ordinary Weierstrass polynomials such that $\Delta(D_f) = \Delta(D_p)$. Assume that $f(Y)$ is irreducible. Then $p(Y)$ is irreducible and the sequences of characteristic exponents of $f(Y)$ and $p(Y)$ are equal. □

Proof. Let $h_1, \dots, h_g$ be the sequence of characteristic exponents of $f(Y)$. By Theorem 6.2 the tree model $T(f)$ is of type $(h_1, \dots, h_g)$. By Theorem 6.3 it is enough to show that $T(p)$ is also a tree of type $(h_1, \dots, h_g)$.

First, we will show that the polynomials $f(Y)$ and $p(Y)$ have the same degree. If $f(Y) = Y^n + a_1 Y^{n-1} + \cdots + a_n$ then $\mathrm{Discr}_Y(f(Y) - V) = d_0 V^{n-1} + d_1 V^{n-2} + \cdots + d_{n-1}$, where $d_0 = (-1)^{(n+2)(n-1)/2} n^n$ (see [25, Lemma 2.1]). It follows that $(\underline{0}, \deg_Y f(Y) - 1)$ is the point of the intersection of $\Delta(D_f)$ with the vertical axis having the smallest last coordinate. Thus the equality of the Newton polytopes $\Delta(D_f)$ and $\Delta(D_p)$ gives $\deg f(Y) = \deg p(Y)$.

Now, let us compute recursively the d-tuples $q(B)$ for $B \in T(f)$. Under the notations of Theorem 6.2 we set $n_0 = 1$ and n_i=$[N_i : N_{i-1}]$ for $i = 1, \dots, g$. By the symmetry of $T(f)$ every bar B of height h_i, where $1 \le i \le g$, has $n_0 \cdots n_{i-1}$ conjugates. Moreover, by Definition 3.1 $q(B)$ is constant on the bars of the same height; we denote $q_i := q(B)$ for such $B \in T(f)$ that $h(B) = h_i$. We have

$$\begin{aligned} q_1 &= n_1 \cdots n_g h_1, \\ q_i &= q_{i-1} + n_i \cdots n_g (h_i - h_{i-1}) \quad \text{for } i = 2, \dots, g. \end{aligned} \tag{5}$$

Hence

$$\Delta_{T(f)} = \sum_{i=1}^{g} \left\{ \frac{n_0 \cdots n_{i-1}(n_i - 1) q_i}{n_0 \cdots n_{i-1}(n_i - 1)} \right\}. \tag{6}$$

By (2)

$$\Delta_{T(p)} = \sum_{B \in \widetilde{T}(p)} \left\{ \frac{(t(B) - 1) q(B)}{t(B) - 1} \right\}. \tag{7}$$

Using the assumption $\Delta(D_f) = \Delta(D_p)$ and Theorem 4.1 we see that polytopes given by (6) and (7) are equal. Hence $\{q(B) : B \in T(p)\} = \{q_1, \dots, q_g\} \cup \{\infty\}$.

Let $H_i = \{B \in T(p) : q(B) = q_i\}$ for $i = 1, \dots, g$. We will show, by induction on i, that the set H_i has $n_0 \cdots n_{i-1}$ elements, the elements of H_i are conjugate and form a partition of $\operatorname{Zer} p$. Moreover, for every $B \in H_i$ we have $h(B) = h_i$, $N(B) = N_{i-1}$ and B has n_i postbars which are conjugate.

Let $B_0 = \operatorname{Zer} p$ be the bar of the tree model $T(p)$ of the minimal height. Clearly $q(B_0) = q_1$ and $H_1 = \{B_0\}$. Since B_0 has $\deg p(Y) = \deg f(Y) = n_1 \cdots n_g$ elements we get from (5) and the formula for $q(B_0)$ (see Definition 3.1) the equality $h(B_0) = h_1$.

Since $\Delta_{T(f)} = \Delta_{T(p)}$ we get from (6) and (7) the equality

$$\left\{ \frac{(t(B_0) - 1) q(B_0)}{t(B_0) - 1} \right\} = \left\{ \frac{(n_1 - 1) q_1}{n_1 - 1} \right\}.$$

Hence B_0 has n_1 postbars. Since $N(B_0) = \mathbf{Z}^d$, we get $n(B_0) = [N(B_0) + \mathbf{Z}h_1 : N(B_0)] = [N_1 : N_0] = n_1$ and by Corollary 5.6 all the postbars of B_0 are conjugate.

Assume that the set H_i has the desired properties. We will prove them for H_{i+1}.

Since $q(B) < q(B')$ for $B \perp B'$, all the elements of H_{i+1} are postbars of the elements of H_i. By the inductive hypothesis all the postbars of the elements of H_i are conjugate under the action of U^d. Hence all of them have the same height and $H_{i+1} = \{\, B' \in T(p) : B \perp B', B \in H_i \,\}$. Since every $B \in H_i$ has n_i postbars, by Corollary 5.6 every postbar B' of B is supported at a nonzero number and $N(B') = N(B) + \mathbf{Z}h_i = N_i$. The set H_{i+1} has $n_0 \cdots n_i$ elements, H_{i+1} is a partition of Zer$p$, and every $B' \in H_{i+1}$ has $n_{i+1} \cdots n_g$ elements.

Since the polytopes given in (6) and (7) are equal, we get

$$\left\{ \frac{n_0 \cdots n_i (n_{i+1} - 1) q_{i+1}}{n_0 \cdots n_i (n_{i+1} - 1)} \right\} = \sum_{B \in H_{i+1}} \left\{ \frac{(t(B) - 1) q(B)}{t(B) - 1} \right\} = n_0 \cdots n_i \left\{ \frac{(t(B') - 1) q(B')}{t(B') - 1} \right\},$$

where B' is a fixed element of H_{i+1}. Consequently B' has n_{i+1} postbars.

By Definition 3.1 we have $q(B') = q(B) + \#B'(h(B') - h(B))$ for $B \perp B'$. If $B \in H_i$ and $B' \in H_{i+1}$, this gives us $q_{i+1} = q_i + (n_{i+1} \cdots n_g)(h(B') - h_i)$. Using formula (5) we get $h(B') = h_{i+1}$.

Once we know the height $h(B')$ we also know that $n(B') = [N_{i+1} : N_i] = n_{i+1}$. Hence by Corollary 5.6 B' has n_{i+1} postbars and all of them are conjugate. ■

8 Arithmetical Test of Irreducibility

In this section, we consider Newton polytopes $\Delta = \sum_{i=1}^{g} \left\{ \frac{L_i}{M_i} \right\} \subset \mathbf{R}_{\geq 0}^{d+1}$, where $\frac{1}{M_1} L_1 < \frac{1}{M_2} L_2 < \cdots < \frac{1}{M_g} L_g$. We associate to Δ the sequences:

1. $H_0 = 1$, $H_i = 1 + M_1 + \cdots + M_i$ for $i \in \{1, \ldots, g\}$,
2. $\gamma_i = \frac{H_{i-1}}{M_i} L_i$ for $i \in \{1, \ldots, g\}$

and the sequence of lattices $W_i = H_g \mathbf{Z}^d + \mathbf{Z}\gamma_1 + \cdots + \mathbf{Z}\gamma_i$, for $i \in \{0, \ldots, g\}$. We say that Δ is an *I-polytope* if and only if $[W_i : W_{i-1}] = H_i / H_{i-1}$ for $i \in \{1, \ldots, g\}$. Note that the I-polytopes for $d = 1$ are called *Merle polygons* in [11].

The reader interested in computing the indices $[W_i : W_{i-1}]$, in an effective way, is encouraged to read Section (5.9) (p. 57) of [23]. For the convenience of the reader we prove this result in Appendix.

Theorem 7.1 allows us to present an arithmetical test of irreducibility for quasi-ordinary Weierstrass polynomials.

Theorem 8.1. Let $f \in \mathbf{K}[[X_1, \ldots, X_d]][Y]$ be a Weierstrass polynomial. Then f is irreducible and quasi-ordinary if and only if $\Delta(D_f)$ is an I-polytope. □

Proof. Let f be an irreducible quasi-ordinary Weierstrass polynomial and let $h_1, \dots, h_g$ be the sequence of its characteristic exponents. By Lemma 6.1 the numbers $n_i = [N_i : N_{i-1}]$, where $N_i = \mathbf{Z}^d + \mathbf{Z}h_1 + \cdots + \mathbf{Z}h_i$, are bigger than 1 for $i = 1, \dots, g$. Consider an auxiliary sequence $\tilde{\gamma}_1, \dots, \tilde{\gamma}_g$ given by recurrence relations

$$\begin{aligned} \tilde{\gamma}_1 &= h_1, \\ \tilde{\gamma}_i &= n_{i-1}\tilde{\gamma}_{i-1} + h_i - h_{i-1} \quad \text{for } i = 2, \dots, g. \end{aligned} \tag{8}$$

Let $n = n_1 \cdots n_g$ and let $\gamma_i = n\tilde{\gamma}_i$. Then it follows from (5) and (6) that

$$\Delta_{T(f)} = \sum_{i=1}^{g} \left\{ \frac{(n_i - 1)\gamma_i}{n_0 \cdots n_{i-1}(n_i - 1)} \right\}. \tag{9}$$

Let L_i and M_i denote the numerator and the denominator of the ith term of (9). It is easy to show by induction that $H_i := 1 + M_1 + \cdots + M_i = n_1 \cdots n_i$ for $i = 1, \dots, g$. Hence $\gamma_i = (H_{i-1}/M_i)L_i$ for $i = 1, \dots, g$.

It follows from (8) that $N_i = \mathbf{Z}^d + \mathbf{Z}h_1 + \cdots + \mathbf{Z}h_i = \mathbf{Z}^d + \mathbf{Z}\tilde{\gamma}_1 + \cdots + \mathbf{Z}\tilde{\gamma}_i$. Since $H_g = n$ and $\gamma_i = n\tilde{\gamma}_i$ for $i = 1, \dots, g$, we get $W_i = nN_i$ for $i = 0, \dots, g$. This gives the arithmetic conditions $[W_i : W_{i-1}] = [N_i : N_{i-1}] = n_i = H_i/H_{i-1}$ for $i = 1, \dots, g$.

It remains to show that $\frac{1}{M_1}L_1 < \frac{1}{M_2}L_2 < \cdots < \frac{1}{M_g}L_g$. Each inequality $(1/M_{i-1})L_{i-1} < (1/M_i)L_i$ can be written in equivalent form $n_{i-1}\gamma_{i-1} < \gamma_i$ which by (8) is equivalent to $h_{i-1} < h_i$. Since characteristic exponents form an increasing sequence, this part of the proof is finished.

We proved that $\Delta_{T(f)}$, which is the Newton polytope of D_f, is an I-polytope.

Now, assume that $\Delta(D_f) = \sum_{i=1}^{g} \left\{\frac{L_i}{M_i}\right\}$ is an I-polytope. Let $n_i = H_i/H_{i-1}$ for $i = 1, \dots g$. Then n_i are integers bigger than 1 and $H_i = n_1 \cdots n_i$ for $i = 1, \dots, g$. We get $M_i = H_i - H_{i-1} = n_1 \cdots n_{i-1}(n_i - 1)$ and $L_i = (M_i/H_{i-1})\gamma_i = (n_i - 1)\gamma_i$ for $i = 1, \dots, g$.

Let $n = n_1 \cdots n_g$ and let $\tilde{\gamma}_i = (1/n)\gamma_i$ for $i = 1, \dots, g$. This time we use the recurrence relations (8) to define the sequence $h_1, \dots, h_g$. As in the first part of the proof we can show that if $N_i = \mathbf{Z}^d + \mathbf{Z}h_1 + \cdots + \mathbf{Z}h_i$ then $W_i = nN_i$. This gives $[N_i : N_{i-1}] = [W_i : W_{i-1}] = n_i > 1$ for $i = 1, \dots, g$. Therefore, $N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_g$.

Again, as in the first part of the proof, we show that the inequalities $\frac{1}{M_1}L_1 < \frac{1}{M_2}L_2 < \cdots < \frac{1}{M_g}L_g$ are equivalent to the inequalities $h_1 < h_2 < \cdots < h_g$. We have shown that $h_1, \dots, h_g$ is a sequence of characteristic exponents of some irreducible quasi-ordinary Weierstrass polynomial f_1. By construction of this sequence and by (9) we get $\Delta_{T(f_1)} = \Delta_{T(f)}$. Hence by Theorem 7.1 f is an irreducible quasi-ordinary Weierstrass polynomial. ∎

Kiyek and Micus [17] introduced the *semigroup* of an irreducible quasi-ordinary hypersurface $f(Y)=0$. Later, González Pérez and Popescu-Pampu introduced again the semigroup in their thesis [13, 26], using different but equivalent definitions. This is the semigroup $\deg f\mathbf{Z}^d_{\geq 0}+\mathbf{Z}_{\geq 0}\gamma_1+\cdots+\mathbf{Z}_{\geq 0}\gamma_g$, where $\gamma_1,\ldots,\gamma_g$ is the sequence defined in Theorem 8.1.

Since the Newton polytope $\Delta(D_f)$, for an irreducible quasi-ordinary polynomial $f(Y)$, determines its semigroup, it also determines the sequence of characteristic exponents (see [13, 26]). Observe that the proof of Theorem 7.1 gives us the sequence of characteristic exponents by using the equalities (8).

Example 8.2 ([2, Example 1])**.** Consider $f_1(Y)=Y^8-2X_1X_2Y^4+X_1^2X_2^2-X_1^3X_2^2\in\mathbf{K}[[X_1,X_2]][Y]$. We get $D_{f_1}(X_1,X_2,V)=-16777216(V-X_1^2X_2^2+X_1^3X^2)^3(V+X_1^3X_2^2)^4$, so

$$\Delta(D_{f_1})=3\left\{\frac{(2,2)}{1}\right\}+4\left\{\frac{(3,2)}{1}\right\}=\left\{\frac{(6,6)}{3}\right\}+\left\{\frac{(12,8)}{4}\right\}.$$

We get $H_0=1$, $H_1=4$, $H_2=8$, $\gamma_1=(2,2)$, and $\gamma_2=(12,8)$. We have $[W_1:W_0]=4=H_1/H_0$ and $[W_2:W_1]=2=H_2/H_1$, and we deduce that f_1 is irreducible. $\square$

Example 8.3 ([2, Example 2])**.** Consider $f_2(Y)=Y^8-2X_1X_2Y^4+X_1^2X_2^2-X_1^4X_2^2-X_1^5X_2^3\in\mathbf{K}[[X_1,X_2]][Y]$. We get $D_{f_2}(X_1,X_2,V)=-16777216(V-X_1^2X_2^2+X_1^4X_2^2+X_1^5X_2^3)^3(V+X_1^4X_2^2+X_1^5X_2^3)^4$, so

$$\Delta(D_{f_2})=3\left\{\frac{(2,2)}{1}\right\}+4\left\{\frac{(4,2)}{1}\right\}=\left\{\frac{(6,6)}{3}\right\}+\left\{\frac{(16,8)}{4}\right\}.$$

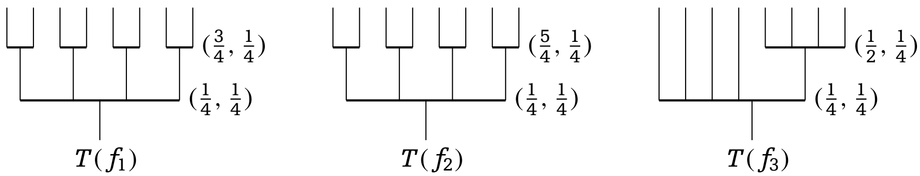
We get $H_0=1$, $H_1=4$, $H_2=8$, $\gamma_1=(2,2)$, and $\gamma_2=(16,8)$. We have $[W_1:W_0]=4=H_1/H_0$ but $[W_2:W_1]=1\neq 2=H_2/H_1$, and we deduce that f_2 is not irreducible. $\square$

Example 8.4. This is Example 3 in [2]. (There is a typo in the equation of this example in [2]. Assi communicated to us the right equation of this example.) Consider $f_3(Y)=Y^8-2X_1X_2Y^4+X_1^3X_2^2-X_1^3X_2^5\in\mathbf{K}[[X_1,X_2]][Y]$. We get $D_{f_3}=-16777216(V+X_1^2X_2^2-X_1^3X_2^2+X_1^3X_2^5)^4(V-X_1^3X_2^2+X_1^3X_2^5)^3$, so

$$\Delta(D_{f_3})=4\left\{\frac{(2,2)}{1}\right\}+3\left\{\frac{(3,2)}{1}\right\}=\left\{\frac{(8,8)}{4}\right\}+\left\{\frac{(9,6)}{3}\right\}.$$

We get $H_0=1$, $H_1=5$, $H_2=8$. Thus H_2/H_1 is not an integer number, so $[W_2:W_1]\neq H_2/H_1$ and we deduce that f_3 is not irreducible. $\square$

Remark 8.5. In general $\Delta(D_f)$ does not determine $T(f)$ as shown in [6] and [21, Proposition 2.2]. But in the above examples it does. To obtain the tree models it is enough to remember that $\Delta(D_f) = \Delta_{T(f)}$ and use Definition 3.1 and Theorem 5.5. The appropriate tree models with indicated heights of bars are drawn below:



□

9 Discriminant of a Y-Regular Power Series

In this section, we generalize the notion of the discriminant $D_f(\underline{X}, V)$, which was previously defined for Weierstrass polynomials, to an arbitrary Y*-regular power* series.

We say that a power series $f(\underline{X}, Y) \in \mathbf{K}[[\underline{X}, Y]]$ is Y-regular of order n if $f(0, Y) = cY^n +$ *higher order terms* with $c \neq 0$.

Assume that $f \in \mathbf{K}[[\underline{X}, Y]]$ is Y-regular of order n. By Weierstrass preparation theorem for every $g \in \mathbf{K}[[\underline{X}, Y, V]]$ there exist a unique $q \in \mathbf{K}[[\underline{X}, Y, V]]$ and $a_0, \dots, a_{n-1} \in \mathbf{K}[[\underline{X}, V]]$ such that

$$g = (f - V)q + \sum_{i=0}^{n-1} a_i Y^i.$$

It follows that the quotient ring $A = \mathbf{K}[[\underline{X}, Y, V]]/(f - V)$ is a free $\mathbf{K}[[\underline{X}, V]]$-module which admits the basis $1, \bar{Y}, \dots, \bar{Y}^{n-1}$, where $\bar{Y}$ is the coset of Y in A. Let $\Phi_g : A \to A$ be the $\mathbf{K}[[\underline{X}, V]]$-endomorphism induced by the multiplication $\mathbf{K}[[\underline{X}, Y, V]] \ni h \to gh \in \mathbf{K}[[\underline{X}, Y, V]]$. We put by definition $\mathbf{D}_f(\underline{X}, V) = \det \Phi_{\frac{\partial f}{\partial Y}}$.

Property 9.1.

(i) If $f(\underline{X}, Y)$ is a Weierstrass polynomial in the variable Y then $\mathbf{D}_f(\underline{X}, V)$ is equal to $D_f(\underline{X}, V)$.

(ii) $\mathbf{D}_f(\underline{X}, V)$ belongs to the ideal $I = (f - V, \frac{\partial f}{\partial Y})\mathbf{K}[[\underline{X}, Y, V]]$. Moreover, the radicals of the ideals $(\mathbf{D}_f)\mathbf{K}[[\underline{X}, V]]$ and $I \cap \mathbf{K}[[\underline{X}, V]]$ are the same.

(iii) Let $g(T, Y) = f(T^{c_1}, \dots, T^{c_d}, Y)$. Then $\mathbf{D}_g(T, V) = \mathbf{D}_f(T^{c_1}, \dots, T^{c_d}, V)$.

(iv) If $f(X, Y) \in \mathbf{K}[[X, Y]]$ is a Y-regular power series in two variables and $\frac{\partial f}{\partial Y}(X, Y) = u(X, Y) \prod_{i=1}^{n-1} [Y - Y_i(X)]$ is a Newton–Puiseux factorization of its partial derivative then $\mathbf{D}_f(X, V) = u'(X, V) \prod_{i=1}^{n-1} [f(X, Y_i(X)) - V]$ where $u'(X, V)$ is a unity in $\mathbf{K}[[X, V]]$.

(v) If $f(X,Y)\in \mathbf{C}\{X,Y\}$ then $\mathbf{D}_f(u,v)=0$ is an equation of the discriminant curve of the holomorphic mapping germ $(\mathbf{C}^2,0)\to(\mathbf{C}^2,0)$, $(u,v)=(x,f(x,y))$ in the sense of Casas-Alvero [5]. □

Proof.

(i) Let n be the Y-degree of f. Then the Y-discriminant of $f-V$ is the determinant of the matrix of $\Phi_{\frac{\partial f}{\partial Y}}$ with respect to the basis $1,\bar{Y},\dots,\bar{Y}^{n-1}$ (see [3, Appendix D.3.6]).

(ii) The mapping $\Phi:=\Phi_{\frac{\partial f}{\partial Y}}$ induces the exact sequence

$$A\xrightarrow{\Phi} A\longrightarrow \mathbf{K}[[\underline{X},Y,V]]/I\longrightarrow 0.$$

By definition (see [15, Section 7.2]), $(\mathbf{D}_f)\mathbf{K}[[\underline{X},V]]$ is the 0th Fitting ideal of the $\mathbf{K}[[\underline{X},V]]$-module $\mathbf{K}[[\underline{X},Y,V]]/I$. On the other hand, $I\cap\mathbf{K}[[\underline{X},V]]$ is the annihilator of $\mathbf{K}[[\underline{X},Y,V]]/I$. By Proposition 20.7 of [7, p. 494] (see also [15, Exercise 7.2.5, p. 388]), we get the equality of the radicals.

(iii) Suppose that f is Y-regular of order n. If

$$Y^i\frac{\partial f}{\partial Y}=\sum_{j=0}^{n-1}m_{ij}(X_1,\dots,X_d,V)Y^j+h_i(\underline{X},Y,V)(f(\underline{X},Y)-V)$$

then

$$Y^i\frac{\partial g}{\partial Y}=\sum_{j=0}^{n-1}m_{ij}(T^{c_1},\dots,T^{c_d},V)Y^j+h_i(T^c,Y,V)(g(T,Y)-V).$$

These relations, for $i=0,\dots,n-1$, imply that $\mathbf{D}_f(T^{c_1},\dots,T^{c_d},V)=\det(m_{ij}(T^{c_1},\dots,T^{c_d},V)_{n\times n})$ is equal to $\mathbf{D}_g(T,V)$.

(iv) Suppose that $Y_i(X)$ are power series for $i=1,\dots,n-1$. Since $\Phi_{gh}=\Phi_g\circ\Phi_h$ we get $\mathbf{D}_f(X,V)=\det\Phi_{\frac{\partial f}{\partial Y}}=\det\Phi_{u(X,Y)}\prod_{i=1}^{n-1}\det\Phi_{Y-Y_i(X)}$. Moreover $\det\Phi_u\cdot\det\Phi_{u^{-1}}=\det(\mathrm{id})=1$. The substitution of $Y_i(X)$ for Y determines an isomorphism between the $\mathbf{K}[X,V]$-modules $\mathbf{K}[[X,Y,V]]/(f(X,Y)-V,Y-Y_i(X))$ and $\mathbf{K}[[X,V]]/(f(X,Y_i(X))-V)$. Hence the ideal generated by $\det\Phi_{Y-Y_i(X)}$, which is the 0-Fitting ideal of both modules, is equal to $(f(X,Y_i(X))-V)\mathbf{K}[[X,V]]$. The proof in this case is finished.

Let us consider the general situation. There exists a natural number m such that $\frac{\partial f}{\partial Y}(T^m,Y)=u(T^m,Y)\prod_{i=1}^{n-1}(Y-Y_i(T^m))$ is a factorization in $\mathbf{K}[[T, Y]]$. Using (iii) and applying (iv), in the case proved before, to $g(T, Y) := f(T^m, Y)$ we get

$$\mathbf{D}_f(T^m, V) = \mathbf{D}_g(T, V) = u'(T, V) \prod_{i=1}^{n-1} (f(T^m, Y_i(T^m)) - V). \tag{10}$$

By definition $\mathbf{D}_f(T^m, V) \in \mathbf{K}[[T^m, V]]$. Denote by $P(T, V)$ the product $\prod_{i=1}^{n-1}(f(T^m, Y_i(T^m)) - V)$ appearing in (10). Let $\epsilon \in \mathbf{K}$ be an mth primitive root of unity. Since $Y_i(T^m) \to Y_i((\epsilon T)^m)$ is a permutation of the roots of the derivative of g, we have $P(\epsilon T, V) = P(T, V)$, and consequently $P(T, V) \in \mathbf{K}[[T^m, V]]$.

We claim that $u'(T, V) = u''(T^m, V)$ for some $u'' \in \mathbf{K}[[X, V]]$. Indeed substituting ϵT for T in (10) we get $u'(\epsilon T, V) = u'(T, V)$ which shows that $u'(T, V) \in \mathbf{K}[[T^m, V]]$. We get $\mathbf{D}_f(X, V) = u''(X, V) \prod_{i=1}^{n-1}(f(X, Y_i(X)) - V)$.

(v) The formula in (iv) determines the equation of the discriminant curve in the sense of Casas-Alvero (see [11, Lemma 5.4] in Appendix). ■

Remark that $\mathbf{D}_f(\underline{X}, V)$ extends, in a natural way, the definition of $D_f(\underline{X}, V)$.

Theorem 9.2. Let $f_1(\underline{X}, Y) \in \mathbf{K}[[\underline{X}]][Y]$ be a Weierstrass polynomial and let $f_2(\underline{X}, Y) = u(\underline{X}, Y) f_1(\underline{X}, Y)$, where $u(\underline{X}, Y)$ is a unit in $\mathbf{K}[[\underline{X}, Y]]$. Then the Newton polytopes of D_{f_1} and $\mathbf{D}_{f_2}$ are equal. □

Proof. Consider the substitution $g_i(T, Y) = f_i(T^{c_1}, \ldots, T^{c_d}, Y)$ for $i = 1, 2$. Later on we assume that $c_j \geq \deg f_1$ for $j = 1, \ldots, d$.

By item (i) of Property 9.1 we have $\mathbf{D}_{f_1} = D_{f_1}$ and $\mathbf{D}_{g_1} = D_{g_1}$. By Corollary 5.3 in [10] and Property 9.1(v) we get $\Delta(\mathbf{D}_{g_1}) = \Delta(\mathbf{D}_{g_2})$. In [10] the above equality was proved in the convergent power series case. Anyway the methods in [10] also work for formal power series.

We finish the proof proceeding as in the proof of Theorem 4.1 replacing $\Delta(D_f)$ by $\Delta(\mathbf{D}_{f_2})$, $\Delta(D_g)$ by $\Delta(\mathbf{D}_{g_2})$, $\Delta_{T(f)}$ by $\Delta(\mathbf{D}_{f_1})$, and $\Delta_{T(g)}$ by $\Delta(\mathbf{D}_{g_1})$. The only difference is that we need to choose a vector $c = (c_1, \ldots, c_d, c_{d+1})$ more carefully to assure that the hyperplanes $H_i = \{x \in \mathbf{R}^{d+1} : \langle c, x\rangle = l(c, \Delta(\mathbf{D}_{f_i}))\}$ support the Newton polyhedra $\Delta(\mathbf{D}_{f_i})$ at exactly one point, for $i = 1, 2$. ■

Corollary 9.3. Let $w(Y)$ be the Weierstrass polynomial of a Y-regular power series $f \in \mathbf{K}[[X_1, \ldots, X_d, Y]]$. Then the following conditions are equivalent:

(i) the polynomial $w(Y)$ is quasi-ordinary,
(ii) the polytope $\Delta(D_w) \cap \mathbf{R}^d \times \{0\}$ has only one vertex,
(iii) the polytope $\Delta(\mathbf{D}_f) \cap \mathbf{R}^d \times \{0\}$ has only one vertex,
(iv) $\mathbf{D}_f(\underline{X}, 0) = u(\underline{X}) \cdot$ monomial, where $u(0) \neq 0$. □

Proof. The Newton polytope of a series $h \in \mathbf{K}[[\underline{X}]]$ has only one vertex if and only if h has a form $u(\underline{X}) \cdot$ *monomial*, where $u(0) \neq 0$. Since $\Delta(D_w) \cap \mathbf{R}^d \times \{0\}$ is the Newton polytope of $D_w(\underline{X}, 0)$ and likewise $\Delta(\mathbf{D}_f) \cap \mathbf{R}^d \times \{0\}$ is the Newton polytope of $\mathbf{D}_f(\underline{X}, 0)$, we get equivalences (i)⇔(ii) and (iii)⇔(iv). The equivalence (ii)⇔(iii) follows from Theorem 9.2. ■

We call a *Y-regular power series f quasi-ordinary* if it satisfies any of equivalent conditions (i)–(iv) of Corollary 9.3. We follow here Lipman who used (i) in [23] as a definition of quasi-ordinary convergent power series with complex coefficients.

Using Theorem 9.2 we may generalize main results of this paper, that is: Theorem 4.1, Corollary 4.5, Theorems 7.1 and 8.1, to Y-regular quasi-ordinary power series.

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Appendix A. Computing Indices

Let $M \subset N$ be lattices in $\mathbf{Z}^d$, that is, additive subgroups of $\mathbf{Z}^d$. In this appendix, we recall a method of computing the *index* of M in N. By definition the index $[N : M]$ is the cardinality of the quotient group N/M. Since $[\mathbf{Z}^d : N] \cdot [N : M] = [\mathbf{Z}^d : M]$ it is enough to compute $[\mathbf{Z}^d : M]$ and $[\mathbf{Z}^d : N]$. The next theorem says how to do it by means of determinants.

Theorem A.1. Let $M = \mathbf{Z}v_1 + \cdots + \mathbf{Z}v_n$ be a sub-lattice of $\mathbf{Z}^d$ of finite index. Then $[\mathbf{Z}^d : M]$ is the greatest common divisor of minors of maximal size of the matrix build from vectors $v_1, \ldots, v_n$. □

Proof. Let $\phi : \mathbf{Z}^n \to \mathbf{Z}^d$ be a group homomorphism given by $\phi(m_1, \ldots, m_n) = m_1 v_1 + \cdots + m_n v_n$. Since every abelian group can be considered as a $\mathbf{Z}$ module, this homomorphism induces the exact sequence of $\mathbf{Z}$ modules

$$\mathbf{Z}^n \xrightarrow{\phi} \mathbf{Z}^d \to \mathbf{Z}^d/M \to 0.$$

As in linear algebra we can associate with the mapping ϕ the matrix A_ϕ whose columns are the vectors $v_1, \ldots, v_n$. The ideal generated in $\mathbf{Z}$ by the minors of maximal size of A_ϕ is by definition the 0th Fitting ideal of the $\mathbf{Z}$ module $\mathbf{Z}^d/M$.

To complete the proof it is enough to show a general statement: for every finite abelian group B, treated as a $\mathbf{Z}$ module, the number of elements of B is the generator of the 0th Fitting ideal of B.

By the structure theorem for finitely generated abelian groups, B is isomorphic to the direct sum $\mathbf{Z}/q_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/q_s\mathbf{Z}$ for some $q_1, \ldots, q_s \in \mathbf{Z}$. Thus B, treated as a $\mathbf{Z}$ module, allows a finite presentation

$$\mathbf{Z}^s \xrightarrow{\phi} \mathbf{Z}^s \to B \to 0$$

where $\phi(n_1, \ldots, n_s) = (q_1 n_1, \ldots, q_s n_s)$. Since A_ϕ is a square matrix, its determinant is the only minor of the maximal size. Thus the 0th Fitting ideal of B is generated by $\det A_\phi$. Notice that the determinant of a diagonal matrix A_ϕ is equal to the product $q_1 \cdots q_s$ which is the cardinality of B. ∎

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