



The valuative tree is the projective limit of Eggers-Wall trees

Evelia R. García Barroso¹ · Pedro D. González Pérez² ·
Patrick Popescu-Pampu³

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Abstract

Consider a germ C of reduced curve on a smooth germ S of complex analytic surface. Assume that C contains a smooth branch L . Using the Newton-Puiseux series of C relative to any coordinate system (x, y) on S such that L is the y -axis, one may define the *Eggers-Wall tree* $\Theta_L(C)$ of C relative to L . Its ends are labeled by the branches of C and it is endowed with three natural functions measuring the characteristic exponents of the previous Newton-Puiseux series, their denominators and contact orders. The main objective of this paper is to embed canonically $\Theta_L(C)$ into Favre and Jonsson's valuative tree $\mathbb{P}(\mathcal{V})$ of real-valued semivaluations of S up to scalar multiplication, and to show that this embedding identifies the three natural functions on $\Theta_L(C)$ as pullbacks of other naturally defined functions on $\mathbb{P}(\mathcal{V})$. As a consequence, we generalize the well-known *inversion theorem* for one branch: if L' is a second smooth branch of C , then the valuative embeddings of the Eggers-Wall trees $\Theta_{L'}(C)$ and $\Theta_L(C)$ identify them canonically, their associated triples of functions being easily expressible in terms of each other. We prove also that the space $\mathbb{P}(\mathcal{V})$ is the projective limit of Eggers-Wall trees over all choices of curves C . As a supplementary result, we explain how to pass from $\Theta_L(C)$ to an associated splice diagram.

Keywords Branch · Characteristic exponent · Contact · Eggers-Wall tree · Newton-Puiseux series · Plane curve singularities · Semivaluation · Splice diagram · Rooted tree · Valuation · Valuative tree

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This paper is dedicated to Felipe Cano and András Némethi.

✉ Patrick Popescu-Pampu
patrick.popescu-pampu@univ-lille.fr
<http://math.univ-lille1.fr/~popescu/>

Extended author information available on the last page of the article

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1 Introduction

In their seminal 2004 book “*The valuative tree*” [11], Favre and Jonsson studied the space of real-valued semivaluations \mathcal{V} on a germ S of smooth complex analytic surface. They proved that the projectivization $\mathbb{P}(\mathcal{V})$ of \mathcal{V} is a compact real tree, called the *valuative tree* of the surface singularity S . They gave several viewpoints on $\mathbb{P}(\mathcal{V})$: as a partially ordered set of normalized semivaluations, as a space of irreducible Weierstrass polynomials and as a universal dual graph of modifications of S .

The main objective of this paper is to present $\mathbb{P}(\mathcal{V})$ as a “*universal Eggers-Wall tree*”, relative to any *smooth* reference branch (that is, germ of irreducible curve) L on S . Namely, we show that $\mathbb{P}(\mathcal{V})$ is the projective limit of the *Eggers-Wall trees* $\Theta_L(C)$ of the reduced germs of curves C on S which contain L .

Given such a germ C , let (x, y) be a coordinate system verifying that L is the y -axis. The tree $\Theta_L(C)$ is rooted at an end labeled by L and its other ends are labeled by the remaining branches of C . Consider the Newton-Puiseux series $(\eta_i(x))_i$ of these branches of C . The tree $\Theta_L(C)$ has marked points corresponding to the characteristic exponents of the series $\eta_i(x)$ and it is endowed with three natural functions: the *exponent* \mathbf{e}_L , the *index* \mathbf{i}_L and the *contact complexity* \mathbf{c}_L (see Definitions 3.9 and 3.19). These functions determine the *equisingularity class* of the germ C with chosen branch L , that is, the oriented topological type of the triple (S, C, L) . In order to emphasize this property, we explain how to get from $\Theta_L(C)$ the minimal splice diagram of C in the sense of Eisenbud and Neumann (see Sect. 5).

The branch L may be seen as an *observer*, defining a coordinate system $(\mathbf{e}_L, \mathbf{i}_L, \mathbf{c}_L)$ on $\Theta_L(C)$. Analogously, an *observer* in the valuative tree $\mathbb{P}(\mathcal{V})$ is either the special point of S , or a smooth branch L , identified with a suitable semivaluation on it. Each observer R determines three functions on the valuative tree, the *log-discrepancy* \mathbf{l}_R , the *self-interaction* \mathbf{s}_R and the *multiplicity* \mathbf{m}_R relative to R (see Definitions 7.4 and 7.14). If one identifies the valuative tree $\mathbb{P}(\mathcal{V})$ with the subspace of \mathcal{V} consisting of those semivaluations which take the value 1 on the ideal defining the observer R , then the functions $\mathbf{l}_R, \mathbf{m}_R, \mathbf{s}_R$ appear as restrictions of functions defined globally on the space of semivaluations.

We describe an embedding of the Eggers-Wall tree $\Theta_L(C)$ inside the valuative tree $\mathbb{P}(\mathcal{V})$. This embedding transforms the exponent plus one $\mathbf{e}_L + 1$ into the log-discrepancy \mathbf{l}_L , the index \mathbf{i}_L into the multiplicity \mathbf{m}_L and the contact complexity \mathbf{c}_L into the self-interaction \mathbf{s}_L (see Theorem 8.11). Our embedding is defined explicitly in terms of Newton-Puiseux series, and is similar to Berkovich’s construction of seminorms on the polynomial ring $K[X]$ extending a given complete non-Archimedean absolute value on a field K , done by maximizing over closed balls of K (see Remark 8.2). Theorem 8.11 generalizes a result of Favre and Jonsson, for a *generic* Eggers-Wall tree relative to the special point (see [11, Prop. D1, page 223]).

If the germ of curve C is contained in another reduced germ C' , then we get a retraction from $\Theta_L(C')$ to $\Theta_L(C)$. These retractions provide an inverse system of continuous maps and we prove, as announced above, that their projective limit is homeomorphic to the valuative

tree $\mathbb{P}(\mathcal{V})$ (see Theorem 8.18 and Corollary 8.20). This is the result alluded to in the title of the paper. It complements the fact that the valuative tree is the projective limit of the dual graphs of all models of S , proved by Favre and Jonsson [11, Theorems 6.22, 6.50], and generalized to higher dimensions by Boucksom, Favre and Jonsson in [5, Theorem 1.13].

We study in which way the triple of functions $(\mathbf{l}_R, \mathbf{s}_R, \mathbf{m}_R)$ changes when the observer R is replaced by another one R' . We provide explicit formulas for this *change of variables* in Propositions 9.1, 9.3 and 9.4. As an application, we prove an *inversion theorem* which shows how to pass from the Eggers-Wall tree $\Theta_L(C)$ relative to a smooth branch L of C to the tree $\Theta_{L'}(C)$ relative to another smooth branch L' of C . Our theorem means that the geometric realization of the Eggers-Wall tree, with the ends labeled by the branches of C , remains unchanged and that one only has to replace the triple of functions $(\mathbf{e}_L, \mathbf{c}_L, \mathbf{i}_L)$ by $(\mathbf{e}_{L'}, \mathbf{c}_{L'}, \mathbf{i}_{L'})$ (see Theorem 4.5). If L and L' are transversal, our result is a geometrization and generalization to the case of several branches of the classical *inversion theorem*, which expresses the characteristic exponents of a branch with respect to a coordinate system (y, x) in terms of those with respect to (x, y) . Abhyankar [1, Theorem 1] and Zariski ([42] Sect. 3) gave proofs of this theorem. In fact, Halphen [21] and Stolz [39] already knew it in the years 1870, as explained in [17]. Our approach, passing by the embeddings of the Eggers-Wall trees in the space of semivaluations, provides a conceptual understanding of these results.

Let us describe briefly the structure of the paper. In Section 2 we state the basic definitions and notions about finite trees and real trees used in the rest of the paper. In Sect. 3 we introduce the definitions of the Eggers-Wall tree and of the exponent, index and contact complexity functions. In Sect. 4 we give the statement of our inversion theorem for Eggers-Wall trees and we prove it using results of later sections. In Sect. 5 we recall basic facts about splice diagrams of links in oriented integral homology spheres of dimension 3 and we explain how to transform the Eggers-Wall tree $\Theta_L(C)$ into the minimal splice diagram of the link of C inside the 3-sphere. The spaces of valuations and semivaluations which play a relevant role in the paper are introduced in Sect. 6. The multiplicity, the log-discrepancy and the self-interaction functions on the valuative tree are introduced in Sect. 7. In Section 8, we prove that the Eggers-Wall trees $\Theta_L(C)$ embed naturally in the valuative tree and we deduce that the valuative tree identifies canonically to the projective limit of Eggers-Wall trees. Moreover, we compute the log-discrepancy and self-interaction of the divisorial valuations corresponding to a rational point P of $\Theta_L(C)$ under this embedding, in terms of $\mathbf{e}_L(P)$, $\mathbf{c}_L(P)$, $\mathbf{i}_L(P)$. In Sect. 9 we describe how the coordinate functions on the valuative tree vary when we change the observer. Finally, in Sect. 10 we give a repertory of the main formulae of the paper, in order to help the reader getting a global vision of them.

2 Finite trees and \mathbb{R} -trees

In this section we introduce the basic vocabulary about *finite trees* used in the rest of the text. Then we define \mathbb{R} -trees, which are more general than finite trees. Our main sources are [11, 23] and [29], even if we do not follow exactly their terminology. We define attaching maps from ambient \mathbb{R} -trees to subtrees (see Definition 2.11) and we recall a criterion which allows to see a given compact \mathbb{R} -tree as the projective limit of convenient families of finite subtrees, when they are connected by the associated attaching maps (see Theorem 2.14). This criterion will be crucial in order to prove in Sect. 8 the theorem stated in the title of the paper.

Intuitively, the finite trees are the connected finite graphs without circuits. As is the case also for graphs, the intuitive idea of tree gets incarnated in several categories: there are

combinatorial, (piecewise) affine and topological trees, with or without a root. Combinatorial trees are special types of abstract simplicial complexes:

Definition 2.1 A finite combinatorial tree \mathcal{T} is formed by a finite set $V(\mathcal{T})$ of vertices and a set $E(\mathcal{T})$ of subsets with two elements of $V(\mathcal{T})$, called edges, such that for any pair of vertices, there exists a unique chain of pairwise distinct edges joining them. The valency $v(P)$ of a vertex P is the number of edges containing it. A vertex P is called a ramification point of \mathcal{T} if $v(P) \geq 3$ and an end vertex (or simply an end) if $v(P) = 1$.

Recall that a geometric realization of a finite combinatorial tree \mathcal{T} in a real vector space W is determined by a bijection $v \mapsto e_v$ from $V(\mathcal{T})$ to a basis of V , by associating to every edge $\{v, v'\} \in E(\mathcal{T})$ the segment $[e_v, e_{v'}] \subset W$.

As a particular case of the general construction performed on any finite abstract simplicial complex, each finite combinatorial tree has a unique geometric realization up to a unique homeomorphism extending the identity on the set of vertices and affine on the edges, which will be called a finite affine tree. If we consider a finite affine tree only up to homeomorphisms, we get the notion of finite topological tree:

Definition 2.2 A topological space homeomorphic to a finite affine tree is called a finite topological tree or, simply, a finite tree. The interior of a finite tree is the set of its points which are not ends. A finite subtree of a given tree is a topological subspace homeomorphic to a finite tree.

The simplest finite trees are reduced to points. Any finite tree is compact. Only the ramification points and the end vertices are determined by the underlying topology. One has to mark as special points the vertices of valency 2 if one wants to remember them. Therefore, we will speak in this case about marked finite trees, in order to indicate that one gives also the set of vertices, which contains, possibly in a strict way, the set of ramification points and of ends. By definition, a subtree \mathcal{T}' of a marked finite tree \mathcal{T} is a finite subtree of the underlying topological space of \mathcal{T} such that its ends are marked points of \mathcal{T} , and its marked points are the marked points of \mathcal{T} belonging to \mathcal{T}' .

A (compact) segment in a finite tree is a connected subset which is homeomorphic to a (compact) real interval. Each pair of points $P, Q \in \mathcal{T}$ is the set of ends of exactly one compact segment, denoted $[P, Q] = [Q, P]$. We speak also about the half-compact and the open segments $(P, Q), [P, Q), (P, Q)$, where $[P, Q) := [P, Q] \setminus \{Q\}$, etc.

We will often deal with sets equipped with a partial order, which are usually called posets. The next definition explains how the choice of a root for a tree endows it with a structure of poset:

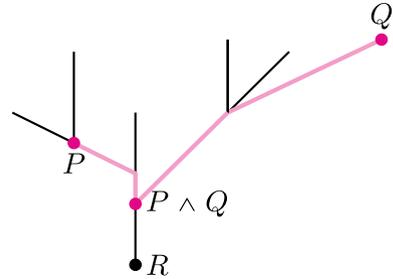
Definition 2.3 A finite rooted tree is a finite (affine or topological) tree with a marked vertex, called the root. In such a tree \mathcal{T} , the ends which are different from the root are called the leaves of \mathcal{T} . If the root is also an end, we say that \mathcal{T} is end-rooted. Each rooted tree with root R may be canonically endowed with a partial order \leq_R in the following way:

$$P \leq_R Q \Leftrightarrow [R, P] \subseteq [R, Q].$$

Each finite marked rooted tree may be seen as a genealogical tree, the individuals with a common ancestor corresponding to the vertices, the elementary filiations to the edges and the common ancestor to the root:

Definition 2.4 Let \mathcal{T} be a marked finite rooted tree, with root R . For each vertex P of \mathcal{T} different from R , its parent $\mathfrak{P}(P)$ is the greatest vertex of \mathcal{T} on the segment $[RP)$. If we define $\mathfrak{P}(R) = R$, we get the parent map $\mathfrak{P} : V(\mathcal{T}) \rightarrow V(\mathcal{T})$.

Fig. 1 The infimum of two elements in a rooted tree



One may generalize in the following way the notion of finite rooted tree by keeping some of the properties of the associated partial order relation:

Definition 2.5 A *rooted \mathbb{R} -tree* is a poset (\mathcal{T}, \preceq) such that:

- (1) There exists an absolute minimum $R \in \mathcal{T}$ (called the *root*).
- (2) For any $P \in \mathcal{T}$, the set $\{Q \in \mathcal{T} \mid Q \preceq P\}$ is isomorphic as a poset to a compact interval of \mathbb{R} (reduced to a point when $P = R$).
- (3) Any totally ordered convex subset of \mathcal{T} is isomorphic as a poset to an interval of \mathbb{R} (a subset K of a poset (P, \preceq) is called *convex* if $c \in K$ whenever $a \preceq c \preceq b$ and $a, b \in K$).
- (4) Every non-empty subset K of \mathcal{T} has an infimum, denoted $\bigwedge_{P \in K} P$.

The rooted \mathbb{R} -tree \mathcal{T} is *complete* if any increasing sequence has an upper bound.

Every finite rooted tree \mathcal{T} is a complete rooted \mathbb{R} -tree, if one works with the partial order \preceq_R determined by its root R (see Definition 2.3).

Remark 2.6 We took Definition 2.5 from Novacoski’s paper [29], where this notion is called instead *rooted non-metric \mathbb{R} -tree*. In fact, Novacoski proved that under the hypothesis that conditions (1) and (2) are both satisfied, the fourth one is equivalent to the condition that any *two* elements have an infimum (see [29, Lemma 3.4]). He emphasized the fact that condition (4) is not implied by the previous ones, because of a possible phenomenon of *double point*. Glue for instance by the identity map along $[0, 1)$ two copies of the segment $[0, 1]$, endowed with the usual order relation on real numbers. One gets then a poset satisfying conditions (1)–(3) but not condition (4). Indeed, the two images of the number 1 do not have an infimum. This subtlety was missed in the book [11], in which an analog of Definition 2.5 was formulated, under the name *rooted nonmetric tree*, only by the conditions (1)–(3) (see [11, Definition 3.1]). Property (4) was nevertheless heavily used in the proofs of [11]. Happily, this invalidates no result of the book, because Novacoski showed that the valuative trees studied by Favre and Jonsson satisfy also the fourth condition (see [29, Theorem 1.1]).

Let \mathcal{T} be a rooted \mathbb{R} -tree. If P, Q are any two points on it and if $P \wedge Q$ is their infimum (see Fig. 1), denote by $[P, Q]$ the *compact segment* joining them, defined by:

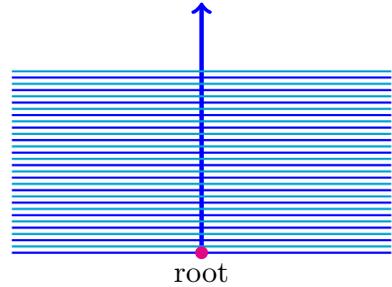
$$[P, Q] := \{A \in \mathcal{T} \mid P \wedge Q \preceq A \preceq P \text{ or } P \wedge Q \preceq A \preceq Q\}.$$

Obviously, $[P, Q]$ is equal to $[Q, P]$.

In the same way as one speaks about affine spaces, which are vector spaces with forgotten origin, we will need the notion of rooted tree with forgotten root:

Definition 2.7 An *\mathbb{R} -tree* is a rooted \mathbb{R} -tree with forgotten root. That is, it is an equivalence class of structures of rooted \mathbb{R} -tree on a fixed set, defining the same compact segments. If \mathcal{T}

Fig. 2 An example of \mathbb{R} -tree



is an \mathbb{R} -tree and $P \in \mathcal{T}$ is an arbitrary point of it, a *direction* at P is an equivalence class of the following equivalence relation \sim_P on $\mathcal{T} \setminus \{P\}$:

$$Q_1 \sim_P Q_2 \iff (P, Q_1] \cap (P, Q_2] \neq \emptyset.$$

The *weak topology* of the \mathbb{R} -tree \mathcal{T} is the minimal one such that all the directions at all points are open subsets of \mathcal{T} .

Any $P \in \mathcal{T}$ induces a partial order \leq_P , as in Definition 2.3. In this way, one recovers the *rooted* \mathbb{R} -tree structure on the set \mathcal{T} with root at P .

The number of directions at a point in a finite tree is equal to its valency. The notion of direction allows to extend to \mathbb{R} -trees \mathcal{T} the notion of ramification point. Namely, a point $P \in \mathcal{T}$ is a *ramification point* if there are at least three directions at P .

Remark 2.8 (1) Definition 2.7 is a reformulation of [11, Definition 3.5]. One may define also a notion of *complete* \mathbb{R} -tree as the equivalence class of a complete rooted \mathbb{R} -tree. This last notion may be defined differently, emphasizing the set of its compact segments (see Jonsson’s [23, Definition 2.2]).

- (2) In [11, Section 3.1.2] the term *tangent vector* is used instead of *direction*. We prefer this last term in order to emphasize the analogy with the usual euclidean space, in which two points Q_1 and Q_2 are said to be *in the same direction* as seen from an observer P if and only if the segments $(P, Q_1]$ and $(P, Q_2]$ are not disjoint.
- (3) Endowed with the weak topology, each \mathbb{R} -tree \mathcal{T} is Hausdorff (see [11, Lemma 7.2]). In that reference a few other tree topologies are defined and studied, but each time starting from supplementary structures on the \mathbb{R} -tree, for instance metrics. We will not need them in this paper.

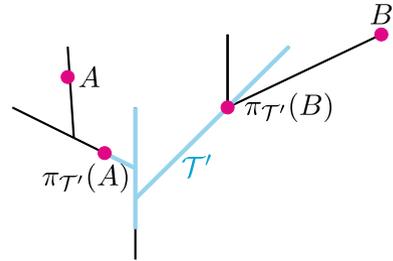
Let us illustrate the previous vocabulary by an example:

Example 2.9 Consider the set $\mathcal{T} := \mathbb{R} \times [0, \infty)$, endowed with the following partial order:

$$(x_1, y_1) \preceq (x_2, y_2) \iff \begin{cases} \text{either } x_1 = 0 \text{ and } y_1 \leq y_2, \\ \text{or } y_1 = y_2, |x_1| \leq |x_2| \text{ and } x_1 \cdot x_2 \geq 0. \end{cases}$$

Its structure is suggested in Fig. 2. This partial order endows \mathcal{T} with a structure of rooted \mathbb{R} -tree. Its root is the point $(0, 0)$. Notice that the segment $[(x_1, y_1), (0, 0)]$ of \mathcal{T} is the union of the segments $[(x_1, y_1), (0, y_1)]$ and $[(0, y_1), (0, 0)]$. The set of ramification points is the vertical half-axis $\{0\} \times [0, \infty)$. At each point of it there are 4 directions (up, down, right and left), with the exception of $(0, 0)$, at which there are only 3 of them (no down one).

Fig. 3 Attaching points on a subtree



Lemma 2.10 *Let T be an \mathbb{R} -tree and let T' be a \mathbb{R} -subtree of T , which is closed for the weak topology. For any $P \in T$, there exists a unique point $Q \in T'$ such that $[Q, P] \cap T' = \{Q\}$.*

This lemma, whose proof is left to the reader, says simply that if we take a point in a tree, then there is a unique minimal segment joining it to a given closed subtree. Note that $Q = P$ if and only if $P \in T'$.

Definition 2.11 We call the point Q characterized in Lemma 2.10 *the attaching point of P on T'* and we denote it $\pi_{T'}(P)$. The map $\pi_{T'} : T \rightarrow T'$ is the *attaching map* of the closed subtree T' .

Notice that the attaching map $\pi_{T'} : T \rightarrow T'$ is a *retraction* onto T' . Indeed:

$$\pi_{T'} \circ \pi_{T'} = \pi_{T'} \text{ and } \text{im}(\pi_{T'}) = T'.$$

Sometimes we consider surjective attaching maps, by replacing the target T by $\text{im}(\pi_{T'})$. The name we chose for $\pi_{T'}$ is motivated by the fact that we think of $\pi_{T'}(P)$ as the point where the smallest segment of T (for the inclusion relation) joining P to T' is *attached* to T' . In the Fig. 3 is represented a tree T and, with heavier lines, a closed subtree T' . We have also represented two points $A, B \in T$ and their attaching points $\pi_{T'}(A), \pi_{T'}(B)$ on T' .

One has the following property:

Lemma 2.12 *Let T be an \mathbb{R} -tree. Then for any $A, B, C \in T$ one has:*

$$\pi_{[A,B]}(C) = \pi_{[B,C]}(A) = \pi_{[A,C]}(B).$$

This point may also be characterized as the intersection of the segments joining pairwise the points A, B, C . If T is rooted at A , then the previous point is equal to $B \wedge C$.

Proof The constructions which allow to define the objects involved in this lemma can be done inside the finite \mathbb{R} -tree which is the union of the segments $[A, B]$, $[B, C]$, and $[C, A]$. Generically, when none of the three points lies on the segment formed by the other two, this tree has the shape of a star with three legs. Otherwise it is a segment. In any of these cases the assertion is clear.

Let us introduce a standard name for the \mathbb{R} -trees determined by three points:

Definition 2.13 If A, B, C are three points of an \mathbb{R} -tree, then the union of the segments $[A, B]$, $[B, C]$, $[C, A]$ is the *tripod* generated by them. Its *center* $\langle A, B, C \rangle$ is the point characterized in Lemma 2.12.

Note that $\langle A, B, C \rangle \in \{A, B, C\}$ if and only if the segment joining two of the points A, B, C contains the third point, which is then equal to $\langle A, B, C \rangle$. In particular, if two of the points are equal, then the center of the tripod coincides with them.

Finite trees are compact for the weak topology. One has the following characterization of the \mathbb{R} -trees which are also compact when endowed with the weak topology in the sense of Definition 2.7 (see [23, Section 2.1]):

Theorem 2.14 *Let T be an \mathbb{R} -tree. Let $(T_J)_{J \in \mathcal{F}}$ be a (possibly infinite) collection of finite subtrees of it. We assume that they form a projective system for the inclusion partial order, that is, for any $J, K \in \mathcal{F}$, there exists $M \in \mathcal{F}$ such that $T_J \subseteq T_M \supseteq T_K$. When $T_J \subseteq T_M$, denote by $\pi_J^M : T_M \rightarrow T_J$ the corresponding attaching map. Then:*

- (1) *the maps π_J^M form a projective system of continuous maps;*
- (2) *their projective limit $\varprojlim T_J$ is compact;*
- (3) *the attaching maps $\pi_J : T \rightarrow T_J$ glue into a continuous map $\pi : T \rightarrow \varprojlim T_J$;*
- (4) *if for any two distinct points $A, B \in T$, there exists a tree T_J such that $\pi_J(A) \neq \pi_J(B)$, then the map π is a homeomorphism onto its image.*
- (5) *T is compact if and only if π is a homeomorphism onto $\varprojlim T_J$.*

This theorem shows also that compact \mathbb{R} -trees may be studied using *sufficiently many* (in the sense of condition (4)) of their finite subtrees.

We will use Theorem 2.14 in order to prove Theorem 8.18, stated briefly in the title of this paper.

3 Curve singularities and their Eggers-Wall trees

In this section we explain the basic notations and conventions used throughout the paper about reduced germs C of curves on smooth surfaces. Then we define the Eggers-Wall tree of such a germ relative to a smooth branch contained in it (see Definition 3.9), as well as three natural real-valued functions defined on it, the exponent, the index and the contact complexity. The intersection numbers of the pairs of different branches of C may be expressed in terms of the values of the last two functions (see Corollary 3.26). Remark 3.18 contains historical comments about the notion of Eggers-Wall tree.

All over the text, S denotes a smooth germ of complex algebraic or analytic surface and O its special point. We denote by \mathcal{O} the formal local ring of S at O (the completion of the ring of germs at O of holomorphic functions on S), by \mathcal{K} its field of fractions, and by \mathcal{M} its maximal ideal.

A *branch* on S is a germ at O of formal irreducible curve drawn on S . A *divisor* on S is an element of the free abelian group generated by the branches on S . A divisor is called *effective* if it belongs to the free abelian monoid generated by the branches.

If $f \in \mathcal{K} \setminus \{0\}$, we denote by $Z(f)$ its *divisor*. This divisor is effective if and only if $f \in \mathcal{O}$. If D is an effective divisor, we denote by $\mathcal{O}(-D)$ the ideal of \mathcal{O} consisting of those functions which vanish along it. As S is smooth, this ideal is principal. Any generator of it is a *defining function* of D . The ring $\mathcal{O}_D := \mathcal{O}/\mathcal{O}(-D)$ is the local ring of D .

A *model* of S is a proper birational morphism $\psi : (\Sigma, E) \rightarrow (S, O)$, where Σ is a *smooth* surface and the restriction $\psi|_{\Sigma \setminus E} : \Sigma \setminus E \rightarrow S \setminus \{O\}$ is an isomorphism. The preimage $E = \psi^{-1}(O)$, seen as a reduced divisor on Σ , is the *exceptional divisor* of the model Σ (or of the morphism ψ). A point of E is called an *infinitely near point* of O . By a theorem of

Zariski, ψ is a composition of blowing ups of points, thus the irreducible components E_j of the exceptional divisor E are projective lines (see [36, Vol.1, Ch. IV.3.4, Thm.5]). We call them the *exceptional prime divisors* of the model Σ .

A *local coordinate system* on S is a pair $(x, y) \in \mathcal{O}$ establishing an isomorphism of \mathbb{C} -algebras, $\mathcal{O} \simeq \mathbb{C}[[x, y]]$, where $\mathbb{C}[[x, y]]$ denotes the \mathbb{C} -algebra of formal power series in the variables x and y .

The \mathbb{C} -algebra $\mathbb{C}[[t]]$ of formal power series in a variable t is endowed with the *order valuation* v_t which associates to every series the lowest exponent of its terms. This ring allows to *parametrize* the branches on S :

Definition 3.1 Let C be a branch on S . A *parametrization* of C is a germ of formal map $(\mathbb{C}, 0) \rightarrow (S, \mathcal{O})$ whose image is C , that is, algebraically speaking, a morphism $\mathcal{O} \rightarrow \mathbb{C}[[t]]$ of \mathbb{C} -algebras whose kernel is the principal ideal $\mathcal{O}(-C)$. The parametrization is called *normal* if this map is a *normalization* of C , that is, if it is of degree one onto its image or, algebraically speaking, if the associated map $\mathcal{O}_D \rightarrow \mathbb{C}[[t]]$ induces an isomorphism at the level of fields of fractions.

Example 3.2 Assume that one works with local coordinates (x, y) . Then the branch $C = Z(y^2 - x^3)$ may be parametrized by $(x = t^2, y = t^3)$ and also by $(x = t^4, y = t^6)$. Only the first parametrization is normal.

A *plane curve singularity* C is a reduced germ of complex analytic curve at O , possibly having several branches $(C_i)_{i \in I}$, which are by definition the irreducible components of C . We think also about C as an effective divisor, which allows us to write $C = \sum_{i \in I} C_i$. We write $C \subseteq D$ if D is another reduced germ containing C . In such a case, $D - C$, thought as a difference of divisors, denotes the union of the branches of D which are not branches of C . We denote by $m_O(C)$ the *multiplicity* of C at O . If C is defined by $f \in \mathcal{O}$, and if a local coordinate system (x, y) is fixed, allowing to express f as a formal series in (x, y) , then the multiplicity $m_O(C)$ is equal to the least total degree of the monomials appearing in this series. One has $m_O(C) = \sum_{i \in I} m_O(C_i)$.

If D_1 and D_2 are two effective divisors through O , we denote by $(D_1 \cdot D_2)$ their *intersection number* at O (also called intersection multiplicity). By definition, it is equal to ∞ if and only if the supports of D_1 and D_2 have a common branch. If $D_k = Z(f_k)$, for $k = 1, 2$ then we have that $(D_1 \cdot D_2) = \dim_{\mathbb{C}} \mathcal{O}/(f_1, f_2)$. If one of the two divisors D_k is a branch, for instance D_1 , then the intersection multiplicity may be computed as the order $v_{t_1}(f_2 \circ \phi_1)$ in t_1 of the series $f_2 \circ \phi_1$, where $\phi_1 : (\mathbb{C}_{t_1}, 0) \rightarrow (S, \mathcal{O})$ is a normal parametrization of D_1 (see [3, Proposition II.9.1]).

Example 3.3 Assume that $D_1 = Z(y^2 - x^3)$ and $D_2 = Z(y^2 - 2x^3)$. Both are branches and $(x = t_1^2, y = t_1^3)$ is a normal parametrization of D_1 . Therefore:

$$(D_1 \cdot D_2) = v_{t_1}((t_1^3)^2 - 2(t_1^2)^3) = v_{t_1}(-t_1^6) = 6.$$

Note that a pair $(x, y) \in \mathcal{O}^2$ defines a local coordinate system on S if and only if the germs $Z(x)$ and $Z(y)$ are *transversal* smooth branches, that is, if and only if $(Z(x) \cdot Z(y)) = 1$.

One can study a reduced germ C , by using Newton-Puiseux series:

Definition 3.4 A *Newton-Puiseux series* η in the variable x is a power series of the form $\psi(x^{1/n})$, where $\psi(t) \in \mathbb{C}[[t]]$ and $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$. For a fixed $n \in \mathbb{N}^*$, they form the ring $\mathbb{C}[[x^{1/n}]]$. Its field of fractions is denoted $\mathbb{C}((x^{1/n}))$. If $\eta \in \mathbb{C}[[x^{1/n}]] \setminus \{0\}$, then its *support* is the set $\mathcal{S}(\eta)$ of exponents of η with non-zero coefficient.

Denote by:

$$\mathbb{C}[[x^{1/\mathbb{N}}]] := \bigcup_{n \in \mathbb{N}^*} \mathbb{C}[[x^{1/n}]]$$

the local \mathbb{C} -algebra of Newton-Puiseux series in the variable x . The algebra $\mathbb{C}[[x^{1/\mathbb{N}}]]$ is endowed with the natural *order* valuation (generalizing the order v_t defined in $\mathbb{C}[[t]]$):

$$v_x : \mathbb{C}[[x^{1/\mathbb{N}}]] \longrightarrow \mathbb{Q}_+ \cup \{\infty\}$$

which associates to each series $\eta = \psi(x^{1/n}) \in \mathbb{C}[[x^{1/n}]]$ the minimum of its support.

The definitions of this section depend on the choice of a smooth branch L . Assume that a coordinate system (x, y) is fixed and that

$$L := Z(x).$$

Let A be a branch on S different from L . Relative to the coordinate system (x, y) , it may be defined by a Weierstrass polynomial $f_A \in \mathbb{C}[[x]][y]$, which is monic, irreducible and of degree $d_A = (L \cdot A)$. For simplicity, we mention only the dependency on A , not on the coordinate system (x, y) .

By the Newton-Puiseux theorem, f_A has d_A roots inside $\mathbb{C}[[x^{1/d_A}]]$. We denote by $\text{Zer}(f_A)$ the set of these roots, which are called the *Newton-Puiseux roots* of A with respect to the coordinate system (x, y) . These roots can be obtained from a fixed one $\eta = \psi(x^{1/d_A})$ by replacing x^{1/d_A} by $\gamma \cdot x^{1/d_A}$, for γ running through the d_A -th roots of 1.

Therefore, all the Newton-Puiseux roots of the branch A have the same exponents. Some of those exponents may be distinguished by looking at the differences of roots:

Definition 3.5 The *characteristic exponents* of the branch $A \neq L$ relative to L are the x -orders $v_x(\eta - \eta')$ of the differences between distinct Newton-Puiseux roots η, η' of A in the coordinate system (x, y) .

The fact that we mention only the dependency on L and not on the full coordinate system (x, y) is explained by Proposition 3.10 below. The characteristic exponents may be read from a given Newton-Puiseux root $\eta \in \mathbb{C}[[x^{1/d_A}]]$ of f_A by looking at the increasing sequence of exponents appearing in η and by keeping those which cannot be written as a quotient of integers with the same smallest common denominator as the previous ones. In this sequence, one starts from the first exponent which is not an integer.

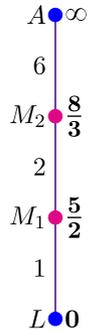
One may find information about the history of the notion of *characteristic exponent* in [16, Section 2].

We keep assuming that A is a branch different from L . The *Eggers-Wall segment of A relative to L* is a geometrical way of encoding the set of characteristic exponents, as well as the sequence of their successive common denominators:

Definition 3.6 The *Eggers-Wall segment* $\Theta_L(A)$ of the branch $A \neq L$ relative to L is a compact oriented segment endowed with the following supplementary structures:

- an increasing homeomorphism $\mathbf{e}_{L,A} : \Theta_L(A) \rightarrow [0, \infty]$, the *exponent function*;
- *marked points*, which are by definition the points whose values by the exponent function are the characteristic exponents of A relative to L , as well as the smallest end of $\Theta_L(A)$, labeled by L , and the greatest end, labeled by A .
- an *index function* $\mathbf{i}_{L,A} : \Theta_L(A) \rightarrow \mathbb{N}$, which associates to each point $P \in \Theta_L(A)$ the index of $(\mathbb{Z}, +)$ in the subgroup of $(\mathbb{Q}, +)$ generated by 1 and the characteristic exponents of A which are strictly smaller than $\mathbf{e}_{L,A}(P)$.

Fig. 4 The Eggers-Wall segment $\Theta_L(A)$ of the series of Example 3.7



The index $i_{L,A}(P)$ may be also seen as the smallest common denominator of the exponents of a Newton-Puiseux root of f_A which are strictly less than $e_{L,A}(P)$.

Example 3.7 Consider the branch A defined by the Newton-Puiseux series $\eta = x^{5/2} + x^{8/3}$. Then its Eggers-Wall segment $\Theta_L(A)$ is represented in Fig. 4. Its marked points are in increasing \leq_L -order L, M_1, M_2, A . Their exponents are $e_{L,A}(L) = 0, e_{L,A}(M_1) = 5/2, e_{L,A}(M_2) = 8/3, e_{L,A}(A) = \infty$. The index is constantly equal to 1 on the segment $[L, M_1]$, constantly equal to 2 on $(M_1, M_2]$ and constantly equal to 6 = $(L \cdot A)$ on $(M_2, A]$. One would obtain the same Eggers-Wall segment if A is defined instead by the series $x^2 - 3x^{5/2} + x^{8/3} + 12x^{7/2} - x^{25/6}$. Indeed, the set of characteristic exponents would be the same as before, that is, $\{5/2, 8/3\}$.

Let us consider now the case of a reduced curve C with several branches. The analog of Eggers-Wall segment associated with C is a tree described in Definition 3.9 below, which depends also on the orders of coincidence of its pairs of branches:

Definition 3.8 If A and B are two distinct branches, which are also distinct from L , then their order of coincidence relative to L is defined by:

$$k_L(A, B) := \max\{v_x(\eta_A - \eta_B) \mid \eta_A \in \text{Zer}(f_A), \eta_B \in \text{Zer}(f_B)\} \in \mathbb{Q}_+^*.$$

Informally speaking, the order of coincidence is the greatest rational number k for which one may find Newton-Puiseux roots of the two branches coinciding up to that number (k excluded).

Note that the order of coincidence is symmetric: $k_L(A, B) = k_L(B, A)$, similarly to the intersection number of the two branches. But, unlike the intersection number, it depends not only on the branches A and B , but also on the choice of branch L . Nevertheless, the two numbers are related, as explained in Theorem 3.23 below.

Here comes the formal definition of the Eggers-Wall tree of a germ consisting of various branches:

Definition 3.9 Let C be a reduced germ of curve on S . Let us denote by \mathcal{I}_C the set of irreducible components of C which are different from L . The Eggers-Wall tree $\Theta_L(C)$ of C relative to L is the rooted tree obtained as the quotient of the disjoint union of the individual Eggers-Wall segments $\Theta_L(A), A \in \mathcal{I}_C$, by the following equivalence relation. If $A, B \in \mathcal{I}_C$, then the gluing of $\Theta_L(A)$ with $\Theta_L(B)$ is done along the initial segments $e_{L,A}^{-1}[0, k_L(A, B)]$ and $e_{L,B}^{-1}[0, k_L(A, B)]$ by:

$$e_{L,A}^{-1}(\alpha) \sim e_{L,B}^{-1}(\alpha), \text{ for all } \alpha \in [0, k_L(A, B)].$$

One endows $\Theta_L(C)$ with the *exponent function* $\mathbf{e}_L : \Theta_L(C) \rightarrow [0, \infty]$ and the *index function* $\mathbf{i}_L : \Theta_L(C) \rightarrow \mathbb{N}$ obtained by gluing the initial exponent functions $\mathbf{e}_{L,A}$ and $\mathbf{i}_{L,A}$ respectively, for A varying among the irreducible components of C different from L . The tree $\Theta_L(L)$ is the trivial tree with vertex set a singleton, whose element is labelled by L . If L is an irreducible component of C , then the marked point $L \in \Theta_L(L)$ is identified with the root of $\Theta_L(A)$ for any $A \in \mathcal{I}_C$. The set of *marked points* of $\Theta_L(C)$ is the union of the set of marked points of the Eggers-Wall segments of the branches of C and of the set of ramification points of $\Theta_L(C)$.

The fact that in the previous notations $\Theta_L(C)$, \mathbf{e}_L , \mathbf{i}_L we mentioned only the dependency on L , and not the whole coordinate system (x, y) , comes from the following fact (see [17, Proposition 26]):

Proposition 3.10 *The Eggers-Wall tree $\Theta_L(C)$, seen as a rooted tree endowed with the exponent function \mathbf{e}_L and the index function \mathbf{i}_L , depends only on the pair (C, L) , where L is defined by $x = 0$.*

When L is generic with respect to C , the Eggers-Wall tree $\Theta_L(C)$ is in fact independent of it (see [41, Theorem 4.3.8]).

Note that the index function \mathbf{i}_L is constant on each segment $(\mathbb{P}(V), V]$ of $\Theta_L(C)$, where V denotes any vertex of the marked tree $\Theta_L(C)$ which is different from the root L , and \mathbb{P} is the parent map introduced in Definition 2.4. Moreover, the set of marked points is determined by the topological structure of $\Theta_L(C)$ and by the knowledge of the index function, as the reader may easily verify:

Lemma 3.11 *The set of marked points of the Eggers-Wall tree $\Theta_L(C)$ is the union of the following sets:*

- the set of ends, consisting of the root L and the leaves $A \in \mathcal{I}_C$;
- the set of ramification points;
- the set of points of discontinuity of the index function.

Any ramification point of $\Theta_L(C)$ is of the form $A \wedge_L B$ for $A, B \in \mathcal{I}_C$. Here, the point $A \wedge_L B$, which has exponent equal to $k_L(A, B)$, is the infimum of the leaves of $\Theta_L(C)$ labeled by A and B , relative to the partial order on the set of vertices of $\Theta_L(C)$ defined by the root L (see Definition 2.3). Note that the first set in Lemma 3.11 is disjoint from the two other ones, but that the second and the third one may have elements in common, as may be seen in Example 3.13, in which 3 of the 4 ramification points are also points of discontinuity of the index function.

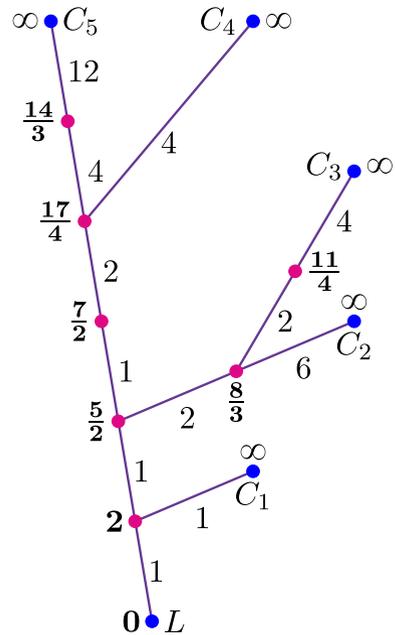
Remark 3.12 By Lemma 3.11, the Eggers-Wall tree $\Theta_L(C)$ is determined by its finite tree equipped with the exponent function and the index function (see Definition 2.1).

Example 3.13 Consider a plane curve singularity $C = \sum_{i=1}^5 C_i$ whose branches C_i are defined by the Newton-Puiseux series η_i , where:

$$\begin{aligned} \eta_1 &= x^2, & \eta_2 &= x^{5/2} + x^{8/3}, & \eta_3 &= -x^{5/2} + x^{11/4}, & \eta_4 &= x^{7/2} + x^{17/4}, \\ \eta_5 &= x^{7/2} + 2x^{17/4} + x^{14/3}. \end{aligned}$$

Note that the branch C_2 coincides with the branch A of Example 3.7. We will denote simply k instead of k_L , where $L = Z(x)$. One has $k(C_1, C_2) = k(C_1, C_3) = k(C_1, C_4) = k(C_1, C_5) = 2, k(C_2, C_4) = k(C_2, C_5) = 5/2, k(C_2, C_3) = 8/3, k(C_3, C_4) = k(C_3, C_5) = 5/2, k(C_4, C_5) = 17/4$ and the Eggers-Wall tree of C relative to L is drawn in Fig. 5. Observe that C_3 admits also as Newton-Puiseux series $\tilde{\eta}_3 := \xi_3(ix^{1/4}) = x^{5/2} - ix^{11/4}$, where $\xi_3(t) = -t^{10} + t^{11}$, and that $k(C_2, C_3) = \nu_x(\eta_2 - \tilde{\eta}_3) > \nu_x(\eta_2 - \eta_3)$.

Fig. 5 The Eggers-Wall tree of Example 3.13



If one considers two reduced germs $C \subseteq C'$, then one has a unique embedding $\Theta_L(C) \subseteq \Theta_L(C')$ such that the restrictions to $\Theta_L(C)$ of the index and of the exponent function on $\Theta_L(C')$ are equal to the corresponding functions on $\Theta_L(C)$.

A point of valency 2 of $\Theta_L(C)$ may become a ramification point of $\Theta_L(C')$. The set of points of $\Theta_L(C)$ which are either ramification points of $\Theta_L(C)$ or become ramification points of a bigger tree $\Theta_L(C')$ is exactly the set of *rational points* of $\Theta_L(C)$ in the following sense:

Definition 3.14 The set of *rational points* $\Theta_L(C)_{\mathbb{Q}}$ of $\Theta_L(C)$ consists of those points $P \in \Theta_L(C)$ such that $\mathbf{e}_L(P) \in \mathbb{Q}_+^*$. The *upper index* $\mathbf{i}_L^+(P)$ of a rational point $P \in \Theta_L(C)_{\mathbb{Q}}$ is the index of $(\mathbb{Z}, +)$ in the subgroup of $(\mathbb{Q}, +)$ generated by 1, $\mathbf{e}_L(P)$ and $\mathbf{e}_L(Q)$, for $Q \prec_L P$ running through the points of discontinuity of the index function \mathbf{i}_L .

The upper index function will be crucial in Proposition 8.16 below.

Lemma 3.15 Let A, B be two branches on S . If the restriction of the index function to the segment $[L, A]$ is continuous at $P := \langle L, A, B \rangle$, then there is a segment $(P, P') \subset [L, B]$ such that the index function is constant on it and $\mathbf{i}_L(P') = \mathbf{i}_L^+(P)$.

Proof We distinguish two cases:

- If the index function of the tree $\Theta_L(A + B)$ is continuous at P , then it follows that a non-zero term with exponent $\mathbf{e}_L(P)$ must appear in the Newton-Puiseux series of the branch A or of the branch B , considered with respect to a variable x such that $Z(x) = L$. Notice that $\mathbf{e}_L(P)$ is not a characteristic exponent of any of them. Then, if $(P, P') \subset [L, B]$ and the index function in restriction to this segment is constant then $\mathbf{i}_L(P') = \mathbf{i}_L^+(P)$ by definition, and $\mathbf{i}_L^+(P) = \mathbf{i}_L(P)$ by continuity.

- Otherwise, the index function in restriction to $[L, B]$ is not continuous at P . This implies that $\mathbf{e}_L(P)$ is a characteristic exponent of the Newton-Puiseux series of B , with respect to

x . Then, if $(P, P') \subset [L, B]$ and the index function in restriction to this segment is constant we get that $\mathbf{i}_L(P') = \mathbf{i}_L^+(P)$, by definition of the index function and of the upper index. \square

We leave as an exercise to prove that the values $\mathbf{i}_L(P)$ and $\mathbf{e}_L(P)$ determine the upper index $\mathbf{i}_L^+(P)$ in the following way:

Proposition 3.16 *Let $P \in \Theta_L(C)_{\mathbb{Q}}$. Then its upper index $\mathbf{i}_L^+(P)$ is equal to the lowest common multiple of $\mathbf{i}_L(P)$ and of the denominator of $\mathbf{e}_L(P)$, when one writes this rational number as an irreducible fraction.*

Example 3.17 Let us come back to Example 3.13 and its associated Eggers-Wall tree, represented in Fig. 5. Then one may see either using Definition 3.14 or using Proposition 3.16 that the upper indices of the marked points lying on the open segment $(L, C_5) \subset \Theta_L(C)$ are, in increasing order, 1, 2, 2, 4, 12.

Remark 3.18 (1) Eggers introduced a slightly different notion of tree in his 1983 paper [9] about the structure of polar curves of a possibly reducible plane curve singularity. Namely, given a reduced germ C , he considered only generic coordinate systems (x, y) , for which $L = Z(x)$ is transversal to all the branches of C . In terms of our notations, he rooted his tree at the minimal marked point different from the root L of the Eggers-Wall tree. He considered only an analog of the exponent function, defined on the set of marked points of the tree. Eggers did not consider the index function. Instead, he used two colors for the edges of his tree, in order to remember for each branch of C which marked points lying on it correspond to its characteristic exponents. Our notion of Eggers-Wall tree is based on Wall’s 2003 paper [40] (which circulated as a preprint since 2000), in which the functions $\mathbf{e}_L, \mathbf{i}_L$ (with different notations) are used for computations adapted to the description of the polar curves of C . The name “Eggers-Wall tree” was introduced by the third author in [32], to honor the previous works of Eggers and Wall.

(2) In previous papers, versions of the notion of Eggers-Wall tree of C with respect to the local coordinates (x, y) were defined under the assumption that L is not a component of C (see [6,9,13–15,20,27,32,33,40]). Allowing L to be a branch of C permits a very easy formulation of the inversion theorem for Eggers-Wall trees (see Theorem 4.5). Note that the third-named author’s paper [33], which presented some of the results of his PhD Thesis [32], introduced an extension of the Eggers-Wall trees to quasi-ordinary power series in several variables, and applied them to the study of polar hypersurfaces of quasi-ordinary hypersurfaces. This study was continued by the first two-named authors in [15].

(3) Corral used in [6] a version of the Eggers-Wall tree to describe the topology of a generic polar curve associated with a generalized curve foliation in $(\mathbb{C}^2, 0)$, with non resonant logarithmic model.

(4) The Eggers-Wall tree may be seen as a Galois quotient of a variant of the tree constructed in 1977 by Kuo and Lu in [26] (see [16, Remark 4.39], as well as [20, Section 2.5]). This variant is defined exactly as the Eggers-Wall tree, but using *all* the Newton-Puiseux roots of C , not only one root for each branch. Therefore, it has as many leaves as the intersection number $(C \cdot L)$. A related construction was performed by Kapranov in his 1993 papers [24] and [25]. He applied it to usual formal power series with complex and real coefficients respectively and he called the resulting rooted trees *Bruhat-Tits trees*.

Let us introduce a third real-valued function \mathbf{e}_L defined on the Eggers-Wall tree. It allows us to compute the pairwise intersection numbers of the branches of the given germ (see

Theorem 3.23 below). It is determined by the knowledge of the exponent function \mathbf{e}_L and of the index function \mathbf{i}_L :

Definition 3.19 Let C be a reduced germ of curve and L a smooth branch on S . The *contact complexity* function

$$\mathbf{c}_L : \Theta_L(C) \rightarrow [0, \infty]$$

at a point $P \in \Theta_L(C)$ is the integral of the piecewise constant function $1/\mathbf{i}_L$ along the segment $[L, P]$ of $\Theta_L(A)$, the measure being determined by the exponent function:

$$\mathbf{c}_L(P) = \int_L^P \frac{d\mathbf{e}_L}{\mathbf{i}_L}. \tag{3.1}$$

In order to compute this integral, it is enough to take a branch A of C such that $P \leq_L A$. Denote by $\alpha_1 < \dots < \alpha_g$ the characteristic exponents of A relative to the smooth germ L . We define conventionally $\alpha_0 = 0$ and $\alpha_{g+1} = \infty$. Let us set $P_j = \mathbf{e}_L^{-1}(\alpha_j)$ for $j = 0, \dots, g + 1$. We denote by \mathbf{i}_j the value of the index function \mathbf{i}_L in restriction to the half-open segment $(P_j, P_{j+1}]$. Denote by $l \in \{0, \dots, g\}$ the unique integer such that $P \in [P_l, P_{l+1})$. Then the contact complexity $\mathbf{c}_L(P)$ is:

$$\mathbf{c}_L(P) = \left(\sum_{j=1}^l \frac{\alpha_j - \alpha_{j-1}}{\mathbf{i}_{j-1}} \right) + \frac{\mathbf{e}_L(P) - \alpha_l}{\mathbf{i}_l}. \tag{3.2}$$

Notice that $\mathbf{c}_L(L) = 0$.

Remark 3.20 Notice also that the knowledge of \mathbf{c}_L and \mathbf{i}_L determines \mathbf{e}_L :

$$\mathbf{e}_L(P) = \int_L^P \mathbf{i}_L d\mathbf{c}_L. \tag{3.3}$$

Or, written in a way which is analogous to the developed expression given in equation (3.2), and keeping the notations of that equation:

$$\mathbf{e}_L(P) = \left(\sum_{j=1}^l \mathbf{i}_{j-1}(c_j - c_{j-1}) \right) + \mathbf{i}_l(\mathbf{c}_L(P) - c_l), \tag{3.4}$$

where $c_j := \mathbf{c}_L(P_j)$ for every $j \in \{0, \dots, g\}$.

Remark 3.21 Formulae (3.1) and (3.3) are inspired by the formulae (3.7) and (3.9) of Favre and Jonsson’s book [11], relating *thinness* and *skewness* as functions on the valuative tree (see Remark 7.2 below).

As the function $\mathbf{i}_L : \Theta_L(A) \rightarrow \mathbb{N}^*$ is increasing along the segment $\Theta_L(A)$ for any branch A of C , formulae (3.1) and (3.3) imply:

Corollary 3.22 (1) If C is a reduced germ of curve on S , then the contact complexity function $\mathbf{c}_L : \Theta_L(C) \rightarrow [0, \infty]$ is a continuous strictly increasing surjection.

(2) In particular, if A is a branch on S different from L , then the contact complexity function $\mathbf{c}_L : \Theta_L(A) \rightarrow [0, \infty]$ is an increasing homeomorphism from the Eggers-Wall segment $[L, A]$ to $[0, \infty]$. Moreover, it is piecewise affine and concave in terms of the parameter \mathbf{e}_L . Conversely, the function \mathbf{e}_L is continuous piecewise affine and convex in terms of the parameter \mathbf{c}_L .

We chose the name of this function motivated by the following theorem, which shows that c_L may be seen as a measure of the contact between the branches of C . In equivalent formulations, this theorem goes back at least to Smith [38, Section 8], Stolz [39, Section 9] and Max Noether [28]. A proof written in current mathematical language may be found in Wall [41, Thm. 4.1.6]:

Theorem 3.23 *Let C be a reduced germ of curve and L a smooth branch on S . Let A and B be two distinct branches of C . Let $P = \langle L, A, B \rangle$ be the center of the tripod determined by L, A, B in the Eggers-Wall tree $\Theta_L(C)$ (see Definition 2.13). Then:*

$$c_L(P) = \frac{(A \cdot B)}{(L \cdot A) \cdot (L \cdot B)}. \tag{3.5}$$

Observe that Theorem 3.23 also holds when L coincides with A or B (using the convention that $a/\infty = 0$ for every $a \in (0, \infty)$).

Remark 3.24 In the paper [31], Płoski proved a theorem which is equivalent to the fact that the function

$$U_L(A, B) := \begin{cases} c_L(\langle L, A, B \rangle)^{-1} & \text{if } A \neq B, \\ 0 & \text{if } A = B, \end{cases}$$

defines an ultrametric distance on the set of branches which are transversal to L . See [17,18] for generalizations of this result to all normal surface singularities (in particular, it is proved in [18, Theorem 1.26] that, given a normal surface singularity S and an arbitrary branch L on it, the function U_L is an ultrametric on the set of branches different from it if and only if S is *arborescent*, that is, the dual graphs of its good resolutions are trees).

Remark 3.25 If D is a branch, then the intersection number $(L \cdot D)$ is equal to the maximum $i_L(D)$ achieved by the index function on the segment $[L, D] = \Theta_L(D)$.

We deduce from Theorem 3.23 and Remark 3.25 that:

Corollary 3.26 (Tripod formula) *Assume that the Eggers-Wall tree $(\Theta_L(C), e_L, i_L)$ of the reduced germ C relative to L is known. If A and B are two branches of C , then their intersection number is determined by:*

$$(A \cdot B) = i_L(A) \cdot i_L(B) \cdot c_L(\langle L, A, B \rangle).$$

The previous equality shows that the intersection number of two branches of C is determined by the indices of the two corresponding leaves and by the contact complexity of the center of the tripod formed by the root of the tree $\Theta_L(C)$ and the two leaves. That is why we call it the *tripod formula*.

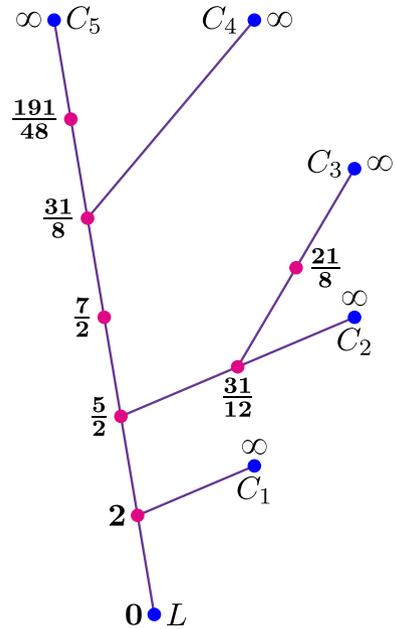
Corollary 3.26 admits an extension for semivaluations (see Proposition 7.18 below).

Example 3.27 Consider again the curve singularity of Example 3.13. Then the contact complexities of the marked points of its Eggers-Wall tree with respect to the given coordinate system are as indicated in Fig. 6. For instance, the contact complexity of the highest marked point on the open segment (L, C_5) is computed in the following way using Formula (3.2):

$$\frac{7}{2} + \frac{1}{2} \left(\frac{17}{4} - \frac{7}{2} \right) + \frac{1}{4} \left(\frac{14}{3} - \frac{17}{4} \right) = \frac{191}{48}.$$

Using Theorem 3.23, we deduce that $(C_1 \cdot C_2) = 12, (C_1 \cdot C_3) = (C_1 \cdot C_4) = 8, (C_1 \cdot C_5) = 24, (C_2 \cdot C_3) = 62, (C_2 \cdot C_4) = 60, (C_2 \cdot C_5) = 180, (C_3 \cdot C_4) = 40, (C_3 \cdot C_5) = 120, (C_4 \cdot C_5) = 186$.

Fig. 6 The values of the contact complexity c_L at the marked points of the tree of Example 3.13



4 An inversion theorem for Eggers-Wall trees

Let C be a curve singularity and let L be a smooth branch on S . Assume that we know the Eggers-Wall tree $\Theta_L(C)$ of C relative to L . How to pass to the Eggers-Wall tree of C relative to another smooth branch L' ? The answer is particularly simple when *both L and L' are branches of C* . Indeed, in this case, we prove that the underlying topological space of the Eggers-Wall tree is unchanged: one has only to modify the exponent and index functions (see Theorem 4.5). This constitutes a geometrization and generalization to the case of several branches of the classical inversion theorem, which expresses the characteristic exponents of a branch with respect to the coordinate system (y, x) , in terms of those with respect to (x, y) (see Remark 4.8 for historical comments about this theorem).

Before stating our inversion theorem, we need some definitions and properties of the Eggers-Wall segments of smooth branches and of their attaching points on Eggers-Wall trees of germs not containing them, in the sense of Definition 2.11.

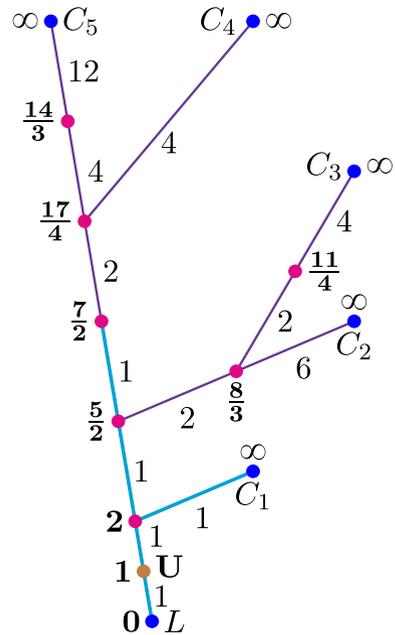
Definition 4.1 Let C be a curve singularity and let L be a smooth branch on S . The *unit subtree* $\Theta_L(C)_1$ of $\Theta_L(C)$ consists of its points of index 1, equipped with the restriction of the exponent function e_L . The *unit point* of the tree $\Theta_L(C)$ is the attaching point of a *generic* smooth branch through O .

The unit point is independent of the choice of generic smooth branch through O , as shown by the following lemma:

Lemma 4.2 *The unit point of $\Theta_L(C)$ is:*

- *the highest end of $\Theta_L(C)_1$, when the exponent function takes only values < 1 in restriction to $\Theta_L(C)_1$ (case in which $\Theta_L(C)_1$ is a segment);*
- *the unique point of $\Theta_L(C)_1$ of exponent 1, otherwise.*

Fig. 7 The unit subtree (in heavier lines) and the unit point of the Eggers-Wall tree (labelled by U)



Proof Consider a smooth branch L' transversal both to L and to the branches of C . Work then in a coordinate system (x, y) such that $L = Z(x)$ and $L' = Z(y)$. Therefore L' has $0 \in \mathbb{C}[[x^{1/\mathbb{N}}]]$ as only Newton-Puiseux series. Our transversality hypothesis implies that for any branch A of C , its Newton-Puiseux series η satisfy $v_x(\eta) \in (0, 1]$. But $v_x(\eta) = v_x(\eta - 0) = k_L(A, L')$. This implies immediately our statements. We are in the first case if $v_x(\eta) < 1$ for all the branches of C and in the second one otherwise.

Example 4.3 In Fig. 7 are represented the unit subtree and the unit point U of the Eggers-Wall tree of Fig. 5.

Let us introduce now special names for the Eggers-Wall segments of smooth branches with respect to a smooth branch L :

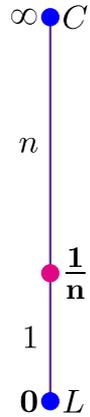
Definition 4.4 Let C be a branch different from L . The Eggers-Wall segment $\Theta_L(C)$ is *simple* if it has no marked points in its interior. It is called *smooth* if it is simple or if it is of the form indicated in Fig. 8. In this last case, the integer $n \geq 2$ is equal to the intersection number $(L \cdot C)$.

If C is a branch, then $\Theta_L(C)$ is smooth if and only if C is smooth, which explains the name.

By Remark 3.12, the Eggers-Wall tree $\Theta_L(C)$ is determined by its geometric realization equipped with the exponent function \mathbf{e}_L and the index function \mathbf{i}_L . Notice also that these two functions determine \mathbf{c}_L . The following *inversion theorem* proves that these functions determine also the Eggers-Wall tree $\Theta_{L'}(C)$ whenever L and L' are branches of C :

Theorem 4.5 Let L and L' be two smooth branches on S , which are components of the reduced germ C . Let us denote by U the unit point of $\Theta_L(C)$ in the sense of Definition 4.1 and by $\pi_{[L, L']}$ the attaching map of the segment $[L, L']$ in the tree $\Theta_L(C)$ in the sense of

Fig. 8 A smooth Eggers-Wall segment with unit point of exponent $1/n$ (see Definition 4.4)



Definition 2.11. Then the finite trees associated with $\Theta_{L'}(C)$ and $\Theta_L(C)$ coincide and the functions $\mathbf{e}_{L'}$, $\mathbf{c}_{L'}$, $\mathbf{i}_{L'}$ are determined in terms of \mathbf{e}_L , \mathbf{c}_L , \mathbf{i}_L by the formulae:

$$\mathbf{e}_{L'} + 1 = \frac{\mathbf{e}_L + 1}{(L \cdot L') \cdot (\mathbf{c}_L \circ \pi_{[L, L']})}, \quad \mathbf{c}_{L'} = \frac{\mathbf{c}_L}{((L \cdot L') \cdot (\mathbf{c}_L \circ \pi_{[L, L']}))^2},$$

$$\mathbf{i}_{L'} = \begin{cases} 1, & \text{on } [\pi_{[L, L']}(U), L'], \\ (L \cdot L'), & \text{on } [L, \pi_{[L, L']}(U)], \\ (L \cdot L') \cdot (\mathbf{c}_L \circ \pi_{[L, L']}) \cdot \mathbf{i}_L, & \text{otherwise.} \end{cases}$$

Moreover, in restriction to the segment $[L, L']$ we have:

$$(L \cdot L') \cdot \mathbf{c}_L = \begin{cases} (L \cdot L') \cdot \mathbf{e}_L & \text{on } [L, \pi_{[L, L']}(U)], \\ \mathbf{e}_L + 1 - \frac{1}{(L \cdot L')} & \text{on } [\pi_{[L, L']}(U), L']. \end{cases}$$

Proof We use here several results developed later in this paper. The idea is to embed the Eggers-Wall tree in the space $\mathbb{P}(\mathcal{V})$ of semivaluations of S and to use formulae about the log-discrepancy, the multiplicity and the self-interaction functions defined on that space.

Denote, as usual, by C_i the branches of C . We will use the valuative embeddings Ψ_L and $\Psi_{L'}$ of Definition 8.19.

By the topological part of Theorem 8.11, the images of both embeddings Ψ_L and $\Psi_{L'}$ are the convex hulls of the ends C_i inside the tree $\mathbb{P}(\mathcal{V})$. Therefore, Ψ_L and $\Psi_{L'}$ are homeomorphisms onto those convex hulls. Consequently, the map:

$$\Psi_{L'}^L := \Psi_{L'}^{-1} \circ \Psi_L : \Theta_L(C) \rightarrow \Theta_{L'}(C) \tag{4.1}$$

is a homeomorphism. By construction, it sends each end C_i of $\Theta_L(C)$ to the end with the same label of $\Theta_{L'}(C)$.

In order to compare $(\mathbf{e}_L, \mathbf{i}_L, \mathbf{c}_L)$ with $(\mathbf{e}_{L'}, \mathbf{i}_{L'}, \mathbf{c}_{L'})$, we use the part of Theorem 8.11 concerning the correspondence between functions, as well as Propositions 9.1, 9.4, 9.3. The statement of our theorem, as well as the one of its Corollary 4.6 are immediate consequences of them (the last assertion of the theorem follows from Definition 3.2). \square

Let us particularize this result to the situation where L and L' are *transversal* smooth branches on S , that is, $(L \cdot L') = 1$. Then, the segment $[L, L'] = \Theta_L(L')$ is a simple Eggers-Wall segment in the sense of Definition 4.4, and the unit point is its point U such that $e_L(U) = 1$.

Corollary 4.6 *Let L and L' be two transversal smooth branches at O which are components of the reduced germ C . We have the relations:*

$$e_{L'} + 1 = \frac{e_L + 1}{e_L \circ \pi_{[L, L']}}, \quad c_{L'} = \frac{c_L}{(e_L \circ \pi_{[L, L']})^2}.$$

and

$$i_{L'} = \begin{cases} 1, & \text{on } [L, L'], \\ (e_L \circ \pi_{[L, L']}) \cdot i_L, & \text{elsewhere.} \end{cases}$$

Remark 4.7 By combining formula (3.1) with Corollary 4.6, we see that in restriction to the segment $[L, L']$, one has the following equalities in the transversal case:

$$c_{L'} = e_{L'} = e_L^{-1} = c_L^{-1}.$$

Remark 4.8 When applied to the case when C is a branch, Corollary 4.6 is a reformulation in terms of Eggers-Wall trees of the *inversion theorem*, which express the characteristic exponents with respect to a coordinate system (y, x) in terms of those with respect to (x, y) . Abhyankar [1, Theorem 1] and Zariski [42, Section 3] gave proofs of this theorem. In fact, Halphen [21] and Stolz [39] respectively, discovered and proved this theorem in the years 1870, as explained in our paper [17].

Remark 4.9 One may try to prove Corollary 4.6 directly from the *inversion theorem* applied to the branches A of C different from L and L' . This provides the functions $e_{L',A}$ and $i_{L',A}$ and $c_{L',A}$. Corollary 3.26 allows to determine the value of $c_{L'}$ on the points (L', A, B) , for A and B two distinct branches of C (different from L'). Formula (3.4) determines the value of the exponent function $e_{L'}$ at the point (L', A, B) , which is equal to $k_{L'}(A, B)$. Then, it remains to prove that the geometric realizations of the trees $\Theta_L(C)$ and $\Theta_{L'}(C)$ are isomorphic by an isomorphism respecting the labelings of the ends by the branches of C . Our approach, using the embeddings of the Eggers-Wall trees in the *valuative tree*, provides a conceptual understanding of these combinatorial operations.

Example 4.10 Consider again the Eggers-Wall tree of Example 3.13. Now we assume that L is a component of C . The Eggers-Wall tree $\Theta_{C_1}(C)$ is represented in the right diagram of Fig. 9. In each one of the two diagrams, we have also indicated the position of the unit point U (which remains unchanged). The roots may be recognized as the only ends with vanishing exponent. In our case, L and $L' := C_1$ are transversal, which means that we may apply Corollary 4.6. This implies that $e_{L'} = \frac{1}{2}(e_L - 1)$ on the union of the segments $[C_i, C_j]$, for $i, j \geq 1$, since in restriction to them $e_L \circ \pi_{[L, L']} = 2$.

Remark 4.11 When L or L' is not a branch of C , we determine the Eggers-Wall tree $\Theta_{L'}(C)$ from $\Theta_L(C)$ by constructing first $\Theta_L(C + L + L')$, by applying then Theorem 4.5 to it in order to get $\Theta_{L'}(C + L + L')$, and by passing finally to its subtree $\Theta_{L'}(C)$.

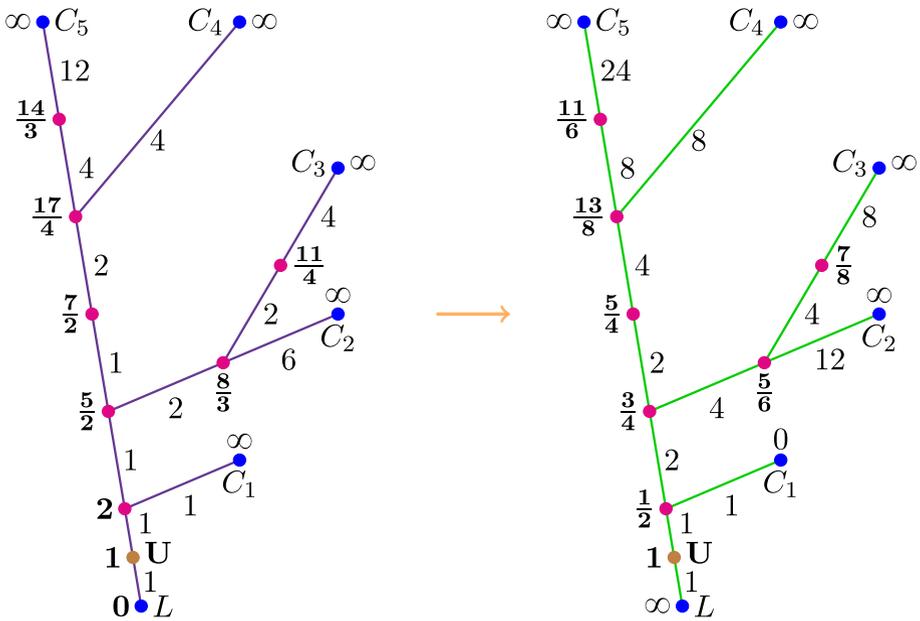


Fig. 9 The Eggers-Wall trees $\Theta_L(C)$ of Example 4.10 on the left, compared with $\Theta_{C_1}(C)$ on the right

5 Eggers-Wall trees and splice diagrams

In this section we recall from Eisenbud and Neumann’s book [10] the topological operation of *splicing* of two oriented links along a pair of their components inside *oriented* integral homology spheres, as well as the associated encoding of *graph links* by *splice diagrams*. Then we particularize this construction to the links of curve singularities inside smooth complex surfaces and we explain how to pass from an Eggers-Wall diagram to a splice diagram (see Theorem 5.14).

A *link* in a 3-dimensional manifold is a closed 1-dimensional submanifold. The link is called a *knot* if it is moreover connected. The *exterior* of a link is the complement of the interior of a compact tubular neighborhood of it in the ambient 3-dimensional manifold.

In this section all ambient 3-dimensional manifolds and all the links considered inside them will be assumed *oriented*. For this reason, we will not mention this hypothesis anymore.

Definition 5.1 An *integral homology sphere* is a closed 3-dimensional manifold Σ which has the same integral homology groups as the 3-dimensional sphere S^3 . Equivalently, it is connected and $H_1(\Sigma, \mathbb{Z}) = 0$.

If K_1 and K_2 are two disjoint knots in an integral homology sphere Σ , then we denote by $lk_\Sigma(K_1, K_2) \in \mathbb{Z}$ their *linking number*. Recall that:

$$lk_\Sigma(K_1, K_2) = lk_\Sigma(K_2, K_1).$$

Definition 5.2 Let K be a knot inside a 3-dimensional integral homology sphere Σ . Denote by U a compact tubular neighborhood of K and by T its boundary, which is a 2-dimensional torus. A *meridian* of K is an oriented simple closed curve M on T which is non-trivial

homologically in T but becomes trivial in U , and satisfies $lk_M(K, M) = 1$. A *longitude* of K is an oriented simple closed curve L on T which is homologous to K in U and satisfies $lk_M(K, L) = 0$.

Note that the constraint that L be homologous to K inside the solid torus U determines its orientation. The condition that $lk_M(K, L) = 0$ means intuitively that L does not spiral around K , seen from the global viewpoint of M . A basic result of 3-dimensional topology is that meridians and longitudes are well-defined up to isotopy on T .

The following topological construction was described by Eisenbud and Neumann [10, Chapter I.1], inspired by previous work of Siebenmann [37] and Bonahon and Siebenmann (by U° we denote the *interior* of the manifold with boundary U):

Definition 5.3 Let Λ_1 and Λ_2 be two links inside the disjoint 3-dimensional integral homology spheres Σ_1 and Σ_2 respectively. Let K_j be a connected component of Λ_j , for each $j \in \{1, 2\}$. Denote by U_j a compact tubular neighborhood of K_j , disjoint from $\Lambda_j \setminus K_j$. We consider longitudes and meridians of K_j on the boundary T_j of U_j . The *splice* of (Σ_1, Λ_1) and (Σ_2, Λ_2) along K_1 and K_2 is the pair (Σ, Λ) defined by:

- Σ is the closed 3-manifold obtained from $\Sigma_1 \setminus U_1^\circ$ and $\Sigma_2 \setminus U_2^\circ$ by identifying their boundaries T_1 and T_2 through a diffeomorphism which permutes (oriented) meridians and longitudes.
- Λ is the link inside Σ obtained by taking the union of the images of $\Lambda_1 \setminus K_1$ and $\Lambda_2 \setminus K_2$ inside Σ .

The basic result about this operation is (see [10, Chapter I.1]):

Proposition 5.4 *The link (Σ, Λ) is well-defined up to an orientation-preserving diffeomorphism which is unique up to isotopy and Σ is again an integral homology sphere.*

Conversely, one may *unsplice* an oriented link (Σ, Λ) inside an integral homology sphere Σ by finding inside $\Sigma \setminus \Lambda$ an embedded 2-torus T , then cutting Σ along T and filling the resulting two manifolds with boundary by solid tori in such a way as to get again integral homology spheres. Inside those two resulting homology spheres, one considers the links which are obtained from Λ by adding central circles of the two solid tori used for performing the two fillings. Remark that the whole process is possible because the complement $\Sigma \setminus T$ is disconnected, as a consequence of the hypothesis that Σ is an integral homology sphere: otherwise, there would exist a simple closed curve intersecting transversely T at one point, which would imply that this curve is not homologous to 0 in Σ .

One has the following result (see [10, page 25]):

Lemma 5.5 *Let (Σ, Λ) be a link inside an integral homology sphere and let T be a 2-torus inside $\Sigma \setminus \Lambda$. Then Λ is the result of a splicing operation along this torus, of two links (Σ_1, Λ_1) and (Σ_2, Λ_2) . If K_i denotes the component of Λ_i along which this operation is done, then the orientations of K_1 and K_2 are well-determined up to a simultaneous reorientation. Moreover, if $\Sigma \simeq \mathbb{S}^3$, then $\Sigma_1 \simeq \mathbb{S}^3$, $\Sigma_2 \simeq \mathbb{S}^3$ and the converse also holds.*

In the sequel we will use integral homology spheres which are Seifert fibred and Seifert links inside them as building blocks in the splicing procedure. Let us start by defining the first notion (see Orlik's book [30], as well as Seifert's original paper [34], or its English translation which appeared as an appendix to Seifert and Threlfall book [35]):

Definition 5.6 A *Seifert fibration* on a compact 3-manifold is a smooth foliation by circles, such that each leaf has a saturated neighborhood (that is, a neighborhood obtained as a union of fibres) which is diffeomorphic by a leaf-preserving diffeomorphism to the quotient of the infinite cylinder $\mathbb{D}^2 \times \mathbb{R}$ by the diffeomorphism:

$$(z, t) \rightarrow (e^{2i\pi q/p}z, t + 1),$$

where:

- q and p are coprime integers, with $p \in \mathbb{N}^*$;
- the quotient is endowed with the projection of the foliation of $\mathbb{D}^2 \times \mathbb{R}$ by the copies $\{z\} \times \mathbb{R}$ of the second factor;
- the initial leaf corresponds to the image of $\{0\} \times \mathbb{R}$ by this quotient map.

When $p \geq 2$, one says that the initial leaf is *singular* and that p is its *multiplicity*. A saturated neighborhood of the previous kind is called a *model neighborhood*. The leaves of the foliation are called its *fibers*.

Let us recall a homological interpretation of the multiplicity p associated to a fiber F_0 of a Seifert fibration. Consider a model neighborhood U of the chosen fiber. Orient all the fibers of this model in a continuous manner. One has $H_1(U, \mathbb{Z}) = \mathbb{Z}[F_0]$, where $[F_0]$ denotes the homology class of F_0 . If F is a fiber contained inside U and different from F_0 , then the homology class $[F]$ of F in $H_1(U, \mathbb{Z})$ is equal to $p[F_0]$. This shows that p is independent on the chosen orientations of F_0 and of the ambient manifold.

In order to get also the number q , one has to consider a meridian disk D of U , whose boundary circle intersects transversally the foliation induced on the 2-torus ∂U . Orient D such that its orientation followed by the orientation of a fiber lying in U gives the ambient orientation. This induces an orientation on ∂D . Consider a fiber F lying on ∂U . It intersects ∂D in p points. Their set may be cyclically ordered by the orientation of ∂D , which allows to identify it canonically with the cyclic group $\mathbb{Z}/p\mathbb{Z}$. The first return map obtained by following F along its chosen orientation is a translation of this group by one of its elements, which is precisely the image of q in $\mathbb{Z}/p\mathbb{Z}$. This shows that q is only well-defined modulo p and that it is changed into its opposite when one changes the ambient orientation.

Having defined Seifert fibrations, we may define *Seifert links* and the more general notion of *graph links*:

Definition 5.7 A *Seifert link* is a link in a closed 3-dimensional manifold whose exterior admits a Seifert fibration. A *graph link* is a link whose exterior may be cut into Seifert fibred manifolds using a finite set of pairwise disjoint tori.

The structure of any graph link inside an integral homology sphere may be expressed using a *splice diagram*. This is a special kind of decorated tree:

Definition 5.8 A *splice diagram* is a marked finite forest (that is, a finite disjoint union of trees) whose vertices are decorated with the signs \pm and whose germs of edges at each *internal vertex* (that is, a vertex which is not an end) are decorated with *pairwise coprime* integers. Some of its ends are distinguished as *arrowhead ends*.

Each splice diagram encodes up to orientation-preserving homeomorphisms a unique graph link inside an integral homology sphere. In order to understand this, we explain it first in the case in which the diagram is *star-shaped*, that is, in which it has exactly one vertex which is not an end. Then the encoding is based on the following proposition (see [10, Chapter II.7]):

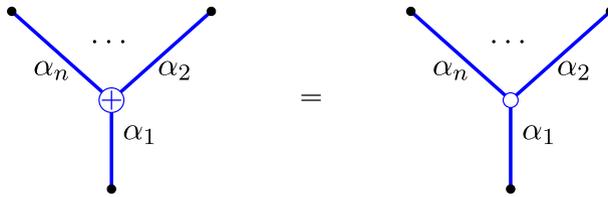


Fig. 10 The splice diagram of the oriented homology sphere $\Sigma(\alpha_1, \dots, \alpha_n)$

Proposition 5.9 *Let $n \geq 2$ and $\alpha_1, \dots, \alpha_n$ be n pairwise coprime non-zero integers. There exists a unique Seifert fibered oriented integral homology sphere $\Sigma(\alpha_1, \dots, \alpha_n)$ endowed with an oriented link $\Lambda := F_1 \cup \dots \cup F_n$ consisting of oriented fibers and with a choice of continuous orientation of the fibers not belonging to Λ , such that:*

- *the link Λ contains all singular fibers of the Seifert fibration;*
- *for every $i \in \{1, \dots, n\}$, the multiplicity of F_i is equal to $|\alpha_i|$;*
- *the orientation of the generic fibers is chosen compatibly with the orientation of F_i if and only if $\alpha_i > 0$;*
- *for every distinct $i, j \in \{1, \dots, n\}$, the linking number $lk_{\Sigma(\alpha_1, \dots, \alpha_n)}(F_i, F_j)$ is equal to the product:*

$$\prod_{k \in \{1, \dots, n\} \setminus \{i, j\}} \alpha_k.$$

In fact, Proposition 5.9 may be extended to the situation where one of the integers α_i is 0 (note that the coprimality condition prohibits having two of them vanishing simultaneously). In order to do this, one must allow still another kind of model neighborhood, in which the nearby fibers turn once around the central fiber (see [10, Lemma 7.1]). The resulting manifold is still an integral homology sphere, but it is Seifert fibered only in the exterior of the link Λ . This explains the mention of such exteriors of links in Definition 5.7.

The Seifert fibered oriented homology sphere $\Sigma(\alpha_1, \dots, \alpha_n)$ may be represented by any of the two star-shaped diagrams of Fig. 10. The diagram on the left specifies the sign attributed to the central node, while that on the right does not mention any sign. This is a general rule:

Notation 5.10 If the internal vertices of a splice diagram do not carry signs, this means by convention that they represent oriented Seifert-fibered homology spheres of the type $\Sigma(\alpha_1, \dots, \alpha_n)$ (see Proposition 5.9). If one replaces the (+)-sign in the diagram on the left of Fig. 10 by a (-)-sign, then one obtains by definition a representation of the oppositely oriented manifold to $\Sigma(\alpha_1, \dots, \alpha_n)$. Denote it simply by $-\Sigma(\alpha_1, \dots, \alpha_n)$.

Each end of the splice diagrams of an oriented integral homology sphere $\pm \Sigma(\alpha_1, \dots, \alpha_n)$ represents by construction an oriented knot in the corresponding manifold. Given two such knots $(\epsilon_1 \Sigma(\alpha_1, \dots, \alpha_n), K_1)$ and $(\epsilon_2 \Sigma(\beta_1, \dots, \beta_m), K_2)$ (where ϵ_i is a sign and K_i is a knot corresponding to an end of the corresponding splice diagram), then one may splice them as explained in Definition 5.3. Graphically, one represents this operation by joining the corresponding edges of the two diagrams. An example is shown in Fig. 11.

It is now easy to understand which integral homology sphere corresponds to a given connected splice diagram. Indeed, it is enough to imagine it obtained by successive joining of simpler diagrams along edges adjacent to ends. Then one performs the corresponding splicing operations, taking into account the fact that the end vertices of a splice diagram

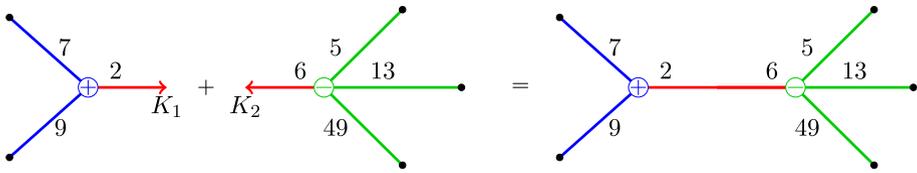


Fig. 11 Splicing two star-shaped diagrams along the knots K_1 and K_2

represent particular oriented knots in the corresponding oriented homology sphere. If one wants to encode not only a manifold, but also a link inside it, then one marks some of the ends of the splice diagram as arrowheads.

If the splice diagram is not connected, then by definition it encodes the connected sum of the links corresponding to its connected components.

A given graph link in an integral homology sphere is representable by an infinite number of diagrams. Among them, one may define the following preferred ones (see [10, Page 72]):

Definition 5.11 A splice diagram is called *minimal* if it minimizes the number of edges among the splice diagrams representing a given graph link.

A minimal splice diagram is unique for a given graph link with all fibers oriented compatibly outside the tori of the splice decomposition (see [10, Corollary 8.3]). There is an algorithmic way to reduce any splice diagram to the minimal one representing the same link (see [10, Theorems 8.1 and 8.2]).

The knowledge of a splice diagram of a graph link Λ inside an oriented integral homology sphere Σ allows to compute very easily the pairwise linking numbers of the components of Λ (see Theorem [10, 10.1]):

Proposition 5.12 *Let $s(\Sigma, \Lambda)$ be a splice diagram for a graph link (Σ, Λ) inside an integral homology sphere Σ . If K_i, K_j are two distinct components of Λ , then the linking number $lk_\Sigma(K_i, K_j)$ is equal to the product of the weights of the germs of edges adjacent to, but not included into the segment of $s(\Sigma, \Lambda)$ which joins the arrowheads corresponding to K_i and K_j , multiplied by the product of the signs of the internal vertices situated on this segment.*

We restrict now to the splice diagrams of the links of reduced germs of curves inside smooth germs of complex surfaces (see [10, Appendix to Chapter I]):

Theorem 5.13 *Let C be a germ of reduced holomorphic curve on the germ of complex analytic smooth surface S . Then its oriented link $\Lambda(C)$ inside the oriented boundary \mathbb{S}^3 of S is a graph link and it has a minimal splice diagram whose vertex signs are all $+$ and whose edge decorations are all strictly positive.*

As explained before, such a totally positive minimal splice diagram of $(\mathbb{S}^3, \Lambda(C))$ is unique. We will call it the *minimal splice diagram of C* . The next theorem explains how to construct it from the Eggers-Wall tree of C relative to a smooth branch L which is transverse to it. It is a more graphical reformulation of Wall’s [41, Theorem 9.8.2] (note that Wall spoke about *Eisenbud-Neumann diagrams* instead of *splice diagrams*). An advantage of speaking about the splice diagram of $C + L$ in the statement below allows a simpler comparison of $\Theta_L(C)$ and of the minimal splice diagram of $C + L$ than in [41], avoiding special cases.

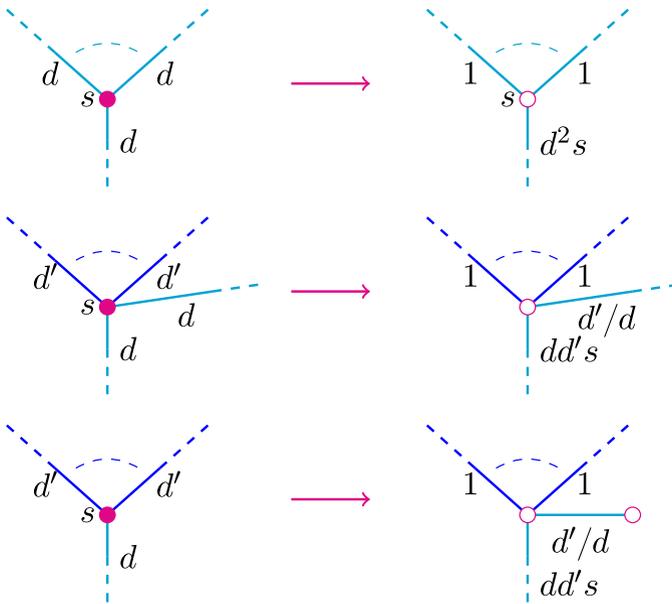


Fig. 12 From the Eggers-Wall tree to the splice diagram

Theorem 5.14 *Let C be a reduced germ of curve and let L be a smooth branch on the smooth germ of surface S , such that L is transversal to C . Then the minimal splice diagram of $C + L$ may be obtained from the Eggers-Wall tree $\Theta_L(C)$ decorated by the contact function c_L and the index functions \mathbf{i}_L by doing the local operations indicated in Fig. 12.*

Proof The topological type of $C + L$ is encoded by either of the following objects (see Wall [41, Proposition 4.3.9, Section 9.8]):

- the collection of characteristic exponents of its branches and of intersection numbers between pairs of branches of $C + L$;
- the Eggers-Wall tree $\Theta_L(C + L)$;
- the minimal splice diagram of $C + L$.

Therefore, in order to prove the theorem it is enough to show that the splice diagram obtained by our construction gives the same characteristic exponents of individual branches and intersection numbers as the starting Eggers-Wall tree. This verification may be done using the description from [10, Appendix to Chapter 1] of the way characteristic exponents are encoded in the splice diagram of a branch and using Proposition 5.12 for the way intersection numbers may be read on a splice diagram of a germ with several branches. Here we use the fact that the intersection number of two distinct branches on S is equal to the linking number of their associated knots in \mathbb{S}^3 .

Let us give now a second proof of the theorem, which furnishes a comparison with Wall’s proof of [41, Theorem 9.8.2]. The transversality hypothesis implies that the tree $\Theta_L(C)$ contains no ramification point of exponent < 1 . We consider another smooth branch L' transversal to the irreducible components of C and to L . The attaching point of L' on the tree $\Theta_L(C)$ is the unit point U of this tree, which has exponent equal to 1. By the inversion theorem 4.5, the Eggers-Wall trees $\Theta_{L'}(C + L)$ and $\Theta_L(C + L')$ have the same exponent and

index functions on the complement of the segment $[L, L']$. We apply the construction of the splice diagram in [41, Theorem 9.8.2] to $\Theta_{L'}(C + L)$. It starts from the *reduced* Eggers-Wall tree $\Theta_{L'}^{red}(C + L)$, which is obtained from $\Theta_{L'}(C + L)$ by removing the segment $[L', U)$ and by unmarking the point U in this tree if this point is not a ramification point of the tree $\Theta_L(C)$ (this corresponds to (i) and (iv) in [41, Theorem 9.8.2]).

In order to make the comparison, Wall considers the *Herbrand function* associated to a branch B of $C + L$, which is a function $H_B : [0, \infty] \rightarrow [0, \infty]$ such that $H_B \circ \mathbf{e}_{B,L'} = \mathbf{c}_{B,L'}$.

The first local operation in Fig. 12 corresponds to point (iii) in Theorem 9.8.2 of [41], when the index function is continuous on the marked point V considered. Wall considers a branch B of $C + L$ through V of multiplicity $m = \mathbf{i}_{L'}(B)$ and such that $P_q \prec_{L'} V \prec_{L'} P_{q+1}$ where P_j are the marked points of the tree $\Theta_{L'}(B)$. Then, the incoming edge at V is marked by $(m^2/e_q^2) \cdot H(\mathbf{e}_{L'}V)$, where $e_q = \mathbf{i}_{L'}(B)/\mathbf{i}_{L'}(V)$. We get the same decoration as in Fig. 12 since:

$$\frac{m^2}{e_q^2} \cdot H_B(\mathbf{e}_{L'}(V)) = (\mathbf{i}_{L'}(V))^2 \mathbf{c}_{L'}(V) = d^2 \cdot s,$$

where we denote $d := \mathbf{i}_{L'}(V)$ and $s := \mathbf{c}_{L'}(V)$.

The second and third local operations in the Fig. below correspond to point (ii) in Theorem 9.8.2 of [41], when the index function is not continuous on the marked point V considered. In the second case, there is a unique branch B_{i_0} of C passing through V such that the index function restricted to this branch is continuous at V . If B_j is any other branch of C , then V is a marked point, say P_q , of the tree $\Theta_{L'}(B_j)$. In terms of Wall's notations, we have $e_q = \mathbf{i}_{L'}(B_j)/\mathbf{i}_{L'}(P_{q+1})$ and $e_{q-1} = \mathbf{i}_{L'}(B_j)/\mathbf{i}_{L'}(P_q)$.

By [41], the outgoing segment at V in the direction of a branch B_i is marked by

$$\frac{e_{q-1}}{e_q} = \frac{\mathbf{i}_{L'}(P_{q+1})}{\mathbf{i}_{L'}(P_q)} = \frac{d'}{d},$$

if $B_i = B_{i_0}$ and by 1 otherwise (where we denoted $d' := \mathbf{i}_{L'}(P_{q+1})$). Let us consider an auxiliary branch K with $(q - 1)$ characteristic exponents, having maximal contact with B_j . By definition, one has $(L' \cdot K) = \mathbf{i}_{L'}(K) = d'$ and $(B_j \cdot L') = e_q \cdot d$. The incoming edge at V is marked by $\bar{\beta}_q/e_q$, where $\{\bar{\beta}_s\}_{s=0}^{q-1}$ denotes the sequence of minimal generators of the semigroup of the branch B_j . By Theorem 3.23, $s = \mathbf{c}_{L'}(V) = (B_j \cdot K)(K \cdot L')^{-1}(B_j \cdot L')^{-1}$, and thus we get the same decoration as in Fig. 12 since:

$$dd's = dd' \frac{(B_j \cdot K)}{(K \cdot L')(B_j \cdot L')} = \frac{(B_j \cdot K)}{e_q} = \frac{\bar{\beta}_q}{e_q}.$$

In the third case, the index function is not continuous on the marked point V considered for all the branches of C containing it. Then, we have to add a side at V marked d'/d to an end vertex, which is not arrow-headed. □

Remark 5.15 If L is not transversal to C , the splice diagram associated to $C + L$ is obtained from the tree $\Theta_L(C + L)$ by doing the local operations indicated in Fig. 12, with respect to the values of the index and contact complexity functions on $\Theta_{L'}(C + L)$, where L' is a smooth branch transversal to $C + L$.

Example 5.16 Consider again our recurrent Example 3.13. Recall that the values of the contact complexity function are represented in Fig. 6. The result of applying Theorem 5.14 is indicated in Fig. 13. One may verify that the application of Proposition 5.12 gives the same values of the intersection numbers $(C_i \cdot C_j)$ as those computed in Example 3.27.

$$(3') \quad \|\lambda\| := \begin{cases} 1 & \text{if } \lambda \in \mathbb{C}^*, \\ 0 & \text{if } \lambda = 0. \end{cases}$$

The term *semivaluation* was introduced as an analog of the more standard term *seminorm*.

If $f \in \mathcal{O}$ defines the divisor D on S and if v is any semivaluation of \mathcal{O} , we set:

$$v(D) := v(f).$$

This definition is independent of the defining function $f \in \mathcal{O}$ of D . Indeed, any other such function is of the form fu , with u a unit of \mathcal{O} . But then $v(u) + v(u^{-1}) = v(1) = 0$, which implies that $v(u) = 0$, as v takes only non-negative values. Therefore one has also $v(fu) = v(f) + v(u) = v(f)$. More generally, if \mathcal{I} is an arbitrary ideal of \mathcal{O} , we set:

$$v(\mathcal{I}) := \min \{v(f) \mid f \in \mathcal{I}\}.$$

This definition generalizes the previous one because the value $v(D)$ computed according to the first definition is equal to the value $v(\mathcal{O}(-D))$ computed according to the second one.

Definition 6.3 Denote by \mathcal{V} the set of semivaluations of \mathcal{O} . We call it the *semivaluation space* of \mathcal{O} or of the germ S . We endow it with the topology of pointwise convergence, that is, with the restriction of the product topology of $[0, \infty]^{\mathcal{O}}$.

The topological space $[0, \infty]^{\mathcal{O}}$ is compact as a product of compact spaces, by Tychonoff’s theorem (see for instance [22, Section 1–10]). The conditions defining semivaluations being closed, we see that:

Proposition 6.4 *The semivaluation space \mathcal{V} is compact.*

Remark 6.5 In contrast to the space \mathcal{V} of semivaluations, the subspace of valuations is not compact. This is the main reason of the importance in our context not only of valuations, but also of semivaluations which are not valuations.

Let us define now the main types of semivaluations which we use in this paper:

Definition 6.6 The *multiplicity valuation* at O , denoted by I^O , is defined by:

$$I^O(f) := m_O(f),$$

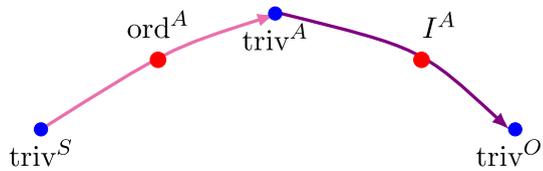
where $m_O(f)$ is the multiplicity of the germ of the curve $f(x, y) = 0$ at O . More generally, if P is an infinitely near point of O , denote by I^P the associated *multiplicity valuation* at P . It may be defined in the following two equivalent ways, starting from a model $(\Sigma, E) \xrightarrow{\psi} (S, O)$ containing P :

- If $f \in \mathcal{O}$, then $I^P(f)$ is the multiplicity of the function $f \circ \psi$ at the point P of the model Σ :

$$I^P(f) := m_P(f \circ \psi).$$

- If $f \in \mathcal{O}$, then $I^P(f)$ is the vanishing order of $f \circ \psi \circ \psi_P$ along $E(P)$, where $\Sigma_P \xrightarrow{\psi_P} \Sigma$ is the blow up of P in Σ and $E(P)$ is the prime exceptional divisor created by it. That is, $I^P(f)$ is the coefficient of $E(P)$ in the divisor of $f \circ \psi \circ \psi_P$.

Fig. 14 The orbits of ord^A and of I^A



Because of this second interpretation, we often denote:

$$\text{ord}^{E(P)} := I^P$$

and we say that $\text{ord}^{E(P)}$ is the *vanishing order* along $E(P)$.

Let A be a branch at O . One has an associated *intersection semivaluation* I^A , defined by:

$$I^A(f) := (A \cdot Z(f)).$$

Note that these are semivaluations which are not valuations, as $I^A(f) = \infty$ precisely for the elements of the principal ideal $\mathcal{O}(-A)$ of the functions vanishing along A .

All the previous examples of semivaluations are centered at O . To any branch A at O is also associated a valuation which is not centered at O , the *vanishing order* ord^A along A :

$$\text{ord}^A(f) := \text{the coefficient of } A \text{ in the divisor of } f.$$

If V is a germ of irreducible subvariety of S through O (that is, either the point O , or a branch A , or S itself), the *trivial semivaluation* triv^V associated to V takes only two values:

$$\text{triv}^V(f) := \begin{cases} \infty & \text{if } f \in \mathcal{O} \text{ vanishes along } V, \\ 0 & \text{otherwise.} \end{cases}$$

Among the trivial semivaluations, only triv^S is a valuation.

Remark 6.7 We have denoted till now by $m_O(C)$ the multiplicity of a germ of curve C at O . We could have chosen to keep this notation, and to write m_P instead of I^P when P is infinitely near O . We have decided not to follow this notational convention, because we will introduce in Definition 7.14 an invariant of semivaluations called *multiplicity*, denoted by \mathbf{m} , and we wanted to avoid the notation “ $\mathbf{m}(m_P)$ ” for the multiplicity of the valuation m_P .

The multiplicative group (\mathbb{R}_+^*, \cdot) acts on the semivaluation space \mathcal{V} by scalar multiplication of the values. We denote by $t\nu \in \mathcal{V}$ the product of $t \in \mathbb{R}_+^*$ and $\nu \in \mathcal{V}$. One may show that this action is continuous. Its orbits allow to relate the three kinds of semivaluations I^A , ord^A and triv^A associated to a branch A at O :

Proposition 6.8 *Let A be any branch through O . Then the orbit of the vanishing order valuation ord^A goes from triv^S to triv^A and the orbit of the intersection semivaluation I^A goes from triv^A to triv^O , that is (see Fig. 14):*

- $\lim_{t \rightarrow 0} (t \text{ord}^A) = \text{triv}^S$ and $\lim_{t \rightarrow \infty} (t \text{ord}^A) = \text{triv}^A$;
- $\lim_{t \rightarrow 0} (t I^A) = \text{triv}^A$ and $\lim_{t \rightarrow \infty} (t I^A) = \text{triv}^O$.

Proposition 6.8 is in fact much more general, as shown by Proposition 6.10 below. Before stating it, let us introduce a new definition.

Definition 6.9 Assume that we work with an arbitrary irreducible analytic or formal germ \mathcal{X} , with local ring \mathcal{R} . The *center* $\mathcal{C}(\nu)$ of a semivaluation ν of \mathcal{R} is the irreducible subvariety of \mathcal{X} defined by the functions $f \in \mathcal{R}$ such that $\nu(f) > 0$. The *support* $\mathcal{S}(\nu)$ of ν is the irreducible subvariety of \mathcal{X} defined by those functions $f \in \mathcal{R}$ such that $\nu(f) = \infty$.

Obviously, one has always the inclusion $\mathcal{C}(\nu) \subseteq \mathcal{S}(\nu)$, with equality precisely when ν is a trivial semivaluation associated to an irreducible subgerm of \mathcal{X} . For instance, one has the following centers and supports of the semivaluations considered in Definition 6.6:

- $\mathcal{C}(I^P) = O, \mathcal{S}(I^P) = S$, for any infinitely near point P of O .
- $\mathcal{C}(I^A) = O, \mathcal{S}(I^A) = A$, for any branch A on S .
- $\mathcal{C}(\text{ord}^A) = A, \mathcal{S}(\text{ord}^A) = S$, for any branch A on S .
- $\mathcal{C}(\text{triv}^O) = O, \mathcal{S}(\text{triv}^O) = O$.
- $\mathcal{C}(\text{triv}^A) = A, \mathcal{S}(\text{triv}^A) = A$, for any branch A on S .
- $\mathcal{C}(\text{triv}^S) = S, \mathcal{S}(\text{triv}^S) = S$.

The announced generalization of Proposition 6.8 is:

Proposition 6.10 *The orbit of ν under scalar multiplication by $t \in \mathbb{R}_+^*$ goes from $\text{triv}^{\mathcal{S}(\nu)}$ to $\text{triv}^{\mathcal{C}(\nu)}$ when t goes from 0 to ∞ .*

Proof Let $f \in \mathcal{R}$ be arbitrary. We have the following possibilities:

- If $\nu(f) = 0$, then $\lim_{t \rightarrow 0} (t \nu)(f) = \lim_{t \rightarrow \infty} (t \nu)(f) = 0$.
- If $\nu(f) \in (0, \infty)$, then $\lim_{t \rightarrow 0} (t \nu)(f) = 0$ and $\lim_{t \rightarrow \infty} (t \nu)(f) = \infty$.
- If $\nu(f) = \infty$, then $\lim_{t \rightarrow 0} (t \nu)(f) = \lim_{t \rightarrow \infty} (t \nu)(f) = \infty$.

The conclusion follows readily from this. □

Let us return to our smooth germ of surface S . In fact, the semivaluations I^A and ord^A associated to the branches A on S may be characterized, up to scalar multiplication, as the only ones whose orbits do not connect triv^S to triv^O :

Proposition 6.11 *Let $\nu \in \mathcal{V}$. If the orbit of ν is not constant and does not go from triv^S to triv^O , then ν is proportional either to I^A (if $\lim_{t \rightarrow 0} (t \nu) = \text{triv}^A$) or to ord^A (if $\lim_{t \rightarrow \infty} (t \nu) = \text{triv}^A$), where A denotes a branch on S .*

Proof This comes from the fact that any irreducible subgerm of S which is distinct from O and S is necessarily a branch A , and that:

- a semivaluation whose center is A is proportional to ord^A ;
- a semivaluation whose support is A is proportional to I^A . □

The other types of semivaluations described in Definition 6.3 do not cover all of \mathcal{V} . One may find concrete descriptions of the remaining possibilities in [11, Sect. 1.5].

The previous considerations show that the quotient of \mathcal{V} under the given action (that is, the space of orbits endowed with the quotient topology), is highly non-Hausdorff, because the closure of any point would contain either the image of triv^S or of triv^O . A way to avoid this is to remove those two trivial semivaluations before doing the quotient. This does still not produce a Hausdorff quotient, because there exist sequences of orbits converging to the union of triv^A and of the orbits of I^A and of ord^A . But this is the only phenomenon which makes the space non-Hausdorff, and if one quotients more, by identifying those three orbits for each branch A , one gets a Hausdorff space:

Definition 6.12 The *projective semivaluation space* or the *valuative tree* $\mathbb{P}(\mathcal{V})$ of \mathcal{O} or of the germ S is the biggest Hausdorff quotient of $\mathcal{V} \setminus \{\text{triv}^S, \text{triv}^O\}$ under the previous action of (\mathbb{R}_+^*, \cdot) . We denote the subspace of non-trivial semivaluations by

$$\mathcal{V}^* := \mathcal{V} \setminus \{\text{triv}^S, \text{triv}^O, \text{triv}^A \mid A \text{ is a branch on } S\}.$$

Let:

$$\pi : \mathcal{V}^* \rightarrow \mathbb{P}(\mathcal{V}) \tag{6.1}$$

be the associated continuous quotient map. We say that an element of $\mathbb{P}(\mathcal{V})$ is a *projective semivaluation* of \mathcal{O} .

The central Theorem 3.14 of [11] implies that (see also [23, Section 7.2]):

Theorem 6.13 $\mathbb{P}(\mathcal{V})$ is a compact \mathbb{R} -tree endowed with its weak topology.

In the sequel it will be important to emphasize the projective semivaluations which are images of vanishing order valuations ord^{E_j} along exceptional prime divisors E_j above $O \in S$ (see Definition 6.6):

Definition 6.14 The *set of rational points* $\mathbb{P}(\mathcal{V})_{\mathbb{Q}} \subset \mathbb{P}(\mathcal{V})$ is the set of images by the projection $\pi : \mathcal{V}^* \rightarrow \mathbb{P}(\mathcal{V})$ of the vanishing orders ord^{E_j} along exceptional prime divisors E_j above $O \in S$.

The name of the set $\mathbb{P}(\mathcal{V})_{\mathbb{Q}}$ is motivated by Proposition 7.19 below.

In Sect. 2 we have not defined \mathbb{R} -trees directly as topological spaces, but as equivalence classes of special partial orders on a set, endowed with a canonically defined “weak” topology. In fact, Favre and Jonsson recognize the structure of \mathbb{R} -tree of $\mathbb{P}(\mathcal{V})$ in the same way, by defining first special partial orders on it. Those partial orders are not defined directly on $\mathbb{P}(\mathcal{V})$, but on sections of the projection π . In turn, those sections are introduced using *normalization* rules relative either to O or to a smooth branch L :

Definition 6.15 A semivaluation $v \in \mathcal{V}$ is *normalized relative to O* if $v(\mathcal{M}) = 1$. Denote by $\mathcal{V}_O \subset \mathcal{V}$ the subspace of semivaluations normalized relative to O . If $v \in \mathcal{V} \setminus \{\text{triv}^O\}$ is centered at O , we denote by $v_O \in \mathcal{V}_O$ the unique semivaluation normalized relative to O which is proportional to v .

Analogously, if L is an arbitrary smooth branch, we define the subspace $\mathcal{V}_L \subset \mathcal{V}$ of semivaluations *normalized relative to L* by the condition $v(L) = 1$, and if $v \in \mathcal{V}$ is not supported by L , we denote by v_L the unique semivaluation in \mathcal{V}_L which is proportional to v .

Notice that we have the following concrete descriptions of the normalizations of a given semivaluation v :

$$v_O = \frac{v}{v(\mathcal{M})}, \quad v_L = \frac{v}{v(L)}. \tag{6.2}$$

Both subspaces \mathcal{V}_O and \mathcal{V}_L are closed inside \mathcal{V} , therefore compact, as \mathcal{V} is compact. On each one of them, one restricts the following partial order on \mathcal{V} :

$$v_1 \leq v_2 \Leftrightarrow v_1(f) \leq v_2(f) \text{ for any } f \in \mathcal{O}. \tag{6.3}$$

Consider also the restrictions to them of the projection π :

$$\pi_O : \mathcal{V}_O \rightarrow \mathbb{P}(\mathcal{V}), \quad \pi_L : \mathcal{V}_L \rightarrow \mathbb{P}(\mathcal{V}). \tag{6.4}$$

What Favre and Jonsson prove in fact is (see [11, Prop. 3.61]):

Theorem 6.16 *Endowed with the restrictions of the previous partial orders, both \mathcal{V}_O and \mathcal{V}_L are compact rooted \mathbb{R} -trees, their roots being I^O and ord^L respectively. The maps π_O and π_L are both homeomorphisms, which induce the same structure of (non-rooted) \mathbb{R} -tree on $\mathbb{P}(\mathcal{V})$. The composed homeomorphism $\pi_L^{-1} \circ \pi_O : \mathcal{V}_O \rightarrow \mathcal{V}_L$ sends I^L to ord^L .*

Let us denote by \leq_O the partial order on $\mathbb{P}(\mathcal{V})$ induced from that of \mathcal{V}_O and by \leq_L the one induced by that of \mathcal{V}_L . Those notations are motivated by the fact that they are the orders induced by the choice of the root at $\pi(I^O)$ and $\pi(\text{ord}^L)$ respectively.

Favre and Jonsson prove in [11] that the multiplicity valuations give by projectivization interior points of $\mathbb{P}(\mathcal{V})$ and that those points are dense inside any finite subtree. They may be characterized as being precisely the ramification points of the tree $\mathbb{P}(\mathcal{V})$. By contrast, the intersection semivaluations are end points. They are not the only ends, but they cannot be characterized purely in terms of the poset or topological structure of the tree $\mathbb{P}(\mathcal{V})$. One needs a supplementary structure on it, a *multiplicity function*. It is one member of a triple of fundamental functions defined on the valuative tree $\mathbb{P}(\mathcal{V})$. The next section is dedicated to them.

7 Multiplicities, log-discrepancies and self-interactions

Either the point O or any smooth branch L may be seen as an *observer* of the valuative tree $\mathbb{P}(\mathcal{V})$. Namely, to each one of them is associated a *coordinate system*, which is a triple of functions defined on $\mathbb{P}(\mathcal{V})$, the *multiplicity*, the *log-discrepancy* and the *self-interaction* relative to that observer. We introduce those functions in Definitions 7.4 and 7.14. In Proposition 7.16 we explain how to express each one of them in terms of the two other ones. Our presentation is a variation on those of Favre and Jonsson [11, Sections 3.3.1, 3.4, 3.6], of Jonsson [23, Section 7] and includes some notions explained in [19].

If E_j is a prime exceptional divisor over $O \in S$, recall that ord^{E_j} denotes the associated vanishing order valuation. Let $\psi : (\Sigma, E) \rightarrow (S, O)$ be the model containing it. We will denote by $(D \cdot D')_\Sigma$ the intersection number of two divisors on Σ without common non-compact branches. Let \check{E}_j be the *dual divisor* of E_j in this model, that is, the only divisor supported by E such that $(\check{E}_j \cdot E_k)_\Sigma = \delta_{j,k}$ for all the components E_k of E .

Definition 7.1 The *log-discrepancy* $\mathbf{l}(\text{ord}^{E_j})$ and the *self-interactions* $\mathbf{s}(\text{ord}^{E_j})$ of the valuation ord^{E_j} are the positive integers defined by:

- $\mathbf{l}(\text{ord}^{E_j}) := 1 + \text{ord}^{E_j}(\psi^*\omega)$, where ω is a non-vanishing holomorphic 2-form on S in the neighborhood of O .
- $\mathbf{s}(\text{ord}^{E_j}) := -(\check{E}_j \cdot \check{E}_j)_\Sigma \geq 1$.

Definition 7.1 is independent of the chosen model containing a representing prime divisor E_j of the valuation ord^{E_j} . This is clear for the log-discrepancy. In the case of the self-interaction, let $\psi : \Xi \rightarrow \Sigma$ be the dominating map between two models. Denote by E_j and E_k two prime exceptional divisors on Σ and by F_j and F_k their strict transforms on Ξ . One can show using the projection formula that $\psi^*\check{E}_j = \check{F}_j$. Then, it follows that $(\check{E}_j \cdot \check{E}_k)_\Sigma = (\check{F}_j \cdot \check{F}_k)_\Xi$ (see for instance [18, Proposition 1.1]). This is the main reason of the importance of the dual divisors \check{E}_j in birational geometry over S . Indeed, the self-intersections $(E_j \cdot E_j)_\Sigma$ are not invariant under blow-ups of points of E_j .

Remark 7.2 We have chosen the letter “I” as the initial of “log-discrepancy” and the letter “s” as initial of “self-interaction”. We think about a self-intersection number as a measure of interaction of an object with itself. See also Proposition 7.10 for another interpretation of this measure of self-interaction. In [11], I is called “thinness” and is denoted “A”, while s is called “skewness” and is denoted “α”. In [23], those names are not used any more, but the notations “A” and “α” remain, “α” being used with an opposite sign convention with respect to [11].

The next proposition shows that the log-discrepancy and the self-interaction may be extended to the whole space \mathcal{V}^* of non-trivial semivaluations introduced in Definition 6.12:

Proposition 7.3 *There exist unique functions $\mathbf{I}, \mathbf{s} : \mathcal{V}^* \rightarrow [0, \infty]$ such that:*

- (1) *In restriction to the valuations ord^{E_j} , one gets the functions introduced in Definition 7.1.*
- (2) *They are continuous in restriction to any subset of the form $\pi^{-1}(T)$, where π is the quotient map (6.1) and T is a finite subtree of $\mathbb{P}(\mathcal{V})$.*
- (3) *\mathbf{I} is homogeneous of degree 1 and \mathbf{s} is homogeneous of degree 2 relative to the action of (\mathbb{R}_+^*, \cdot) .*

Proposition 7.3 is a consequence of Favre and Jonsson’s [11, Sections 3.3, 3.3, 3.9.2] and Jonsson’s [23, Sections 7.4 and 7.5]. Note that the statements about \mathbf{I} were extended by Favre [12, Proposition 1.6] to arbitrary normal surface singularities. See also Remark 7.20 below about the domain of definition of the two functions.

Definition 7.4 Let $\mathbf{I}, \mathbf{s} : \mathcal{V}^* \rightarrow (0, \infty]$ be the functions characterized in Proposition 7.3. If $v \in \mathcal{V}^*$, then $\mathbf{I}(v)$ is called the *log-discrepancy* of v and $\mathbf{s}(v)$ is called its *self-interaction*.

The self-interaction function may be seen as the quadratic function associated to the $(1, 1)$ -bihomogeneous function described by the following proposition, similar to Proposition 7.3 (see [11, Sections 3.9.4 and 7.12], and [19]):

Proposition 7.5 *There exists a unique function $\langle \cdot, \cdot \rangle : \mathcal{V}^* \times \mathcal{V}^* \rightarrow [0, \infty]$ such that:*

- (1) *$\langle \text{ord}^{E_j}, \text{ord}^{E_k} \rangle = -(\check{E}_j \cdot \check{E}_k)_\Sigma$ for any model $\psi : (\Sigma, E) \rightarrow (S, O)$ containing both E_j and E_k .*
- (2) *It is continuous in restriction to any subset of the form $\pi^{-1}(T) \times \pi^{-1}(T)$, where T is a finite subtree of $\mathbb{P}(\mathcal{V})$.*
- (3) *It is bihomogeneous of degree $(1, 1)$ relative to the action of (\mathbb{R}_+^*, \cdot) on both entries.*

The following terminology is taken from [18, Definition 1.6]:

Definition 7.6 If $v_1, v_2 \in \mathcal{V}^*$, we say that $\langle v_1, v_2 \rangle \in \mathbb{R}$ is the *bracket* of v_1 and v_2 .

The bracket is obviously symmetric, and $\mathbf{s}(v) = \langle v, v \rangle$ for any $v \in \mathcal{V}^*$. Proposition 7.10 below gives an alternative description of it for divisorial valuations in terms of the intersection of *curvetas*, which are branches on S defined in the following way:

Definition 7.7 A *curvetta* K_j for a prime divisor E_j over O is the projection to S of a smooth germ of curve K'_j transversal to E_j at a smooth point of the exceptional divisor E of the model $\psi : (\Sigma, E) \rightarrow (S, O)$ containing it.

Remark 7.8 It is folklore knowledge among specialists of singularities that the term “curvetta” was used by Deligne, who took it from the italian geometers of the beginning of the XXth century. This term was used by Deligne in his paper [8], but he called like this the projective line created by blowing up a smooth point of a surface (see [8, Page 13])! In the introduction, he mentioned the 1949 book [7] of Defrise as a source for the italian geometers’ vocabulary concerning point blow ups. One finds indeed in [7, Page 13] the following footnote:

“*Nous avons conservé le mot italien curvetta pour la facilité. Signalons toutefois que ce mot a été employé (ENRIQUES) pour désigner l’ensemble des points voisins du premier ordre [...] d’un point.*”

That is, in our translation:

“*We kept the italian term curvetta for easiness. Let us mention nevertheless that this word was used (ENRIQUES) to denote the set of infinitely near points of the first order [...] of a point.*”

Notice how the use of the term “curvetta” nowadays has evolved into a different meaning (compare with Definition 7.7).

Lemma 7.9 *Let K_j be a curvetta for E_j , and $\psi : (\Sigma, E) \rightarrow (S, O)$ a model as in Definition 7.7. Then:*

$$\psi^* K_j = K'_j - \check{E}_j.$$

In particular, if F is any divisor on Σ supported on E , we have:

$$(K'_j \cdot F)_\Sigma = (\check{E}_j \cdot F)_\Sigma. \tag{7.1}$$

Proof As on a smooth surface the intersection number of a compact divisor with a principal one is 0, we have:

$$(E_k \cdot \psi^* D)_\Sigma = 0$$

for any component E_k of E and for any effective divisor D on S .

Let us apply this fact to $D := K_j$. We decompose $\psi^* K_j = (\psi^* K_j)_{ex} + K'_j$, where $(\psi^* K_j)_{ex}$ is the exceptional part supported on E . We get:

$$0 = (E_k \cdot \psi^* K_j)_\Sigma = (E_k \cdot ((\psi^* K_j)_{ex} + K'_j))_\Sigma = (E_k \cdot (\psi^* K_j)_{ex})_\Sigma + \delta_{k,j}.$$

This equality being valid for all the components E_k of E , we see that $(\psi^* K_j)_{ex} = -\check{E}_j$. \square

Proposition 7.10 *Let E_j and E_l be two prime divisors over O , which are not necessarily distinct and let $\psi : (\Sigma, E) \rightarrow (S, O)$ be a model containing both of them. Consider curvette K_j and K_l for E_j and E_l respectively in this model, If $E_j = E_l$, we assume that the the strict transforms of K_j and K_l by ψ do not pass through the same point of E_j . Then we have:*

$$\langle \text{ord}^{E_j}, \text{ord}^{E_l} \rangle = (K_j \cdot K_l).$$

Proof We apply Lemma 7.9 taking into account the hypothesis $(K'_j \cdot K'_l)_\Sigma = 0$:

$$\begin{aligned}
 (K_j \cdot K_l) &= (\psi^* K_j \cdot \psi^* K_l)_\Sigma \\
 &\stackrel{(7.9)}{=} (K'_j \cdot \psi^* K_l)_\Sigma - (\check{E}_j \cdot \psi^* K_l)_\Sigma \\
 &\stackrel{(7.9)}{=} (K'_j \cdot (K'_l - \check{E}_k))_\Sigma \\
 &= -(K'_j \cdot \check{E}_k)_\Sigma \\
 &\stackrel{(7.1)}{=} -(\check{E}_j \cdot \check{E}_k)_\Sigma \\
 &\stackrel{(7.5)}{=} \langle \text{ord}^{E_j}, \text{ord}^{E_l} \rangle,
 \end{aligned}$$

where we have used again that the intersection of a compact divisor with a principal one on a smooth surface is zero. □

There is also an alternative description of the bracket in the case when one of the semi-valuations is the intersection semivaluation of a branch or the multiplicity valuation at O :

Proposition 7.11 *Let A be a branch on S and $v \in \mathcal{V}^*$. Then:*

$$\langle v, I^A \rangle = v(A).$$

In particular, if A, B are distinct branches at O , one gets $\langle I^A, I^B \rangle = (A \cdot B)$. Analogously:

$$\langle v, I^O \rangle = v(\mathcal{M}).$$

In order to prove this proposition, one uses the viewpoint on the bracket explained in [18, Proposition 2.7, Definitions 2.9 and 2.11].

The log-discrepancy \mathbf{l} and the self-interaction \mathbf{s} are functions defined on \mathcal{V}^* . One may push them down to $\mathbb{P}(\mathcal{V})$ using images of sections of the quotient map $\pi : \mathcal{V}^* \rightarrow \mathbb{P}(\mathcal{V})$. As mentioned in Theorem 6.16, the maps $\pi_O : \mathcal{V}_O \rightarrow \mathbb{P}(\mathcal{V})$ and $\pi_L : \mathcal{V}_L \rightarrow \mathbb{P}(\mathcal{V})$ are homeomorphisms (where L denotes an arbitrary smooth branch), which shows that \mathcal{V}_O and \mathcal{V}_L are such images. This motivates the following definition:

Definition 7.12 The functions $\mathbf{l}_O, \mathbf{s}_O : \mathbb{P}(\mathcal{V}) \rightarrow [0, \infty]$ and $\langle \cdot, \cdot \rangle_O : \mathbb{P}(\mathcal{V}) \times \mathbb{P}(\mathcal{V}) \rightarrow [0, \infty]$ are defined by:

$$\mathbf{l}_O := \mathbf{l} \circ \pi_O^{-1}, \quad \mathbf{s}_O := \mathbf{s} \circ \pi_O^{-1}, \quad \langle \cdot, \cdot \rangle_O := \langle \pi_O^{-1}(\cdot), \pi_O^{-1}(\cdot) \rangle.$$

That is, they are the push-forwards of the functions $\mathbf{l}, \mathbf{s}, \langle \cdot, \cdot \rangle$ by the homeomorphism π_O . They are called the *log-discrepancy relative to O* , the *self-interaction relative to O* and the *bracket relative to O* . One defines analogously three functions $\mathbf{l}_L, \mathbf{s}_L, \langle \cdot, \cdot \rangle_L$ relative to L .

We think of the irreducible subvariety O or L of S as an *observer* of the topological space $\mathbb{P}(\mathcal{V})$, carrying with itself a coordinate system. That is:

Definition 7.13 An *observer* of the valuative tree $\mathbb{P}(\mathcal{V})$ is either the point O or a smooth branch L . The set of observers is considered embedded inside $\mathbb{P}(\mathcal{V})$ through the map $R \rightarrow \pi(I^R)$, which will allow us to write simply R instead of $\pi(I^R)$ abusing slightly of notation.

We work also with a third kind of function on $\mathbb{P}(\mathcal{V})$, relative to an observer, this time with values in $\mathbb{N}^* \cup \infty$:

Definition 7.14 Let R be an observer of $\mathbb{P}(\mathcal{V})$. The *multiplicity relative to R* is the function denoted $\mathbf{m}_R : \mathbb{P}(\mathcal{V}) \rightarrow \mathbb{N}^* \cup \{\infty\}$ and defined by:

$$\mathbf{m}_R(P) := \min\{\langle I^R, I^A \rangle \mid \text{for } A \text{ a branch on } \mathcal{S} \text{ such that } P \preceq_R A\}. \tag{7.2}$$

Here \preceq_R is the partial order relation defined on the tree $\mathbb{P}(\mathcal{V})$ by choosing the root at R .

The triple $(\mathbf{l}_R, \mathbf{s}_R, \mathbf{m}_R)$ is the *coordinate system* on the space $\mathbb{P}(\mathcal{V})$ determined by the observer R . We list the essential properties of the coordinate system associated to any observer in the following three propositions (see [11, Sections 3.3, 3.4, 3.6, 3.9]):

Proposition 7.15 *Let R be an observer of $\mathbb{P}(\mathcal{V})$. Consider the tree $\mathbb{P}(\mathcal{V})$ as a poset with the order relation \preceq_R . Then the following functions are increasing, surjective and continuous on finite subtrees:*

- $\mathbf{l}_R : \mathbb{P}(\mathcal{V}) \rightarrow [\mathbf{l}_R(R), \infty]$, where $\mathbf{l}_O(O) = 2$ and $\mathbf{l}_L(L) = 1$ for any smooth branch L .
- $\mathbf{s}_R : \mathbb{P}(\mathcal{V}) \rightarrow [\mathbf{s}_R(R), \infty]$, where $\mathbf{s}_O(O) = 1$ and $\mathbf{s}_L(L) = 0$ for any smooth branch L .

The multiplicity function $\mathbf{m}_R : \mathbb{P}(\mathcal{V}) \rightarrow \mathbb{N}^* \cup \{\infty\}$ is lower semi-continuous, surjective and increasing, when $\mathbb{N}^* \cup \{\infty\}$ is endowed with the divisibility order relation (in which, by definition, any positive integer divides ∞).

Proposition 7.16 *Let R be an observer of $\mathbb{P}(\mathcal{V})$. One has the following differential relation for $P \in \mathbb{P}(\mathcal{V}) \setminus \{R\}$:*

$$\mathbf{m}_R(P) = \lim_{P_- \rightarrow P, P_- \prec_R P} \frac{\mathbf{l}_R(P) - \mathbf{l}_R(P_-)}{\mathbf{s}_R(P) - \mathbf{s}_R(P_-)}. \tag{7.3}$$

That is, one has in integral form:

$$\mathbf{l}_R(P) - \mathbf{l}_R(R) = \int_R^P \mathbf{m}_R(p) d \mathbf{s}_R(p), \tag{7.4}$$

$$\mathbf{s}_R(P) - \mathbf{s}_R(R) = \int_R^P \frac{1}{\mathbf{m}_R(p)} d \mathbf{l}_R(p). \tag{7.5}$$

Remark 7.17 We could have written the relation (7.3) more concisely as:

$$d \mathbf{l}_R = \mathbf{m}_R d \mathbf{s}_R. \tag{7.6}$$

We will write it sometimes in this way, even if this has, strictly speaking, no meaning in the usual interpretation of differential geometry, as there is no differentiable structure on $\mathbb{P}(\mathcal{V})$ for which \mathbf{l}_R and \mathbf{s}_R are both differentiable.

Proposition 7.18 *(Generalized tripod formulae, see [11], Lemma 3.56 and Lemma 3.69) Let R be an observer of $\mathbb{P}(\mathcal{V})$ and $P, Q \in \mathbb{P}(\mathcal{V})$ be arbitrary. Recall that $\langle R, P, Q \rangle$ denotes the center of the tripod determined by R, P, Q in the tree $\mathbb{P}(\mathcal{V})$ (see Definition 2.13). Then:*

$$\mathbf{s}_R(\langle R, P, Q \rangle) = \langle P, Q \rangle_R. \tag{7.7}$$

Equivalently:

$$\mathbf{s}_R(\langle R, P, Q \rangle) = \frac{\langle v^P, v^Q \rangle}{\langle I^R, v^P \rangle \langle I^R, v^Q \rangle}, \tag{7.8}$$

where $v^P, v^Q \in \mathcal{V}^*$ are arbitrary semivaluations centered at O , which represent P and Q respectively.

Proposition 7.18 generalizes the tripod formula of Proposition 3.26. This is not obvious, as that proposition dealt with contact complexities and the previous one deals with self-interactions. In fact, both functions \mathbf{c}_L and \mathbf{s}_L coincide if one embeds naturally the Eggers-Wall tree Θ_L in the space \mathcal{V}_L of semivaluations normalized relative to L . This embedding is the subject of next section, the coincidence of the two functions being part of the content of its Theorem 8.11.

Note that the set $\mathbb{P}(\mathcal{V})_{\mathbb{Q}}$ of rational points of $\mathbb{P}(\mathcal{V})$, introduced in Definition 6.14 (see [11, Appendix C]), may be characterized in the following way:

Proposition 7.19 *The set $\mathbb{P}(\mathcal{V})_{\mathbb{Q}}$ is equal to $\mathbf{I}_R^{-1}(\mathbb{Q}) = \mathbf{s}_R^{-1}(\mathbb{Q})$, for any observer R .*

We end this section with the following remark which illustrates that the function \mathbf{s} cannot be extended to all trivial semivaluations triv^A for A a branch in S , in such a way that we get a continuous function on $\pi^{-1}(A) \cup \{\text{triv}^A\}$ (compare with the second item in Proposition 7.3).

Remark 7.20 Let us consider a smooth branch L as an observer in $\mathbb{P}(\mathcal{V})$. Its preimage in the space of non-trivial semivaluations \mathcal{V}^* (see Definition 6.12) is equal to the union of two disjoint sets:

$$\pi^{-1}(L) = \{t \cdot \text{ord}^L \mid t \in \mathbb{R}_+^*\} \cup \{t \cdot I^L \mid t \in \mathbb{R}_+^*\}.$$

By Proposition 7.15, we have that $\mathbf{s}(\text{ord}^L) = \mathbf{s}_L(L) = 0$ and $\mathbf{s}(I^L) = \mathbf{s}_O(L) = \infty$. By the homogeneity of the function \mathbf{s} , we get:

$$\mathbf{s}(t \cdot \text{ord}^L) = t^2 \mathbf{s}(\text{ord}^L) = 0, \quad \text{and} \quad \mathbf{s}(t \cdot I^L) = t^2 \mathbf{s}_O(L) = \infty$$

for every $t \in \mathbb{R}_+^*$ (see Proposition 7.3). Proposition 6.8 implies that there is no continuous extension of the function \mathbf{s} to the set $\pi^{-1}(L) \cup \{\text{triv}^L\}$.

8 The valuative embedding of the Eggers-Wall tree

In this section we explain the construction and some properties of a canonical embedding of the Eggers-Wall tree $\Theta_L(C)$ into the valuative tree $\mathbb{P}(\mathcal{V})$ (see Definition 8.9 and Theorem 8.11). Then we prove the result announced in the title of the paper (see Theorem 8.18). We conclude with formulae allowing to compute the log-discrepancy and the self-interaction of a divisorial valuation of the form ord^{E_j} , where E_j is a prime exceptional divisor, in terms of the functions $(\mathbf{e}_L, \mathbf{s}_L, \mathbf{i}_L, \mathbf{i}_L^+)$ evaluated at the associated rational point of $\Theta_L(C)$ (see Proposition 8.16).

As usual, (x, y) is a local coordinate system on S , such that $Z(x) = L$. If $\xi \in \mathbb{C}[[x^{1/\mathbb{N}}]]$ with $\xi(0) = 0$ and $\alpha \in (0, \infty]$, consider the set of Newton-Puiseux series which coincide with ξ up to the exponent α (but not including α):

$$\mathcal{NP}_x(\xi, \alpha) := \{\eta \in \mathbb{C}[[x^{1/\mathbb{N}}]] \mid v_x(\eta - \xi) \geq \alpha\}. \tag{8.1}$$

Let $\xi \in \mathbb{C}[[x^{1/\mathbb{N}}]]$ and $\alpha \in (0, \infty]$ be fixed. Define the map

$$v^{\xi, \alpha} : \mathcal{O} \rightarrow [0, \infty]$$

by:

$$v^{\xi, \alpha}(f) := \inf\{v_x(f(x, \eta)) \mid \eta \in \mathcal{NP}_x(\xi, \alpha)\}. \tag{8.2}$$

Define also the map $v^{\xi, 0} : \mathcal{O} \rightarrow [0, \infty]$ by:

$$v^{\xi, 0} := \text{ord}^L. \tag{8.3}$$

Remark 8.1 The infimum in the definition (8.2) is not always a minimum. For instance, if $\xi = 0$, $\alpha \in (0, 1)$ is irrational and $f(x, y) = y$, then $\nu_x(f(x, \eta)) = \nu_x(\eta)$ may take any rational value in the interval $[\alpha, \infty]$ when η varies in $\mathcal{N}\mathcal{P}_x(\xi, \alpha) = \{\eta \in \mathbb{C}[[x^{1/\mathbb{N}}]] \mid \nu_x(\eta) \geq \alpha\}$. In fact, as an immediate consequence of Proposition 8.4 below, one may prove that the infimum is a minimum precisely when α is rational.

Remark 8.2 If one sets $\|\eta\| := e^{-\nu_x(\eta)}$, one gets a multiplicative non-archimedean norm on the \mathbb{C} -algebra $\mathbb{C}[[x^{1/\mathbb{N}}]]$. Then $\mathcal{N}\mathcal{P}_x(\xi, \alpha)$ is simply the closed ball of center ξ and radius $e^{-\alpha}$ in this normed complex vector space. The definition of the function $\mathcal{N}\mathcal{P}_x(\xi, \alpha)$ parallels Berkovich’s construction of semi-norms on the K -algebra $K[X]$, where K is any non-archimedean field, associating to each element of $K[X]$ its supremum on a given closed ball of K (see Berkovich [4, Section 1.4.4] and Baker and Rumely [2, Page xvi]).

We will see in Proposition 8.7 that the map $\nu^{\xi, \alpha}$ is a semivaluation for any choice of ξ and α . Let us understand first in terms of Eggers-Wall trees what is the value $\nu^{\xi, \alpha}(f)$ and for which series $\eta \in \mathcal{N}\mathcal{P}_x(\xi, \alpha)$, the number $\nu_x(f(x, \eta)) \in [0, \infty]$ achieves it.

Notation 8.3 If $\eta \in \mathbb{C}[[x^{1/\mathbb{N}}]]$ is a Newton-Puiseux series such that $\eta(0) = 0$, we denote by C_η the branch defined by the minimal polynomial of η in $\mathbb{C}[[x]][[y]]$. If A is a branch, recall from Definition 2.13 that $\langle L, C_\eta, A \rangle$ denotes the center of the tripod generated by the ends L, C_η, A of the Eggers-Wall tree $\Theta_L(C_\eta + A)$. If $f \in \mathcal{O}$ and $A = Z(f)$, we denote $\nu^{\xi, \alpha}(f)$ also as $\nu^{\xi, \alpha}(A)$.

Lemma 8.4 Let $f \in \mathcal{O}$ be irreducible and $\eta \in \mathbb{C}[[x^{1/\mathbb{N}}]]$. Then:

$$\nu_x(f(x, \eta)) = \begin{cases} (L \cdot Z(f)) \cdot \mathbf{c}_L(\langle L, C_\eta, Z(f) \rangle) & \text{if } Z(f) \neq L, \\ 1 & \text{if } Z(f) = L. \end{cases} \tag{8.4}$$

Proof The formula is clearly true when $Z(f) = L$.

If $Z(f) \neq L$, notice that $(L \cdot C_\eta) = \mathbf{i}_L(C_\eta)$, where \mathbf{i}_L denotes the index function on $\Theta_L(C_\eta)$, and C_η is viewed as the leaf of this Eggers-Wall tree. One has $\eta = \tilde{\eta}(x^{1/\mathbf{i}_L(C_\eta)})$, where $\tilde{\eta}(t) \in \mathbb{C}[[t]]$. Therefore:

$$\begin{aligned} \nu_x(f(x, \eta)) &= \frac{1}{\mathbf{i}_L(C_\eta)} \cdot \nu_t(f(t^{\mathbf{i}_L(C_\eta)})), \\ \tilde{\eta}(t) &= \frac{(Z(f) \cdot C_\eta)}{(L \cdot C_\eta)} = (L \cdot Z(f)) \cdot \mathbf{c}_L(\langle L, C_\eta, Z(f) \rangle), \end{aligned}$$

the last equality being a consequence of Theorem 3.23. The proof is finished in all cases.

Note that when $Z(f) = C_\eta$, we have $f(x, \eta) = 0$ and $\langle L, C_\eta, Z(f) \rangle = C_\eta$, which shows that both sides of the equality (8.4) are ∞ . □

Proposition 8.5 Let $\xi \in \mathbb{C}[[x^{1/\mathbb{N}}]]$ be a Newton Puiseux series such that $\xi(0) = 0$, let $\alpha \in [0, \infty]$ and let A be a branch. Denote by $P(\alpha) \in \Theta_L(C_\xi)$ the unique point with exponent α . Then:

$$\nu^{\xi, \alpha}(A) = \begin{cases} (L \cdot A) \cdot \mathbf{c}_L(\langle L, P(\alpha), A \rangle) & \text{if } A \neq L, \\ 1 & \text{if } A = L. \end{cases} \tag{8.5}$$

Proof If $A = L$, then the equality results from the fact that $\nu^{\xi, \alpha}(x) = 1$ for all $\alpha \in [0, \infty]$.

We assume from now on that $A \neq L$.

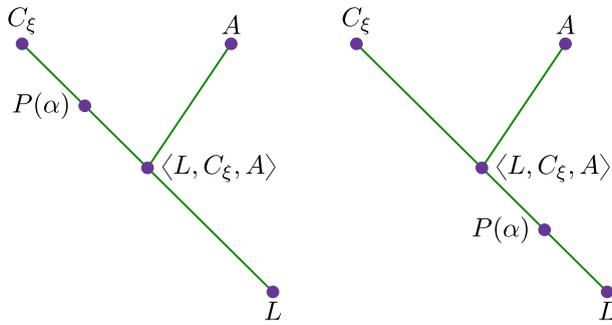


Fig. 15 One has to compare $P(\alpha)$ and $\langle L, C_\xi, A \rangle$

- Suppose first that $\alpha = 0$. Then, by definition 8.3, $v^{\xi,0} = \text{ord}^L$. As we assumed that $A \neq L$, this implies that $v^{\xi,0}(A) = 0$. But the right-hand side is also 0, because $P(0) = L$, $\langle L, P(0), A \rangle = L$, $\mathbf{c}_L(L) = 0$, and $(L \cdot A) < +\infty$.
 - Suppose now that $\alpha > 0$. The condition $\eta \in \mathcal{NP}_x(\xi, \alpha)$ implies that the attaching point $\pi_{[L, C_\xi]}(C_\eta) = \langle L, C_\xi, C_\eta \rangle$ of C_η in $\Theta_L(C_\xi)$ belongs to the segment $[P(\alpha), C_\xi]$. We will consider two cases, according to the position of $\langle L, C_\xi, A \rangle$ relative to $P(\alpha)$ on the segment $[L, C_\xi]$.
- Assume that $\langle L, C_\xi, A \rangle <_L P(\alpha)$ (see the tree on the left of Fig. 15). This implies the equality

$$\langle L, C_\eta, A \rangle = \langle L, P(\alpha), A \rangle$$

for all $\eta \in \mathcal{NP}_x(\xi, \alpha)$. We deduce the assertion

$$v^{\xi,\alpha}(f) = (L \cdot A) \cdot \mathbf{c}_L(\langle L, P(\alpha), A \rangle)$$

from Formula (8.4).

- Assume that $\langle L, C_\xi, A \rangle \geq_L P(\alpha)$ (see the tree on the right in Fig. 15). When η varies in $\mathcal{NP}_x(\xi, \alpha)$, the point $\langle L, C_\eta, A \rangle$ varies surjectively in the set of rational points of the segment $[P(\alpha), \langle L, C_\xi, A \rangle]$. Since those points are dense in this segment, we deduce from Formula (8.4) that:

$$\begin{aligned} v^{\xi,\alpha}(f) &= (L \cdot A) \cdot \inf\{\mathbf{c}_L(P) \mid P \in [P(\alpha), \langle L, C_\xi, A \rangle] \text{ is rational}\} \\ &= (L \cdot A) \cdot \mathbf{c}_L(P(\alpha)) \\ &= (L \cdot A) \cdot \mathbf{c}_L(\langle L, P(\alpha), A \rangle). \end{aligned}$$

By combining the results of the two cases, we get the announced conclusion for $\alpha > 0$. \square

We need also the following lemma in order to prove in Proposition 8.7 that the map $v^{\xi,\alpha}$ is a semi-valuation:

Lemma 8.6 *Let us fix $\xi \in \mathbb{C}[[x^{1/\mathbb{N}}]]$ and $\alpha \in (0, \infty]$. If $\eta_1, \eta_2 \in \mathcal{NP}_x(\xi, \alpha)$ and if $f_1, f_2 \in \mathcal{O}$, then there exists $\eta \in \mathcal{NP}_x(\xi, \alpha)$ such that:*

$$v_x(f_i(x, \eta)) \leq v_x(f_i(x, \eta_i)), \quad \text{for } i = 1, 2.$$

Proof Let us denote by Θ the Eggers-Wall tree relative to L of the reduced effective divisor whose branches are $C_\xi, C_{\eta_1}, C_{\eta_2}, Z(f_1)$ and $Z(f_2)$. By definition, if $\eta_i \in \mathcal{NP}_x(\xi, \alpha)$,

then the point $P_i = \langle L, C_\xi, C_{\eta_i} \rangle$ is $\succeq_L P(\alpha)$ in the tree Θ for $i = 1, 2$. The segment $[P(\alpha), \min\{P_1, P_2\}]$ contains a rational point P since its right hand extremity is rational. Since $P \succeq_L P(\alpha)$ is a rational point, there exists a Newton-Puiseux series $\eta \in \mathcal{N}\mathcal{P}_x(\xi, \alpha)$ such that the attaching point of the branch C_η on the Eggers-Wall tree Θ is P . Let us check that η verifies the assertion.

If $Z(f_i) = L$ for some $i \in \{1, 2\}$, then the inequality of the statement trivially holds. Assume then that f_i is irreducible and $Z(f_i) \neq L$ for all $i \in \{1, 2\}$. Set $Q_i = \langle L, C_\xi, Z(f_i) \rangle$. We get from the definition of the tree Θ that $\langle L, C_{\eta_i}, Z(f_i) \rangle = \min\{P_i, Q_i\}$. Similarly, the point $\langle L, C_\eta, Z(f_i) \rangle$ is equal to $\min\{P, Q_i\}$. Since $P \preceq \min\{P_1, P_2\}$, we obtain the inequality:

$$\langle L, C_\eta, Z(f_i) \rangle \preceq_L \langle L, C_{\eta_i}, Z(f_i) \rangle.$$

In this case, the assertion follows from this and Formula (8.4), taking into account that the function \mathbf{c}_L is increasing.

In the general case, by applying the previous argument to the irreducible components $f_{i,j}$ of $f_i = \prod_j f_{i,j}$, we see that:

$$v_x(f_{i,j}(x, \eta)) \leq v_x(f_{i,j}(x, \eta_i)).$$

Since v_x is a valuation on $\mathbb{C}[[x^{1/\mathbb{N}}]]$, we get:

$$v_x(f_i(x, \eta)) = \sum_j v_x(f_{i,j}(x, \eta)) \leq \sum_j v_x(f_{i,j}(x, \eta_i)) = v_x(f_i(x, \eta_i)).$$

□

Proposition 8.7 *The map $v^{\xi, \alpha} : \mathcal{O} \rightarrow [0, \infty]$ belongs to the set \mathcal{V}_L of semivaluations normalized relative to $L = Z(x)$, introduced in Definition 6.15.*

Proof If $\alpha = 0$, the statement is clear, because $v^{\xi, 0} = \text{ord}^L$.

Consider from now on the case $\alpha > 0$. Let us prove successively the three conditions (1), (2), (3) of Definition 6.1, ensuring that $v^{\xi, \alpha}$ is a semivaluation on \mathcal{O} .

- *Proof of condition (1).* Consider two functions $f, g \in \mathcal{O}$. As v_x is a valuation of $\mathbb{C}[[x^{1/\mathbb{N}}]]$, we have:

$$v_x(f(x, \eta) \cdot g(x, \eta)) = v_x(f(x, \eta)) + v_x(g(x, \eta))$$

for all $\eta \in \mathcal{N}\mathcal{P}_x(\xi, \alpha)$. But, by the definition of $v^{\xi, \alpha}$: $v_x(f(x, \eta)) \geq v^{\xi, \alpha}(f)$ and $v_x(g(x, \eta)) \geq v^{\xi, \alpha}(g)$. This implies that: $v_x(f(x, \eta) \cdot g(x, \eta)) \geq v^{\xi, \alpha}(f) + v^{\xi, \alpha}(g)$. Passing to the infimum of the left-hand-sides over $\eta \in \mathcal{N}\mathcal{P}_x(\xi, \alpha)$, we get the inequality:

$$v^{\xi, \alpha}(f \cdot g) \geq v^{\xi, \alpha}(f) + v^{\xi, \alpha}(g).$$

We want now to show that in fact this is an equality. We will prove this by showing that one has always also the converse inequality:

$$v^{\xi, \alpha}(f \cdot g) \leq v^{\xi, \alpha}(f) + v^{\xi, \alpha}(g). \tag{8.6}$$

Let us consider $\eta_1, \eta_2 \in \mathcal{N}\mathcal{P}_x(\xi, \alpha)$. By Lemma 8.6, there exists a series $\eta \in \mathcal{N}\mathcal{P}_x(\xi, \alpha)$ such that

$$\begin{aligned} v_x(f(x, \eta)) &\leq v_x(f(x, \eta_1)), \\ v_x(g(x, \eta)) &\leq v_x(g(x, \eta_2)). \end{aligned}$$

By summing both inequalities, we get:

$$v_x((f \cdot g)(x, \eta)) \leq v_x(f(x, \eta_1)) + v_x(g(x, \eta_2)).$$

Therefore:

$$v_x^{\xi, \alpha}(f \cdot g) \leq v_x(f(x, \eta_1)) + v_x(g(x, \eta_2)).$$

This being true for all $\eta_1, \eta_2 \in \mathcal{NP}_x(\xi, \alpha)$, we may take the infimum over those choices, and get the desired converse inequality (8.6).

- *Proof of condition (2).* Consider again two functions $f, g \in \mathcal{O}$. As v_x is a valuation of $\mathbb{C}[[x^{1/\mathbb{N}}]]$, we have:

$$v_x(f(x, \eta) + g(x, \eta)) \geq \min\{v_x(f(x, \eta)), v_x(g(x, \eta))\}$$

for all $\eta \in \mathcal{NP}_x(\xi, \alpha)$. This implies, as in the previous reasoning, that:

$$v_x(f(x, \eta) + g(x, \eta)) \geq \min\{v_x^{\xi, \alpha}(f), v_x^{\xi, \alpha}(g)\}.$$

Passing to the infimum of the left-hand-sides over $\eta \in \mathcal{NP}_x(\xi, \alpha)$, we get the desired inequality:

$$v_x^{\xi, \alpha}(f + g) \geq \min\{v_x^{\xi, \alpha}(f), v_x^{\xi, \alpha}(g)\}.$$

- *Proof of condition (3).* This is immediate from the definition.

Finally notice that $v_x^{\xi, \alpha}(x) = 1$, thus the semivaluation $v_x^{\xi, \alpha}$ is normalized relative to L . \square

Remark 8.8 It is clear from the definition that if $0 < \alpha < \infty$, then the semivaluation $v_x^{\xi, \alpha}$ is actually a valuation centered at \mathcal{O} in the sense of Definition 6.1. We know that $v_x^{\xi, 0} = \text{ord}^L$, while by Proposition 8.5 one has $v_x^{\xi, \infty} = I_L^{C_\xi}$ (see Definition 6.6 and Formula (6.2)). This is because $\mathcal{NP}_x(\xi, \infty) = \{\xi\}$ and for any irreducible element $f \in \mathcal{O}$, we have:

$$v_x^{\xi, \infty}(f) = v_x(f(x, \xi)) \stackrel{(8.4)}{=} (L \cdot Z(f)) \cdot \mathbf{c}_L(\langle L, C_\xi, Z(f) \rangle) \stackrel{(3.5)}{=} \frac{(C_\xi \cdot Z(f))}{(L \cdot C_\xi)} = I_L^{C_\xi}(f).$$

Definition 8.9 Let C be a reduced germ of curve on S and L be a smooth branch. Its associated *valuative map* is defined by:

$$\begin{aligned} V_L : \Theta_L(C) &\rightarrow \mathcal{V}_L \\ P &\rightarrow V_L^P := v_x^{\xi, \alpha}, \end{aligned} \tag{8.7}$$

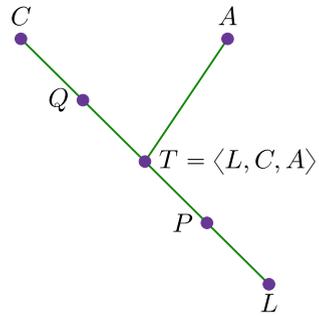
being P the point of exponent α in the segment $[L, C_\xi]$ of $\Theta_L(C)$, where C_ξ is a component of C .

The map V_L is well-defined, in the sense that it depends neither on the local coordinate system (x, y) such that $Z(x) = L$, nor on the choice of the component C_ξ of C verifying $P \in [L, C_\xi]$ and of Newton-Puiseux series ξ defining it. This results from the following proposition which allows to compute the values taken by V_L^P on any branch (hence on any divisor, by the additivity property (1) in the Definition 6.1 of valuations):

Proposition 8.10 *Let C be a reduced germ of curve on S and A be any branch on S . Fix a smooth reference branch L . If $P \in \Theta_L(C)$, then:*

$$V_L^P(A) = \begin{cases} (L \cdot A) \cdot \mathbf{c}_L(\langle L, P, A \rangle) & \text{if } A \neq L, \\ 1 & \text{if } A = L. \end{cases}$$

Fig. 16 The case when P and Q are comparable



Proof This follows by reformulating Proposition 8.5 in terms of Definition 8.9. □

We state now the embedding theorem of the Eggers-Wall tree in the \mathbb{R} -tree of normalized semivaluations:

Theorem 8.11 *The map V_L is an increasing embedding of rooted trees, which sends the root L of $\Theta_L(C)$ onto the root ord^L of \mathcal{V}_L and the end C_i of $\Theta_L(C)$ onto the end $I_L^{C_i}$ of \mathcal{V}_L for each branch C_i of C . Under this embedding, the function $1 + \mathbf{e}_L$ is identified with the relative log-discrepancy \mathbf{l}_L , the index function \mathbf{i}_L with the relative multiplicity \mathbf{m}_L and the contact complexity \mathbf{c}_L with the self-interaction \mathbf{s}_L .*

Proof We will prove successively the various statements of the theorem.

- *The map V_L is increasing.* Consider two points $P, Q \in \Theta_L(C)$, with $P \preceq_L Q$. Therefore, there exists a branch C_i of C such that $P, Q \in \Theta_L(C_i)$.

Consider an arbitrary function $f \in \mathcal{O}$. By the definition of the order relation \preceq_L on \mathcal{V}_L , we want to show that $V_L^P(f) \leq V_L^Q(f)$. It is enough to prove this inequality when f is irreducible, because it extends then to arbitrary f by the additivity property (1) in the Definition 6.1 of semivaluations.

Assume therefore that f is irreducible. Let A be the branch defined by it. By Proposition 8.10, the inequality is equivalent to $\mathbf{c}_L(\langle L, P, A \rangle) \leq \mathbf{c}_L(\langle L, Q, A \rangle)$. But this is obvious, as $P \preceq_L Q$ implies $\langle L, P, A \rangle \preceq_L \langle L, Q, A \rangle$, and the function \mathbf{c}_L is increasing.

- *The map V_L is injective.* Let us consider two distinct points $P, Q \in \Theta_L(C)$. We want to show that there exists a branch A such that $V_L^P(A) \neq V_L^Q(A)$. We will consider two cases, according to the comparability or incomparability of P and Q for the partial order relation \preceq_L .

- Assume that P and Q are comparable for \preceq_L , say $P \prec_L Q$.

By restricting C to a suitable branch of it, we can suppose that C is irreducible and that $P, Q \in \Theta_L(C)$. Let T be a rational point of the open segment (P, Q) of $\Theta_L(C)$, and let A be a branch on S whose attaching point $\langle L, C, A \rangle$ in $\Theta_L(C)$ is T (see Fig. 16).

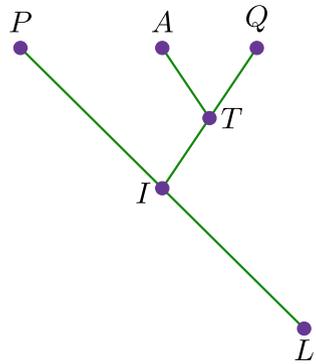
We have then:

$$\langle L, P, A \rangle = P, \quad \langle L, Q, A \rangle = T.$$

As the function \mathbf{c}_L is strictly increasing on $\Theta_L(C)$ and $P \prec_L T$, we deduce that $\mathbf{c}_L(P) < \mathbf{c}_L(T)$. By Proposition 8.10, we conclude that $V_L^P(A) < V_L^Q(A)$.

- Assume that P and Q are incomparable for \preceq_L .

Fig. 17 The case when P and Q are incomparable



Denote $I := P \wedge_L Q = \langle L, P, Q \rangle$. We have the strict inequalities $I \prec_L P, I \prec_L Q$. Choose a rational point $T \in (I, Q)$. Therefore there exists a branch A on S such that its attaching point in $\Theta_L(C)$ is the point T (see Fig. 17).

We deduce that:

$$\langle L, P, A \rangle = I, \text{ and } \langle L, Q, A \rangle = T.$$

As the function \mathbf{c}_L is strictly increasing on $\Theta_L(C)$ and $I \prec_L T$, we deduce that $\mathbf{c}_L(I) < \mathbf{c}_L(T)$. By Proposition 8.10, we conclude that $V_L^P(A) < V_L^Q(A)$.

• *The map V_L is continuous.* It is enough to prove that V_L is continuous when C is a branch. By the definition of the weak topology on the semivaluation space \mathcal{V} , this amounts to proving the continuity of the following map:

$$\begin{cases} \Theta_L(C) & \rightarrow [0, \infty] \\ P & \rightarrow V_L^P(A) \end{cases}$$

for any fixed branch A . But this is an immediate consequence of Proposition 8.10.

- *The map V_L sends L to ord^L and C_i to $I_L^{C_i}$.* This follows from Remark 8.8.
- *The map V_L identifies \mathbf{c}_L with \mathbf{s}_L .* We will prove this in restriction to the rational points of $\Theta_L(C)$. Such a point is the center $\langle L, C_i, A \rangle$ of a tripod, where C_i is a branch of C and A is a certain branch on S . We may assume as before that C is irreducible (therefore $C_i = C$), and that we look at the point $\langle L, C, A \rangle$. Then the fact that V_L is continuous, injective and increasing implies that $V_L^{\langle L, C, A \rangle} = \langle \text{ord}^L, I_L^C, I_L^A \rangle$.

By Theorem 3.23, we have:

$$\mathbf{c}_L(\langle L, C, A \rangle) = \frac{(C \cdot A)}{(L \cdot C)(L \cdot A)}.$$

By Theorem 7.18, we also have:

$$\mathbf{s}_L(\langle \text{ord}^L, I_L^C, I_L^A \rangle) = \frac{\langle I^C, I^A \rangle}{\langle I^L, I^C \rangle \langle I^L, I^A \rangle}.$$

Proposition 7.11 shows then that the right-hand sides of the two previous equalities coincide.

As the statement is true for the rational points, which are dense in $\Theta_L(C)$, and both \mathbf{c}_L and \mathbf{s}_L are continuous, we deduce that the statement is true for all points.

- *The map V_L identifies \mathbf{i}_L with \mathbf{m}_L .* We reason analogously, by first proving the statement for rational points of $\Theta_L(C)$. Let P be such a point. We may choose a branch A such

that $P \in \Theta_L(A)$ and $\mathbf{i}_L(P) = \mathbf{i}_L(A)$. Since the index function is increasing relative to \preceq_L , one has:

$$\mathbf{i}_L(A) = \min\{\mathbf{i}_L(B) \mid \text{for } B \text{ a branch on } S \text{ such that } P \preceq_R B\}. \tag{8.8}$$

If B is a branch, then $\langle I^L, I^B \rangle = (L \cdot B)$ by Proposition 7.11 and $\mathbf{i}_L(B) = (L \cdot B)$ by Remark 3.25. Using these equalities, the equation (8.8) is reformulated as:

$$\langle I^L, I^A \rangle = \min\{\langle I^L, I^B \rangle \mid \text{for } B \text{ a branch on } S \text{ such that } P \preceq_L B\}.$$

By definition, this last minimum is $\mathbf{m}_L(V_L^P)$. The conclusion follows.

- *The map V_L identifies $1 + \mathbf{e}_L$ with $\mathbf{1}_L$.* As a direct consequence of the differential relations $d\mathbf{1}_L = \mathbf{m}_L ds_L$ and $d\mathbf{e}_L = \mathbf{i}_L d\mathbf{c}_L$ (see Remark 7.17), we see that there exists a constant $a \in \mathbb{R}$ such that $\mathbf{e}_L + a$ is sent to $\mathbf{1}_L$ by the map V_L . As $\mathbf{e}_L(L) = 0$ and $\mathbf{1}_L(\text{ord}^L) = 1$, we deduce that $a = 1$. □

Remark 8.12 As it was the case with the Eggers-Wall tree $\Theta_L(C)$ itself, the map V_L depends only on C and on the smooth branch L . Namely, for every branch C_i of C such that $P \in \Theta_L(C_i)$, V_L^P is the unique semivaluation of the segment $[\text{ord}^L, I_L^{C_i}] \subset \mathcal{V}_L$ whose self-interaction is equal to $\mathbf{c}_L(P)$.

Remark 8.13 A variant of the map V_L was already defined by Favre and Jonsson in [11, Prop. D1, page 223]. They started from a generic Eggers-Wall tree and a generic version of the exponent function. They associated to any point of it of exponent α , situated on the segment $[O, C_i]$, the unique point of the segment $[I^O, I_O^{C_i}]$ with log-discrepancy $1 + \alpha$ relative to O . They did not give another interpretation of that map, for instance analogous to our definition (8.2).

Theorem 8.11 implies that the embedding V_L identifies the set $\Theta_L(C)_{\mathbb{Q}}$ of rational points of $\Theta_L(C)$ in the sense of Definition 3.14 with the set of semivaluations of its image in \mathcal{V}_L which have rational log-discrepancies. By Proposition 7.19, we know that those are exactly the semivaluations of the image $V^L(\Theta_L(C))$ which are proportional to order of vanishing valuations ord^{E_j} along prime exceptional divisors E_j above $O \in S$.

Our aim now is to explain how to compute the log-discrepancies and the self-interactions of such order of vanishing valuations in terms of the Eggers-Wall tree $\Theta_L(C)_{\mathbb{Q}}$. We get the formulae of Proposition 8.16. Before proving them, we introduce the following terminology for those valuations and we prove a lemma about the computation of intersection numbers of branches on S using models of S .

Definition 8.14 Let P be a rational point of $\Theta_L(C)$. Denote by E_P the unique prime exceptional divisor (up to birational transformations) such that ord^{E_P} is proportional to V_L^P . We call it the *representing divisor* of P .

Recall from Definition 7.7 the notion of *curvetta* used in the following lemma:

Lemma 8.15 *Let P be a rational point of $\Theta_L(C)$ and K_P be a curvetta for its representing divisor E_P . Consider a model $\pi : (\Sigma, E) \rightarrow (S, O)$ containing the divisor E_P . Let A, B be two branches on S such that the strict transform B' of B intersects E_P at a smooth point of the exceptional divisor E of π . Then:*

$$(A \cdot B) = (A' \cdot B')_{\Sigma} + \text{ord}^{E_P}(A) \cdot (E_P \cdot B')_{\Sigma}. \tag{8.9}$$

In particular, if B is a curvetta for E_P , then:

$$(A \cdot B) = (A' \cdot B')_{\Sigma} + \text{ord}^{E_P}(A). \tag{8.10}$$

Proof We have $\pi^*A = (\pi^*A)_{\text{ex}} + A'$, where A' denotes the strict transform of A by π and $(\pi^*A)_{\text{ex}} = \sum_j \text{ord}^{E_j}(A)E_j$ is a divisor supported on the irreducible components E_j of the exceptional divisor E of π . Similarly, we denote $\pi^*B = (\pi^*B)_{\text{ex}} + B'$. Then, using the fact that $(\pi^*A \cdot (\pi^*B)_{\text{ex}})_{\Sigma} = 0$, we get:

$$\begin{aligned} (A \cdot B) &= (\pi^*A \cdot \pi^*B)_{\Sigma} \\ &= (((\pi^*A)_{\text{ex}} + A') \cdot B')_{\Sigma} \\ &= (A' \cdot B')_{\Sigma} + ((\pi^*A)_{\text{ex}} \cdot B')_{\Sigma} \\ &= (A' \cdot B')_{\Sigma} + \text{ord}^{E_P}(A) \cdot (E_P \cdot B')_{\Sigma}, \end{aligned}$$

which proves equation (8.9). The last equality is a consequence of the hypothesis that B' intersects E_P at a smooth point of the exceptional divisor E of π . In the special case in which B is a curvetta for E_P , we get equation (8.10) using the fact that, by definition, $(E_P \cdot B')_{\Sigma} = 1$. □

The following proposition explains how to compute the log-discrepancy and the self-interaction of ord^{E_P} in terms of the functions $(\mathbf{e}_L, \mathbf{i}_L, \mathbf{c}_L)$ evaluated at $P \in \Theta_L(C)_{\mathbb{Q}}$. Recall that the upper index function \mathbf{i}_L^+ was introduced in Definition 3.14.

Proposition 8.16 *Let P be a rational point of $\Theta_L(C)$. Then the following formulae are valid for its representing divisor E_P :*

- (1) $\text{ord}^{E_P}(L) = \mathbf{i}_L^+(P)$.
- (2) $\mathbf{I}_L(\text{ord}^{E_P}) = \mathbf{i}_L^+(P) \cdot (1 + \mathbf{e}_L(P))$.
- (3) $-\check{E}_P \cdot \check{E}_P = \mathbf{s}_L(\text{ord}^{E_P}) = \mathbf{i}_L^+(P)^2 \cdot \mathbf{c}_L(P)$.

Proof Let us prove successively the three formulae. Note that in our proofs the initial germ C whose Eggers-Wall tree $\Theta_L(C)$ contains the rational point P plays no role. We choose in fact a branch A_0 on S such that $P \in \Theta_L(A_0)$ and $\mathbf{i}_L(P) = \mathbf{i}_L(A_0)$.

(1) Consider a model $\pi : (\Sigma, E) \rightarrow (S, O)$ containing E_P . If B is any branch on S , we denote by B' its strict transform on Σ . By replacing the model if it is necessary, we can assume that $(L' \cdot A'_0)_{\Sigma} = 0$. Choose a curvetta K_P for E_P such that $(L' \cdot K'_P)_{\Sigma} = 0$ and also:

$$(A'_0 \cdot K'_P)_{\Sigma} = 0. \tag{8.11}$$

By Lemma 8.15 applied to $A := L$ and $B := K_P$, we obtain that

$$(L \cdot K_P) = \text{ord}^{E_P}(L). \tag{8.12}$$

Let D be a branch such that $P <_L D$. By applying (8.10) to $A := D$ and $B := K_P$, we get:

$$\text{ord}^{E_P}(D) = (D \cdot K_P) - (D' \cdot K'_P)_{\Sigma}.$$

Therefore:

$$\begin{aligned} V_L^P(D) &= \frac{\text{ord}^{E_P}(D)}{\text{ord}^{E_P}(L)} \\ &= \frac{(D \cdot K_P)}{(L \cdot K_P)} - \frac{(D' \cdot K'_P)_{\Sigma}}{(L \cdot K_P)} \stackrel{(3.5)}{=} (L \cdot D) \left(\mathbf{c}_L(\langle L, A, K_P \rangle) - \frac{(D' \cdot K'_P)_{\Sigma}}{(L \cdot D)(L \cdot K_P)} \right). \end{aligned}$$

Proposition 8.10 gives us a second expression for $V_L^P(D)$:

$$V_L^P(D) = (L \cdot D) \cdot \mathbf{c}_L(\langle L, P, D \rangle).$$

By comparing the two expressions, we get:

$$c_L(\langle L, K_P, D \rangle) = c_L(\langle L, P, D \rangle) + \frac{(D' \cdot K'_P)_\Sigma}{(L \cdot D)(L \cdot K_P)}.$$

Taking into account that the contact complexity function c_L is strictly increasing on the poset $(\Theta_L(K_P + D), \preceq_L)$, that $\langle L, P, D \rangle = P$ (since $P \prec_L D$) and that P and $\langle L, K_P, D \rangle$ are \preceq_L -comparable (since they both belong to the segment $[L, D] \subseteq \Theta_L(K_P + D)$), we get:

$$\langle L, K_P, D \rangle \succeq_L P, \text{ with equality if and only if } (D' \cdot K'_P)_\Sigma = 0. \tag{8.13}$$

Let us apply relation (8.13) to the branch $D := A_0$. Since $(A'_0 \cdot K'_P)_\Sigma = 0$ by the hypothesis (8.11) we must have $P = \langle L, K_P, A_0 \rangle$. This means that P is the attaching point of K_P to the tree $\Theta_L(A_0)$. Take a segment $(P, P') \subset [P, K_P]$ such that the restriction of the index function to it is constant. Then, by Lemma 3.15 we have $\mathbf{i}_L(P') = \mathbf{i}_L^+(P)$. This implies that:

$$(L \cdot K_P) = \mathbf{i}_L(K_P) \geq \mathbf{i}_L^+(P). \tag{8.14}$$

Let us apply now relation (8.13) to $D := G$, where G is a branch on S such that

$$P' = \langle L, K_P, G \rangle \succ_L P \tag{8.15}$$

and $\mathbf{i}_L(G) = \mathbf{i}_L^+(P)$. By relation (8.13) we have the inequality $(G' \cdot K'_P)_\Sigma > 0$, which allows us to apply Lemma 8.15 with $A := L$ and $B := G$. It follows that:

$$\mathbf{i}_L^+(P) = \mathbf{i}_L(G) = (L \cdot G) = \text{ord}^{E_P}(L) \cdot (E_P \cdot G')_\Sigma \stackrel{(8.12)}{\geq} (L \cdot K_P). \tag{8.16}$$

By combining the relations (8.12), (8.16) and (8.14), we see that:

$$\text{ord}^{E_P}(L) = (L \cdot K_P) = \mathbf{i}_L(K_P) = \mathbf{i}_L(G) = \mathbf{i}_L^+(P).$$

This ends the proof of (1).

(2) By Definition 8.14, the valuations V_L^P and ord^{E_P} are proportional. As $V_L^P \in \mathcal{V}_L$, we deduce that:

$$V_L^P = \frac{\text{ord}^{E_P}}{\text{ord}^{E_P}(L)}.$$

By point (1) of the present proposition, we get:

$$V_L^P = \frac{\text{ord}^{E_P}}{\mathbf{i}_L^+(P)}.$$

Using also the equality $\mathbf{I}_L(V_L^P) = 1 + \mathbf{e}_L(P)$, which is a consequence of the formula $V_L^* \mathbf{I}_L = 1 + \mathbf{e}_L$ of Theorem 8.11, and the homogeneity of degree 1 of the log-discrepancy \mathbf{I} (see Proposition 7.3), we get:

$$\begin{aligned} \mathbf{I}_L(\text{ord}^{E_P}) &= \mathbf{i}_L^+(P) \cdot \mathbf{I}_L\left(\frac{\text{ord}^{E_P}}{\mathbf{i}_L^+(P)}\right) = \\ &= \mathbf{i}_L^+(P) \cdot \mathbf{I}_L(V_L^P) = \\ &= \mathbf{i}_L^+(P) \cdot (1 + \mathbf{e}_L(P)). \end{aligned}$$

(3) Reasoning analogously, using the formula $V_L^* \mathbf{s}_L = \mathbf{c}_L$ and the homogeneity of degree 2 of the self-interaction \mathbf{s} , we get:

$$\begin{aligned} \mathbf{s}_L(\text{ord}^{E_P}) &= \mathbf{i}_L^+(P)^2 \cdot \mathbf{s}_L\left(\frac{\text{ord}^{E_P}}{\mathbf{i}_L^+(P)}\right) = \\ &= \mathbf{i}_L^+(P)^2 \cdot \mathbf{s}_L(V_L^P) = \\ &= \mathbf{i}_L^+(P)^2 \cdot \mathbf{c}_L(P). \end{aligned}$$

Now that we have embedded the Eggers-Wall trees $\Theta_L(C)$ into \mathcal{V}_L . We are almost ready to prove that those embeddings allow to see \mathcal{V}_L as their projective limit, once one relates them by their canonical attaching maps. Before doing that, we need one more lemma, stating that if ν and ν' are two different semivaluations in \mathcal{V}_L , then there exists a branch A such that the attaching points of ν and ν' on the segment $[\text{ord}^L, I_L^A]$ are different.

Lemma 8.17 (1) *Let ν and ν' be two different semivaluations in \mathcal{V}_L . Then, there exists a branch A such that $\nu(A) \neq \nu'(A)$.*

(2) *If A is such a branch, denote by $P := \langle \text{ord}^L, I_L^A, \nu \rangle$ the center of the tripod determined by the normalized semivaluations ord^L, I_L^A and ν on the tree \mathcal{V}_L , and denote similarly $P' = \langle \text{ord}^L, I_L^A, \nu' \rangle$. Then, we have that $P \neq P'$.*

Proof Assume that $\nu(A) = \nu'(A)$ for any branch A . Then, if $h \in \mathcal{O}$, we may write h as a finite product $h = \prod_l h_l$, with all h_l irreducible. By hypothesis, the semivaluations ν and ν' have the same value on the branch $A_l = Z(h_l)$. It follows that $\nu(h) = \sum_l \nu(A_l) = \nu'(h)$. This proves the first statement.

By the tripod formula (7.8), we get the relations:

$$\mathbf{s}_L(P) = \frac{\langle \nu, I^A \rangle}{\langle I^L, \nu \rangle \langle I^L, I^A \rangle} \quad \text{and} \quad \mathbf{s}_L(P') = \frac{\langle \nu', I^A \rangle}{\langle I^L, \nu' \rangle \langle I^L, I^A \rangle}. \tag{8.17}$$

By Proposition 7.11, we have that:

$$\begin{aligned} \langle \nu, I^L \rangle &= \nu(L), & \langle \nu', I^L \rangle &= \nu'(L), \\ \langle \nu, I^A \rangle &= \nu(A), & \langle \nu', I^A \rangle &= \nu'(A), \end{aligned}$$

and $\langle I^L, I^A \rangle = (L \cdot A)$. In addition, $\nu(L) = \nu'(L) = 1$ since ν and ν' belong to \mathcal{V}_L . As $\nu(A) \neq \nu'(A)$ by hypothesis, it follows that $L \neq A$, hence $(L \cdot A) \in \mathbb{N}^*$. Since $\nu(A) \neq \nu'(A)$, we deduce from (8.17) that $\mathbf{s}_L(P) \neq \mathbf{s}_L(P')$. Since the restriction to \mathbf{s}_L to the segment $[\text{ord}^L, I_L^A]$ is strictly increasing and P, P' belong to this segment, it follows that $P \neq P'$, which proves the second statement. \square

We prove now that the semivaluation space \mathcal{V}_L is the projective limit of the Eggers-Wall trees $\Theta_L(C)$ of reduced plane curves, embedded by the map V_L .

Theorem 8.18 *Let us denote by \mathcal{B} the set of branches on S and by $\mathcal{F}(\mathcal{B})$ the set consisting of finite subsets of \mathcal{B} . For any $J \in \mathcal{F}(\mathcal{B})$, we denote by C_J the reduced plane curve singularity whose branches are the elements of the set J . Denote by $\mathcal{V}_{L,J}$ the subtree $V_L(\Theta_L(C_J))$ of \mathcal{V}_L . The collection $(\mathcal{V}_{L,J})_{J \in \mathcal{F}(\mathcal{B})}$ forms a projective system for the inclusion partial order. If $\mathcal{V}_{L,J} \subseteq \mathcal{V}_{L,M}$, we denote by $\pi_{L,J}^M : \mathcal{V}_{L,M} \rightarrow \mathcal{V}_{L,J}$ the corresponding attaching map. Then:*

(1) *The maps $\pi_{L,J}^M$ form a projective system of continuous maps.*

(2) The attaching maps $\pi_{L,J} : \mathcal{V}_L \rightarrow \mathcal{V}_{L,J}$ glue into a homeomorphism $\pi_L : \mathcal{V}_L \rightarrow \varprojlim \mathcal{V}_{L,J}$.

Proof The collection $(\mathcal{V}_{L,J})_{J \in \mathcal{F}(\mathcal{B})}$ form a projective system for the inclusion partial order, since for any $J, K \in \mathcal{F}(\mathcal{B})$ there exists $M \in \mathcal{F}(\mathcal{B})$ such that $\mathcal{V}_{L,J} \subset \mathcal{V}_{L,M}$ and $\mathcal{V}_{L,K} \subset \mathcal{V}_{L,M}$ (one may simply take $M := J \cup K$).

Notice that if $\mathcal{V}_{L,J} \subset \mathcal{V}_{L,M}$, we can understand the attaching map $\pi_{L,J}^M : \mathcal{V}_{L,M} \rightarrow \mathcal{V}_{L,J}$ by using the embedding V_L , since for any $P \in \Theta_L(C_K)$ we have:

$$\pi_{L,J}^M(V_L(P)) = V_L(\pi_{\Theta_L(C_J)}^{\Theta_L(C_M)}(P)),$$

where $\pi_{\Theta_L(C_J)}^{\Theta_L(C_M)} : \Theta_L(C_M) \rightarrow \Theta_L(C_J)$ is the surjective attaching map of Definition 2.11 (whose image is $\Theta_L(C_J) \subset \Theta_L(C_M)$). This implies that the maps $\pi_{L,J}^M$ form a projective system of continuous maps.

Now we apply Theorem 2.14 in this setting:

- The only hypothesis we need to check is point (4) in Theorem 2.14. This hypothesis holds by Lemma 8.17.
- Recall that the semivaluation space \mathcal{V}_L is compact. Therefore, Theorem 2.14, applied to the projective system $\pi_{L,J}^M$, implies that the map $\pi_L : \mathcal{V}_L \rightarrow \varprojlim \mathcal{V}_{L,J}$ is a homeomorphism. □

In order to be able to compare the points of Eggers-Wall trees of various curves relative to various smooth branches considered as their roots, we embed them also in the fixed valuative tree $\mathbb{P}(\mathcal{V})$, instead of doing it in the varying trees \mathcal{V}_L , as in Definition 8.9:

Definition 8.19 The *valuative embedding* of the Eggers-Wall tree $\Theta_L(C)$ is the map $\Psi_L := \pi_L \circ V_L : \Theta_L(C) \rightarrow \mathbb{P}(\mathcal{V})$.

Point (2) of Theorem 8.18 yields then the following precise formulation of the theorem stated in the title of the paper:

Corollary 8.20 *The valuative tree $\mathbb{P}(\mathcal{V})$ is the projective limit of the trees $\Theta_L(C)$, for varying reduced germs $C \hookrightarrow S$, the maps $\mathbb{P}(\mathcal{V}) \rightarrow \Theta_L(C)$ being the attaching maps of the embeddings $\Psi_L : \Theta_L(C) \rightarrow \mathbb{P}(\mathcal{V})$.*

9 Change of observer on the semivaluation space

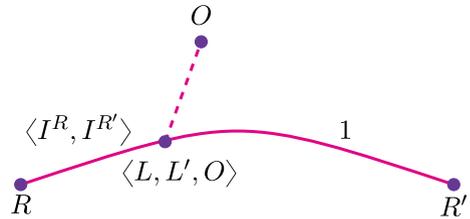
It is important to know how to change coordinates when one changes the observer. The aim of this section is to prove formulae expressing the functions $(\mathbf{1}_{R'}, \mathbf{m}_{R'}, \mathbf{s}_{R'})$ in terms of the functions $(\mathbf{1}_R, \mathbf{m}_R, \mathbf{s}_R)$, whenever R and R' are two distinct observers of the valuative tree $\mathbb{P}(\mathcal{V})$, in the sense of Definition 7.13. Combined with the embedding theorem 8.11, these formulae of changes of coordinates are the main ingredients of the proof of the generalized inversion theorem 4.5.

The following proposition is an immediate consequence of Definition 7.12:

Proposition 9.1 *Let R, R' be two observers of $\mathbb{P}(\mathcal{V})$. Then one has the following formulae of change of coordinates from R to R' :*

$$\mathbf{1}_{R'} = \gamma_{R'}^R \cdot \mathbf{1}_R, \tag{9.1}$$

Fig. 18 Some values of $\mathbf{m}_{R'}$



$$\mathbf{s}_{R'} = (\gamma_{R'}^R)^2 \cdot \mathbf{s}_R, \tag{9.2}$$

where:

$$\gamma_{R'}^R(P) = \frac{\langle v^P, I^R \rangle}{\langle v^P, I^{R'} \rangle} \tag{9.3}$$

for any projective semivaluation $P \in \mathbb{P}(\mathcal{V})$. Here $v^P \in \mathcal{V}^*$ is an arbitrary semivaluation centered at O and representing P .

Remark 9.2 Notice that if R is a smooth branch, then $\gamma_{R'}^R(R) = \frac{\langle I^R, I^R \rangle}{\langle I^R, I^{R'} \rangle} = \infty$. If R' is a smooth branch, we have similarly that $\gamma_{R'}^R(R') = \frac{\langle I^{R'}, I^{R'} \rangle}{\langle I^{R'}, I^R \rangle} = 0$. Seen as functions on $\mathbb{P}(\mathcal{V}) \setminus \{R, R'\}$, one has $\gamma_{R'}^R \cdot \gamma_R^{R'} = 1$.

The function $\gamma_{R'}^R$ is expressed in the following way in terms of the relative interaction and self-interaction functions:

Proposition 9.3 Assume that R, R' are distinct observers of $\mathbb{P}(\mathcal{V})$. Then:

$$\gamma_{R'}^R(P) = (\langle I^R, I^{R'} \rangle \cdot \mathbf{s}_R(\langle R, R', P \rangle))^{-1} \text{ for any } P \in \mathbb{P}(\mathcal{V}).$$

In particular, if R and R' are transversal smooth branches, we have:

$$\gamma_{R'}^R(P) = \mathbf{s}_R(\langle R, R', P \rangle)^{-1} = (\mathbf{1}_R(\langle R, R', P \rangle) - 1)^{-1} \text{ for any } P \in \mathbb{P}(\mathcal{V}).$$

Proof The first equality is an immediate consequence of the tripod formula (7.8). The second equality is a consequence of formula (7.5) and of the fact that \mathbf{m}_R is identically equal to 1 on the segment $[L, L'] \subset \mathbb{P}(\mathcal{V})$, when R and R' are transversal smooth branches. This last fact is a consequence of the Definition 7.14 of the relative multiplicity function. \square

There is also a formula of change of coordinates for the relative multiplicity functions of Definition 7.14:

Proposition 9.4 Let R, R' be two distinct observers of $\mathbb{P}(\mathcal{V})$. Then (see Fig. 18):

$$\mathbf{m}_{R'} = \begin{cases} 1 & \text{on } [R', \langle R, R', O \rangle], \\ \langle I^R, I^{R'} \rangle & \text{on } (\langle R, R', O \rangle, R], \\ \gamma_{R'}^R \cdot \mathbf{m}_R & \text{on } \mathbb{P}(\mathcal{V}) \setminus [R', R]. \end{cases} \tag{9.4}$$

Proof We prove the formulae when both observers are smooth branches $R = L, R' = L'$, leaving to the reader the analogous reasoning in the remaining case when one of the observers is the point O . We will consider successively the three possibilities listed in formula (9.4) for the position of the point $P \in \mathbb{P}(\mathcal{V})$ relative to the tripod determined by O, L, L' .

Fig. 19 The case $P \in [L', \langle L, L', O \rangle]$

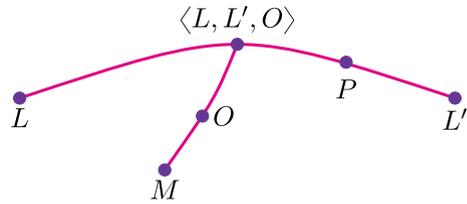
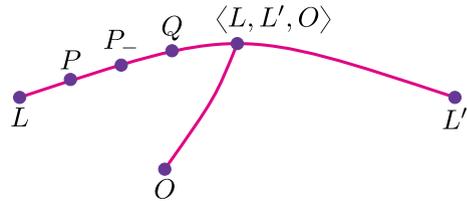


Fig. 20 The case $P \in \langle (L, L', O), L \rangle]$



• Assume that $P \in [L', \langle L, L', O \rangle]$. Consider a third smooth branch M , transversal to L' (see Fig. 19). Then $O \in [L', M]$ and $\mathbf{m}_{L'}$ is constantly equal to 1 on $[L', M]$. As $[L', \langle L, L', O \rangle] \subset [L', O]$, we deduce the desired relation $\mathbf{m}_{L'}(P) = 1$.

• Assume that $P \in \langle (L, L', O), L \rangle]$. Apply then formula (7.3) to $\mathbf{m}_{L'}(P)$:

$$\mathbf{m}_{L'}(P) = \lim_{P_- \rightarrow P, P_- \prec_{L'} P} \frac{\mathbf{I}_{L'}(P) - \mathbf{I}_{L'}(P_-)}{\mathbf{s}_{L'}(P) - \mathbf{s}_{L'}(P_-)}. \tag{9.5}$$

In order to compute the limit (9.5) we can assume that $P_- \in \langle (L, L', O), L \rangle]$. Take then an auxiliary point $Q \in \langle (L, L', O), L \rangle]$ (see Fig. 20). Our choice implies that $\langle O, L', Q \rangle = \langle L, L', O \rangle$.

By Proposition 9.3 and the tripod formula (7.8) we deduce that:

$$\gamma_{L'}^O(Q) = \langle (I^O, I^{L'}) \cdot \mathbf{s}_O(\langle O, L', Q \rangle) \rangle^{-1} = \mathbf{s}_O(\langle L, L', O \rangle)^{-1} = (L \cdot L')^{-1}. \tag{9.6}$$

We pass now from the observer L' to O . That is, we apply the formulae (9.1) and (9.2) to (9.5), with $R' = L'$ and $R = O$.

By (9.6) the value of $\gamma_{L'}^O$ is constant on the segment $\langle (L, L', O), L \rangle]$ and we may factor it when computing the limit in (9.5). We get:

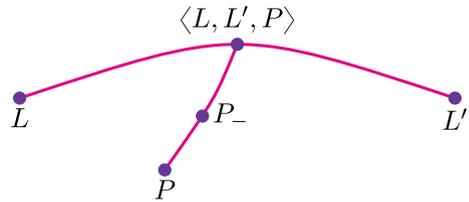
$$\mathbf{m}_{L'}(P) = (L \cdot L') \lim_{P_- \rightarrow P, P_- \prec_{L'} P} \frac{\mathbf{I}_O(P) - \mathbf{I}_O(P_-)}{\mathbf{s}_O(P) - \mathbf{s}_O(P_-)}. \tag{9.7}$$

Since $P_- \in \langle (L, L', O), L \rangle]$, we have that $P_- \prec_{L'} P$ is equivalent to $P_- \prec_O P$. Therefore, the limit (9.7) is equal to $\mathbf{m}_O(P)$. Notice that m_O is constantly equal to 1 on $[O, L] \supset \langle (L, L', O), P \rangle$, thus $\mathbf{m}_O(P) = 1$. By Proposition 7.11, we get the desired equality $\mathbf{m}_{L'}(P) = (L \cdot L') = \langle I^L, I^{L'} \rangle$.

• Assume that $P \in \mathbb{P}(\mathcal{V}) \setminus [L, L']$. Here the reasoning is analogous to the one done in the previous case, but instead of changing coordinates by replacing the observer L' with O , one replaces it with L . The main point is that one may compute the limit (9.5) by restricting the points P_- to the segment $\langle (L, L', P), P \rangle$. This implies that $\langle L, L', P_- \rangle = \langle L, L', P \rangle$ (see Fig. 21).

In particular, by Proposition 9.3, we get that $\gamma_{L'}^L(P_-) = \gamma_{L'}^L(P)$. Therefore, one may factor $\gamma_{L'}^L(P)$ in the numerator and $(\gamma_{L'}^L(P))^2$ in the denominator of the fraction in formula (9.5), which implies by Remark 9.2 that:

Fig. 21 The case $P \in \mathbb{P}(\mathcal{V}) \setminus [L, L']$



$$\mathbf{m}_{L'}(P) = \gamma_{L'}^{L'}(P) \cdot \lim_{P_- \rightarrow P, P_- \prec_{L'} P} \frac{\mathbf{l}_L(P) - \mathbf{l}_L(P_-)}{\mathbf{s}_L(P) - \mathbf{s}_L(P_-)}.$$

But one has also the inequality $P_- \prec_L P$, as $P_- \in \langle L, L', P \rangle, P$. By (7.3), this implies that the last limit is equal to $\mathbf{m}_L(P)$. We get the desired relation $\mathbf{m}_{L'}(P) = \gamma_{L'}^{L'}(P) \cdot \mathbf{m}_L(P)$. \square

10 Repertory of formulae

In this section we gather the main formulae of this paper, which relate the triple of functions $(\mathbf{e}_L, \mathbf{i}_L, \mathbf{c}_L)$ defined on the Eggers-Wall trees $\Theta_L(C)$ to the triple of functions $(\mathbf{l}_L, \mathbf{m}_L, \mathbf{s}_L)$ defined on the valuative tree $\mathbb{P}(\mathcal{V})$. We hope that this will help the reader getting a global vision of the structure of the paper.

- The formula of Proposition 3.16, concerning any $P \in \Theta_L(C)_{\mathbb{Q}}$:

$$\mathbf{i}_L^+(P) = \text{the lowest common multiple of } \mathbf{i}_L(P) \text{ and of the denominator of } \mathbf{e}_L(P).$$

- Formula (3.1), concerning any $P \in \Theta_L(C)$:

$$\mathbf{c}_L(P) = \int_L^P \frac{d \mathbf{e}_L}{\mathbf{i}_L}.$$

- Formula (3.3), concerning any $P \in \Theta_L(C)$:

$$\mathbf{e}_L(P) = \int_L^P \mathbf{i}_L d \mathbf{c}_L.$$

- The tripod formula of Corollary 3.26. If A, B are two branches on S , then:

$$(A \cdot B) = \mathbf{i}_L(A) \cdot \mathbf{i}_L(B) \cdot \mathbf{c}_L(\langle L, A, B \rangle).$$

- The first formula of Corollary 4.6, relating the functions \mathbf{e}_L and $\mathbf{e}_{L'}$, when L and L' are transversal:

$$\mathbf{e}_{L'} + 1 = \frac{\mathbf{e}_L + 1}{\mathbf{e}_L \circ \pi_{[L, L']}}.$$

- The second formula of Corollary 4.6, relating the functions \mathbf{c}_L and $\mathbf{c}_{L'}$, when L and L' are transversal:

$$\mathbf{c}_{L'} = \frac{\mathbf{c}_L}{(\mathbf{e}_L \circ \pi_{[L, L']})^2}.$$

- The third formula of Corollary 4.6, relating the functions \mathbf{i}_L and $\mathbf{i}_{L'}$, when L and L' are transversal:

$$\mathbf{i}_{L'} = \begin{cases} 1, & \text{on } [L, L'], \\ (\mathbf{e}_L \circ \pi_{[L, L']}) \cdot \mathbf{i}_L, & \text{elsewhere.} \end{cases}$$

- The definition of log-discrepancy (see Definition 7.1):

$$\mathbf{l}(\text{ord}^{E_j}) := 1 + \text{ord}^{E_j}(\omega),$$

where ω is a non-vanishing holomorphic 2-form on S in the neighborhood of O .

- The definition of self-interaction (see Definition 7.1):

$$\mathbf{s}(\text{ord}^{E_j}) := -(\check{E}_j \cdot \check{E}_j).$$

- The formula of Proposition 8.10, expressing the value taken by the valuation V_L^P on a branch A , for any $P \in \Theta_L(C)$:

$$V_L^P(A) = \begin{cases} (L \cdot A) \cdot \mathbf{c}_L(\langle L, P, A \rangle) & \text{if } A \neq L, \\ 1 & \text{if } A = L. \end{cases}$$

- The first formula of Theorem 8.11:

$$V_L^* \mathbf{l}_L = 1 + \mathbf{e}_L.$$

- The second formula of Theorem 8.11:

$$V_L^* \mathbf{m}_L = \mathbf{i}_L.$$

- The third formula of Theorem 8.11:

$$V_L^* \mathbf{s}_L = \mathbf{c}_L.$$

- The first formula of Proposition 8.16, concerning any $P \in \Theta_L(C)_{\mathbb{Q}}$:

$$\text{ord}^{E_P}(L) = \mathbf{i}_L^+(P).$$

- The second formula of Proposition 8.16, concerning any $P \in \Theta_L(C)_{\mathbb{Q}}$:

$$\mathbf{l}(\text{ord}^{E_P}) = \mathbf{i}_L^+(P) \cdot (1 + \mathbf{e}_L(P)).$$

- The third formula of Proposition 8.16, concerning any $P \in \Theta_L(C)_{\mathbb{Q}}$:

$$\mathbf{s}(\text{ord}^{E_P}) = \mathbf{i}_L^+(P)^2 \cdot \mathbf{c}_L(P).$$

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Affiliations

Evelia R. García Barroso¹  · Pedro D. González Pérez²  ·
Patrick Popescu-Pampu³ 

Evelia R. García Barroso
ergarcia@ull.es
<http://ergarcia.webs.ull.es/>

Pedro D. González Pérez
pgonzalez@mat.ucm.es
<http://www.mat.ucm.es/~pdperez/>

¹ Departamento de Matemáticas, Estadística e I.O. Sección de Matemáticas, Universidad de La Laguna, Calle Astrofísico Francisco Sánchez, 38200 San Cristóbal de La Laguna, Tenerife, Spain

² Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28070 Madrid, Spain

³ Département de Maths, Université Lille, Bâtiment M2, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France