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ULTRAMETRIC PROPERTIES FOR VALUATION SPACES OF NORMAL SURFACE SINGULARITIES

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Abstract. Let L be a fixed branch – that is, an irreducible germ of curve – on a normal surface singularity X. If A, B are two other branches, define $u_L(A,B) := \dfrac{(L \cdot A)\,(L \cdot B)}{A \cdot B}$, where $A \cdot B$ denotes the intersection number of A and B. Call X *arborescent* if all the dual graphs of its good resolutions are trees. In a previous paper, the first three authors extended a 1985 theorem of Płoski by proving that whenever X is arborescent, the function u_L is an *ultrametric* on the set of branches on X different from L. In the present paper we prove that, conversely, if u_L is an ultrametric, then X is arborescent. We also show that for *any* normal surface singularity, one may find arbitrarily large sets of branches on X, characterized uniquely in terms of the topology of the resolutions of their sum, in restriction to which u_L is still an ultrametric. Moreover, we describe the associated tree in terms of the dual graphs of such resolutions. Then we extend our setting by allowing L to be an arbitrary *semivaluation* on X and by defining u_L on a suitable space of semivaluations. We prove that any such function is again an ultrametric if and only if X is arborescent, and without any restriction on X we exhibit special subspaces of the space of semivaluations in restriction to which u_L is still an ultrametric.

Contents

Introduction

Let X be a *normal surface singularity*, which will mean for us throughout the paper a germ of normal complex analytic surface. A *branch* on it is an irreducible germ of formal curve on X. In his 1985 paper [40], Płoski proved a theorem which may be reformulated in the following way.

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Theorem. *If X is smooth, then the map which associates to any pair (A, B) of branches on it the quotient $\frac{m(A)\, m(B)}{A \cdot B}$ of the product of their multiplicities by their intersection number is an ultrametric on the set of branches on X.*

The first three authors proved in [19, Theorem 4.18] that this result generalizes to the case of *arborescent singularities*, which are the normal surface singularities whose good resolutions (with simple normal crossing exceptional divisors) have trees as dual graphs.

Theorem. *Let X be an arborescent singularity and let L be a fixed branch on it. Then the map u_L which associates to any pair (A, B) of branches on X the quotient $\frac{(L \cdot A)\,(L \cdot B)}{A \cdot B}$ is an ultrametric on the set of branches on X distinct from L.*

Note that on arbitrary normal surface singularities the intersection numbers are defined in the sense of Mumford [38] and may take non-integral (but still rational) values.

One may recover Płoski's theorem as a particular case of the previous one. Indeed, smooth germs X are arborescent, and the ultrametric property of the quotients involved in Płoski's theorem may be tested on any finite set of branches. Then it is enough to choose a smooth branch L which is transversal to all the branches in a fixed finite set.

The main aspect of the approach of [19] was to express the intersection numbers of branches on a normal surface singularity X in terms of intersection numbers of exceptional divisors on a resolution X_π of X. What made ultimately everything work was the following inequality between the intersection numbers of the divisors of the basis $(\check{E}_u)_u$ of the vector space of real exceptional divisors of X_π which is dual to the basis formed by the prime exceptional divisors $(E_u)_u$. This inequality (see Proposition 1.18) was generalized by Gignac and the fourth author in [21, Proposition 1.10].

Proposition. *Let X be a normal surface singularity and let X_π be a good resolution of it. Let E_u, E_v, and E_w be not necessarily distinct exceptional prime divisors of X_π. Then one has the inequality*

$$(-\check{E}_u \cdot \check{E}_v)(-\check{E}_v \cdot \check{E}_w) \leq (-\check{E}_v \cdot \check{E}_v)(-\check{E}_u \cdot \check{E}_w),$$

with equality if and only if v separates u and w in the dual graph of X_π.

This inequality is also crucial in this paper and has an intriguing reformulation in terms of spherical geometry (see Proposition 1.19).

Our paper has two main sections. Section 1 treats the case of the functions u_L restricted to finite sets of branches. In Section 2 we show how the results of the first section can be extended to the space of normalized semivaluations of X. Let us summarize our main results.

We prove a converse of one of the main theorems of [19], which stated that u_L is an ultrametric whenever X is arborescent (see Theorem 1.46).

Theorem A. *The normal surface singularity X is arborescent if and only if either one or all of the functions u_L for varying branches L on X are ultrametrics.*

More generally, if X is a normal surface singularity and $\mathcal{F}$ is a finite set of branches on X containing a fixed branch L, we show that u_L is an ultrametric on

$\mathcal{F} \setminus \{L\}$ whenever the dual graph of the total transform of the sum of the branches in $\mathcal{F}$ in an arbitrary embedded resolution of it satisfies a topological condition. Its formulation uses the notion of *brick-vertex tree* $\mathcal{BV}(G)$ of a finite connected graph G. It is a finite tree, containing the vertices of G and other vertices called *brick vertices*, which encodes the way the vertices of G get separated by an arbitrary one of them (see Subsection 1.4). We prove that (see Theorem 1.42):

Theorem B. *If the convex hull* $\mathrm{Conv}(\mathcal{F})$ *of the branches of* $\mathcal{F}$ *in the brick-vertex tree of the dual graph of the chosen embedded resolution does not contain brick vertices of valency at least* 4 *in* $\mathrm{Conv}(\mathcal{F})$*, then* u_L *is an ultrametric in restriction to* $\mathcal{F} \setminus \{L\}$*. Moreover, in this case the rooted tree of* u_L *restricted to* $\mathcal{F} \setminus \{L\}$ *is isomorphic to* $\mathrm{Conv}(\mathcal{F})$*, rooted at the vertex corresponding to* L*.*

Note that this result does not involve intersection numbers or genera of prime exceptional divisors. It is always satisfied when X is arborescent, which allows us to recover [19, Theorem 4.18].

Let us pass to the *semivaluations* of X considered in Section 2. Compared to valuations, they may achieve the value $+\infty$ on elements of the local ring of X other than simply 0. Allowing us to work not only with valuations but also with semivaluations has the advantage that any branch on X has an associated semivaluation, which associates to an element of the local ring of X the intersection number of its divisor with L. Also, any prime exceptional divisor of a normal crossings resolution of X has an associated semivaluation, which is in fact a valuation. Therefore, the vertices of the dual graphs of the total transforms of the sums of finite sets of branches on X embed naturally in the space of semivaluations of X. In fact, this embedding can be extended to the whole dual graph, seen as a topological space. It is more convenient to our purpose, as it was in the model case of smooth X treated in Favre and Jonsson's book [14], to consider a space of *normalized* semivaluations. The normalization condition is simply to consider only semivaluations which take the value 1 on the maximal ideal of the local ring of X. It ensures that one gets a topological space of dimension 1.

We generalize Theorem 1.46 to arbitrary semivaluations on X (cf. Theorem 2.19). Namely, we replace the branch L, seen as a particular semivaluation by an arbitrary normalized semivaluation λ on X, and we consider an analog u_λ of the function u_L, defined this time on the space of normalized semivaluations which are distinct from λ. We prove that:

Theorem C. *The normal surface singularity* X *is arborescent if and only if either one or all the functions* u_λ *for varying semivaluations* λ *of* X *are ultrametrics.*

We generalize Theorem 1.42 to arbitrary semivaluations on X (see Theorem 2.53). Namely, we prove that for any normal surface singularity X, any normalized semivaluation λ on it, and any set $\mathcal{F}$ (not necessarily finite) of normalized semivaluations containing λ, the function u_λ is an ultrametric in restriction to $\mathcal{F}$ whenever $\mathcal{F}$ satisfies a suitable topological condition in the space of normalized semivaluations of X. The topological conditions involved in the statements of Theorems 1.42 and 2.53 are analogous, involving finite graphs in the first case and special types of *infinite graphs* in the second case. Let us compare both cases.

We show that the space of normalized semivaluations has a structure of *connected graph of* $\mathbb{R}$*-trees of finite type* (see Proposition 2.51). We extend the notion of *brick-vertex tree* to such spaces (see Subsection 2.6). In the case of the space of normalized semivaluations, there is only a finite number of brick vertices which correspond bijectively to those of the dual graph of any normal crossings resolution of X. Using the brick-vertex tree of the space of normalized semivaluations of X, we prove analogs for the functions u_λ of the results formulated in terms of brick-vertex trees of finite graphs for the functions u_L (see Subsection 2.7). In fact, the bricks are precisely the non-punctual *cyclic elements* of the space of normalized semivaluations. In Remark 2.50 we give historical details about the *topological theory of cyclic elements*.

In the whole paper, we deal for simplicity with *complex* normal surface singularities. But our approach works also for singularities which are spectra of normal 2-dimensional local rings defined over fields of arbitrary characteristic. Indeed, our treatment is ultimately based on the fact that the intersection matrix of a resolution of the singularity is negative definite; see Theorem 1.2 below, a theorem which is true in this greater generality, as shown by Lipman [33, Lemma 14.1]. For the description of semivaluation spaces associated to regular surface singularities over fields of any characteristic, we refer to Jonsson's paper [29, Section 7]; see in particular its section 7.11 for a discussion of the specificities of non-algebraically closed base fields. Jonsson's approach can be directly generalized to any normal surface singularity defined over arbitrary fields by applying his constructions to the sets of semivaluations centered at smooth points in any good resolution of the given singularity.

1. Ultrametric distances on finite sets of branches

Let X be a normal surface singularity and let L be a finite branch on it. Let u_L be the function introduced by the first three authors in [19], which associates to every pair (A, B) of branches on X which are different from L the number $(L \cdot A)\,(L \cdot B)(A \cdot B)^{-1}$. In this first part of the paper we study its behavior on finite sets of branches on X. Our main results are that u_L is an ultrametric on any such set if and only if X is *arborescent* (see Theorem 1.46) and that even when X is not arborescent, it is still an ultrametric in restriction to arbitrarily large sets of branches, which may be characterized topologically in terms of their total transform on any good resolution of their sum (see Theorem 1.42). These theorems need a certain amount of preparation, which explains the need for a subdivision of this section into six subsections. The content of each subsection is briefly described at its beginning.

1.1. **Mumford's intersection number of divisors.** In this subsection we recall *Mumford's definition of intersection number* of Weil divisors on a normal surface singularity X (see Definition 1.10). This definition passes through an intermediate definition of *total transform* of such a divisor by a resolution of the singularity (see Definition 1.7), which in turn uses basic properties of the intersection form on such a resolution. That is why we begin the subsection by recalling the needed theorems about the intersection theory on resolutions of X (see Theorem 1.2 and Propositions 1.1, 1.4, 1.5). We also introduce many of the notions used elsewhere in the paper. The most important one for what follows is that of *bracket* $\langle u, v\rangle$ of two prime divisorial valuations u, v on X (see Definition 1.6), which may be interpreted as Mumford's intersection number of a pair of branches adapted to the two valuations (see Proposition 1.11).

In the whole paper, we fix a **normal surface singularity** (X, x_0), that is, a germ of complex analytic normal surface. In particular, the germ is irreducible and has a representative which is smooth outside x_0. In order to shorten the notation, most of the time we will write simply X instead of (X, x_0). We will denote by $\mathcal{O}_X$ the local ring of X.

A **branch** on X is a germ at x_0 of irreducible formal curve lying on X. The set of branches on X will be denoted by $\boxed{\mathcal{B}(X)}$.

By a **divisor** on X we will mean an integral Weil divisor, that is, an element of the free abelian group generated by the branches on X. As usual, a **principal divisor** is the divisor (f) of a formal meromorphic function f on X, that is, of an element of the fraction field of the completion of $\mathcal{O}_X$ relative to its maximal ideal.

A **resolution** of X is a proper bimeromorphic morphism $\pi \colon X_\pi \to X$ of complex analytic spaces such that X_π is smooth and π is an isomorphism over $X \setminus \{x_0\}$. If $\pi \colon X_\pi \to X$ is a resolution of X, we will say that X_π is a **model** of X. The **reduced exceptional divisor of the resolution** π will be denoted by $\boxed{E(\pi)}$, and its set of irreducible components by $\boxed{\mathcal{P}(\pi)}$. By **an exceptional divisor** on X_π we mean, depending on the context, either an element of the abelian group $\boxed{\mathcal{E}(\pi)_\mathbb{Z}}$ freely generated by the elements of $\mathcal{P}(\pi)$, of the associated $\mathbb{Q}$-vector space $\boxed{\mathcal{E}(\pi)_\mathbb{Q}}$, or of the associated $\mathbb{R}$-vector space $\boxed{\mathcal{E}(\pi)_\mathbb{R}}$.

The irreducible components of the reduced exceptional divisors of the various resolutions of X will be called **prime exceptional divisors**. By associating to a prime exceptional divisor its corresponding integer-valued valuation on the local ring $\mathcal{O}_X$ (that is, the vanishing order along the divisor), we may identify $\mathcal{P}(\pi)$ with a set of divisorial valuations on the local ring $\mathcal{O}_X$ (see section 2.1). Therefore, denoting by $\boxed{E_u}$ the prime divisor on X_π corresponding to $u \in \mathcal{P}(\pi)$, we may think that u also denotes the corresponding divisorial valuation on $\mathcal{O}_X$. Whenever we reason with several models at the same time, we will denote by E_u^π instead of E_u the prime divisor on the model X_π corresponding to the divisorial valuation u. But when we work with a fixed model, for simplicity we will drop from the notation this dependency on the model.

We will say that the divisorial valuations u on $\mathcal{O}_X$ associated to prime divisors E_u are **prime divisorial valuations**. We will denote by $\boxed{\mathcal{P}(X)}$ the set of prime divisorial valuations. It is the union of the subsets $\mathcal{P}(\pi)$ of the set of divisorial valuations of X when π varies among the resolutions of X. If $u \in \mathcal{P}(X)$ and X_π is a model such that $u \in \mathcal{P}(\pi)$, we say that u **appears on the model** X_π.

Given a resolution π of X, the intersection number of exceptional divisors of X_π defines a symmetric bilinear form on the vector space $\mathcal{E}(\pi)_\mathbb{R}$, called its **intersection form**. For simplicity, we will denote by $\boxed{D_1 \cdot D_2}$ the intersection number of the exceptional divisors D_1 and D_2 without mentioning the morphism π explicitly. This convention may be motivated by the classical fact that **the intersection number is birationally invariant** in the following sense.

Proposition 1.1. *If the model X_{π_2} dominates the model X_{π_1}, then the intersection number of two divisors of X_{π_1} is equal to the intersection number of their total transforms on X_{π_2}.*

Proof. Let $\psi : X_{\pi_2} \to X_{\pi_1}$ be the domination morphism between the two models. Recall the *projection formula*, comparing intersection numbers on the two models (see Hartshorne [25, Appendix A.1]):

$$D_2 \cdot \psi^* D_1 = \psi_* D_2 \cdot D_1 \tag{1}$$

for every $D_1 \in \mathcal{E}(\pi_1)_{\mathbb{R}}$ and $D_2 \in \mathcal{E}(\pi_2)_{\mathbb{R}}$ (the left-hand side being computed on X_{π_2} and the right hand side on X_{π_1}). Here $\psi^* D_1$ denotes the total transform of D_1 by the morphism ψ, and $\psi_* D_2$ denotes the direct image of D_2 by the same morphism. Consider now two divisors A, B on X_{π_1}. Then

$$\psi^* A \cdot \psi^* B = (\psi_* \psi^* A) \cdot B = A \cdot B,$$

the first equality being a consequence of the projection formula (1) applied to $D_1 = B$, $D_2 = \psi^* A$ and the second equality being a consequence of the fact that $\psi_* \psi^* A = A$. □

Note that the previous assertion does not remain true if one replaces total transforms of divisors by strict transforms. In particular, for fixed $u, v \in \mathcal{P}(X)$, the intersection number $E_u^\pi \cdot E_v^\pi$ depends on the model X_π on which E_u^π and E_v^π appear. Compare this fact with Proposition 1.5 below.

One has the following fundamental theorem concerning the intersection form on a fixed model (see Du Val [9] and Mumford [38] in what concerns point (1) and Zariski [58, Lemma 7.1] in what concerns point (2)).

Theorem 1.2. *Let X_π be a model of the normal surface singularity X.*

(1) *The intersection form on the vector space $\mathcal{E}(\pi)_{\mathbb{R}}$ is negative definite.*
(2) *If $D \in \mathcal{E}(\pi)_{\mathbb{R}} \setminus \{0\}$ is such that $D \cdot H \geq 0$ for all effective divisors $H \in \mathcal{E}(\pi)_{\mathbb{R}}$, then $-D$ is effective and it is of full support in the basis $(E_u)_{u \in \mathcal{P}(\pi)}$; that is, all the coefficients of its decomposition in this basis are positive.*

The second statement is a consequence of the following theorem of linear algebra, which will be used in the proof of Proposition 1.18 (one may verify easily that Zariski's proof in [58, Lemma 7.1] transcribes immediately to a proof of it).

Proposition 1.3. *Let $\mathcal{E}$ be a Euclidean finite-dimensional vector space. Consider a basis $\mathcal{B}$ of $\mathcal{E}$ such that the plane angles generated by any pair of its vectors are right or obtuse. Assume moreover that $\mathcal{B}$ cannot be partitioned into two non-empty subsets orthogonal to each other. Denote by σ the cone generated by $\mathcal{B}$ and let $\check{\sigma}$ be the cone generated by the dual basis. Then $\check{\sigma} \setminus 0$ is included in the interior of σ.*

In order to get Theorem 1.2(2) from Proposition 1.3, one takes as Euclidean vector space $\mathcal{E}$ the space of exceptional divisors $\mathcal{E}(\pi)_{\mathbb{R}}$, endowed with the opposite of the intersection form and with the basis $(E_u)_{u \in \mathcal{P}(\pi)}$. The hypothesis on the angles is satisfied because $E_u \cdot E_v \geq 0$ for all $u \neq v$. The hypothesis on the impossibility to partition the basis into two orthogonal non-empty subsets is equivalent to the connectedness of the exceptional divisor $E(\pi)$. In turn, this is a consequence of the hypothesis that X is normal, as a special case of the so-called *Zariski main theorem* (see [25, Corollary 11.4]).

If $D \in \mathcal{E}(\pi)_{\mathbb{R}}$ is a divisor such that $-D$ is effective, we will say that D is **anti-effective**. If $D \cdot H \geq 0$ for all effective divisors $H \in \mathcal{E}(\pi)_{\mathbb{R}}$, we will say that D is **nef (numerically eventually free)**. Usually one says in this case that D is **nef relative to the morphism** π, but in order to be concise we will drop the reference to π.

If E_u is an exceptional prime divisor on the model X_π, we denote by $\boxed{\check{E}_u} \in \mathcal{E}(\pi)_{\mathbb{Q}}$ the dual divisor with respect to the intersection form. It is defined by

$$\check{E}_u \cdot E_v = \delta_{u,v} \quad \text{for all } v \in \mathcal{P}(\pi), \tag{2}$$

where $\delta_{u,v}$ denotes Kronecker's delta. The existence and uniqueness of this dual basis is a consequence of Theorem 1.2(1). The fact that it lives in $\mathcal{E}(\pi)_{\mathbb{Q}}$ follows from the fact that all the intersection numbers $E_u \cdot E_v$ are integers. One has the following immediate consequence of formulae (2):

$$D = \sum_{v \in \mathcal{P}(\pi)} \left(D \cdot \check{E}_v\right) E_v \tag{3}$$

for all $D \in \mathcal{E}(\pi)_{\mathbb{R}}$.

As an immediate consequence of Theorem 1.2(2) and of formula (3) applied to the nef divisors $\check{E}_u$, we get:

Proposition 1.4. *The divisors $\check{E}_u$ are anti-effective with full support in the basis $(E_u)_{u \in \mathcal{P}(\pi)}$; that is, $\check{E}_u \cdot \check{E}_v < 0$ for all $u, v \in \mathcal{P}(\pi)$.*

In contrast with the fact that the intersection numbers $E_u \cdot E_v$ depend on the model on which they are computed, one has the following classical invariance property.

Proposition 1.5. *Let $u, v \in \mathcal{P}(X)$. Then the intersection number $\check{E}_u \cdot \check{E}_v$ does not depend on the model on which it is computed.*

Proof. Let $\psi : X_{\pi_2} \to X_{\pi_1}$ be the domination morphism between two models of X. In this proof we will not drop the reference to the model on which one works, using the notation $E_u^{\pi_i}, \check{E}_u^{\pi_i}$ for $i \in \{1, 2\}$. In view of Proposition 1.1, it is enough to show that if $u \in \mathcal{P}(\pi_1)$, then the divisor $\check{E}_u^{\pi_2}$ is the total transform of the divisor $\check{E}_u^{\pi_1}$.

By the projection formula (1), one has $E_v^{\pi_2} \cdot \psi^* \check{E}_u^{\pi_1} = 0$ for all $v \in \mathcal{P}(\pi_2) \setminus \{u\}$ and $E_u^{\pi_2} \cdot \psi^* \check{E}_u^{\pi_1} = \psi_* E_u^{\pi_2} \cdot \check{E}_u^{\pi_1} = E_u^{\pi_1} \cdot \check{E}_u^{\pi_1} = 1$. This shows that one has indeed $\psi^* \check{E}_u^{\pi_1} = \check{E}_u^{\pi_2}$. $\square$

The following definition is inspired by the approaches of Favre-Jonsson in [16, Appendix A] and Jonsson [29, section 7.3.6].

Definition 1.6. Let u, v be two possibly equal prime divisorial valuations of X. Their **bracket** is defined by

$$\boxed{\langle u, v \rangle} := -\check{E}_u \cdot \check{E}_v \in \mathbb{Q}_+^*.$$

Here E_u and E_v denote the representing divisors on a model on which both of them appear.

By Proposition 1.5, the bracket is independent of the choice of a model on which both u and v appear. We get in this way a function

$$\langle \cdot, \cdot \rangle \colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{Q}_+^*.$$

Till now we have worked with total transforms of divisors living on models of X, that is, on smooth surfaces. Let us consider now the case of a divisor A on X. If A is a principal divisor, then one may define its total transform $\pi^* A$ by a resolution π as the divisor of the pull-back of a defining function of A. The total transform is independent of the choice of defining function. Moreover, as a consequence of the projection formula (1), which is still true if one works with a proper birational morphism between normal surfaces, the intersection number of the total transform of A with any exceptional divisor on X_π is 0. This property was converted by Mumford [38] into a *definition* of the total transform of a not necessarily principal divisor on X.

Definition 1.7. Let A be a divisor on (X, x_0) and $\pi : X_\pi \to X$ a resolution of X. The **total transform** of A on X^π is the $\mathbb{Q}$-divisor $\pi^* A = A_\pi + A_\pi^{ex}$ on X^π such that:

(1) $\boxed{A_\pi}$ is the **strict transform** of A on X^π. Its support is the closure of $\pi^{-1}(|A| \setminus \{x_0\})$ in X_π, each one of its irreducible components being endowed with the same coefficient as its image in X.
(2) The support of the **exceptional transform** $\boxed{A_\pi^{ex}}$ of A on X^π is included in the exceptional divisor $E(\pi)$.
(3) $\pi^* A \cdot E_u = 0$ for each irreducible component E_u of $E(\pi)$.

The fact that such a divisor exists and is unique comes from the fact that condition (3) of the definition may be written as a square linear system of equations whose unknowns are the coefficients of A_π^{ex} in the basis $(E_u)_{u \in \mathcal{P}(\pi)}$ of $\mathcal{E}(\pi)_\mathbb{R}$ and whose matrix is the intersection matrix $(E_u \cdot E_v)_{u,v \in \mathcal{P}(\pi)}$. This matrix is nonsingular by Theorem 1.2(1). Note that we make here a slight abuse of language, as one gets a matrix only after having chosen a total order on the set $\mathcal{P}(\pi)$.

Note also that in Definition 1.7, one allows X_π to be *any* model of X without imposing that it be adapted in any sense to the divisor A. We say that π is an **embedded resolution** of A if the total transform $\pi^* A$ is a divisor with normal crossings. In this case, each branch of A has a strict transform on X_π which intersects *transversally* a *unique* prime exceptional divisor. Therefore, one has the following immediate consequence of Definition 1.7.

Proposition 1.8. *Assume that A is a branch and that π is an embedded resolution of it. Let $E_a \in \mathcal{P}(\pi)$ be the unique prime exceptional divisor which intersects the strict transform of A. Then*

$$A_\pi^{ex} = -\check{E}_a.$$

Let us introduce the following denomination for the divisor E_a.

Definition 1.9. Let A be a branch on X and let π be an embedded resolution of it. The unique prime exceptional divisor $E_a \in \mathcal{P}(\pi)$ which intersects the strict transform of A on X_π is called the **representing divisor** of A on X_π.

Using the notion of total transform of divisors from Definition 1.7, Mumford defined in the following way in [38] the intersection number of two divisors without common branches on X.

Definition 1.10. Let A, B be two divisors on X without common branches. Then their **intersection number** $\boxed{A \cdot B} \in \mathbb{Q}$ is defined by

$$A \cdot B := \pi^* A \cdot \pi^* B$$

for any resolution π of X.

This definition is independent of the resolution. In the special case in which both A and B are branches, we get the following interpretation of the *bracket*.

Proposition 1.11. *Let A, B be two distinct branches on X. Consider an embedded resolution X_π of the divisor $A + B$. If E_a and E_b are the possibly coinciding representing divisors of A and B on X_π, then*

$$A \cdot B = \langle a, b\rangle.$$

Proof. According to Definition 1.10, we have $A \cdot B = \pi^* A \cdot \pi^* B$. By bilinearity of the intersection product, $\pi^* A \cdot \pi^* B = \pi^* A \cdot B_\pi + \pi^* A \cdot B_\pi^{ex}$. The second term of this sum vanishes by the projection formula (1): $\pi^* A \cdot B_\pi^{ex} = A \cdot \pi_* B_\pi^{ex} = A \cdot 0 = 0$. Hence, we get $A \cdot B = \pi^* A \cdot B_\pi = A_\pi \cdot B_\pi + A_\pi^{ex} \cdot B_\pi$. The first term of this last sum vanishes, because our hypothesis that π is an embedded resolution of the divisor $A + B$ shows that the strict transforms A_π and B_π are disjoint. Consider now the relation $A_\pi^{ex} \cdot \pi^* B = 0$, symmetrical to the relation $\pi^* A \cdot B_\pi^{ex} = 0$ used before. Using again the bilinearity of the intersection product, it may be written $A_\pi^{ex} \cdot B_\pi + A_\pi^{ex} \cdot B_\pi^{ex} = 0$. Therefore

$$A \cdot B = A_\pi^{ex} \cdot B_\pi = -A_\pi^{ex} \cdot B_\pi^{ex} = -\check{E}_a \cdot \check{E}_b = \langle a, b\rangle, \tag{4}$$

the penultimate equality being a consequence of Proposition 1.8, and the last one being just the definition of the bracket. □

Notice that the case $a = b$ in Proposition 1.11 may occur when the strict transforms A_π and B_π intersect the same irreducible component of $E(\pi)$.

The next consequence of Proposition 1.11 will be used in the proof of Proposition 1.45.

Corollary 1.12. *Let π be a resolution of X. Let A, B be two distinct branches on X such that the strict transforms A_π and B_π are disjoint. Then*

$$A \cdot B = -A_\pi^{ex} \cdot B_\pi^{ex}.$$

Proof. This results from the proof of Proposition 1.11, which uses the fact that the modification π is an embedded resolution of $A + B$ only in the last two equalities in (4), what precedes them needing only the hypothesis of disjointness of the strict transforms. □

1.2. **The angular distance.** In this subsection we recall the notion of *angular distance* ρ of prime divisorial valuations (see Definition 1.13), introduced in greater generality by Gignac and the last author in [21] and by the first three authors in a slightly different form in [19] for the restricted class of arborescent singularities. The definition uses the bracket of Definition 1.6. The fact that ρ is indeed a distance depends on a crucial inequality of Gignac and the last author, which we recall in Proposition 1.18. We conclude the section with a list of reformulations of this inequality (see Proposition 1.19).

Let X_π be a model of X and let $u, v \in \mathcal{P}(\pi)$ be two prime divisorial valuations appearing on it. By Theorem 1.2(1), the intersection form on $\mathcal{E}(\pi)_\mathbb{R}$ is negative definite. Let us apply the Cauchy-Schwartz inequality to its opposite bilinear form and to the vectors $\check{E}_u, \check{E}_v \in \mathcal{E}(\pi)_\mathbb{R}$. Using Proposition 1.4 and Definition 1.6, we get the following inequalities:

$$0 < \langle u, v\rangle^2 \leq \langle u, u\rangle \cdot \langle v, v\rangle, \tag{5}$$

with equality if and only if $u = v$. This allows us to define the following.

Definition 1.13. The **angular distance** of the prime divisorial valuations $u, v \in \mathcal{P}(X)$ is

$$\boxed{\rho(u,v)} := -\log \frac{\langle u,v\rangle^2}{\langle u,u\rangle \cdot \langle v,v\rangle} \in [0,\infty). \tag{6}$$

As an immediate consequence of inequality (5) and of the characterization of the case of equality, one gets:

Proposition 1.14. *For every pair of prime divisorial valuations (u, v) of X, one has $\rho(u, v) \geq 0$, with equality if and only if $u = v$.*

Remark 1.15. A slightly different notion was introduced before by the first three authors in [19, Definition 4.11], in the special case of arborescent normal surface singularities. It was introduced almost simultaneously by the last author and Gignac for arbitrary semivaluations of X in [21, Definition 2.39].

As indicated by the name chosen in Definition 1.13, ρ is indeed a metric on the set $\mathcal{P}(X)$ (see Proposition 1.19(II) below). But this fact is not immediate. It is a consequence of an inequality of Gignac and the last author (see Proposition 1.18 below). In order to state this inequality, we need the following graph-theoretical notion (see section 1.4 for our vocabulary concerning graphs).

Definition 1.16. Let a, b, c be three not necessarily pairwise distinct vertices of the connected graph Γ. One says that c **separates** a **from** b in Γ if:

- either $c \in \{a, b\}$
- or a and b belong to distinct connected components of the topological space $\Gamma \setminus \{c\}$.

We apply the previous notion of separation to the *dual graphs* of the *good models* of X.

Definition 1.17. Let $\pi : X_\pi \to X$ be a resolution of X. The resolution π and the model X_π are called **good** if their exceptional divisor has normal crossings and its prime components are smooth. The **dual graph** $\boxed{\Gamma_\pi}$ of a good model X_π has vertex set $\mathcal{P}(\pi)$ and set of edges between any two vertices $u, v \in \mathcal{P}(\pi)$ in bijection with the intersection points on X_π between the associated prime divisors E_u and E_v.

Here is the announced inequality of Gignac and the last author (see [21, Proposition 1.10]), which is crucial for the present paper.

Proposition 1.18 ([21, Proposition 1.10])**.** *Let X_π be a good model of the normal surface singularity X, and let E_u, E_v, and E_w be not necessarily distinct exceptional prime divisors of π. Then one has the inequality*

$$(-\check{E}_u \cdot \check{E}_v)(-\check{E}_v \cdot \check{E}_w) \leq (-\check{E}_v \cdot \check{E}_v)(-\check{E}_u \cdot \check{E}_w), \tag{7}$$

with equality if and only if v separates u and w in the dual graph Γ_π of X_π.

Proof. Let us sketch a slight variant of the original proof. We work with the opposite of the intersection form, which is positive definite. Denote therefore $\langle V_1, V_2\rangle := -V_1 \cdot V_2$ for any $V_1, V_2 \in \mathcal{E}(\pi)_{\mathbb{R}}$. Inequality (7) may be rewritten as

$$\langle \check{E}_u - \frac{\langle \check{E}_u, \check{E}_v\rangle}{\langle \check{E}_v, \check{E}_v\rangle}\check{E}_v \ , \ \check{E}_w\rangle \geq 0. \tag{8}$$

Using equation (3), we see that the truth of the previous inequality for all $w \in \mathcal{P}(\pi)$ and fixed $u, v \in \mathcal{P}(\pi)$ is equivalent to the following statement:

$$\text{the divisor } \check{E}_u - \frac{\langle \check{E}_u, \check{E}_v \rangle}{\langle \check{E}_v, \check{E}_v \rangle} \check{E}_v \text{ is effective.} \tag{9}$$

The key to the proof of (9) is to understand geometrically the previous expressions. Consider the linear hyperplane $\mathcal{H}_w$ of $\mathcal{E}(\pi)_{\mathbb{R}}$ spanned by the vectors E_a for $a \in \mathcal{P}(\pi) \setminus \{w\}$. Those vectors form a basis of the hyperplane $\mathcal{H}_w$. Look at the dual basis relative to the restriction of $\langle \cdot, \cdot \rangle$ to $\mathcal{H}_w$. As can be verified by an immediate computation, the vector corresponding to E_u in this dual basis is exactly the vector occurring in (9). Now let us apply Proposition 1.3 to the Euclidean space $(\mathcal{H}_w, \langle \cdot, \cdot \rangle)$ and the basis $(E_a)_{a \in \mathcal{P}(\pi) \setminus \{w\}}$. We deduce that the coefficients of the elements of its dual basis in the starting basis are non-negative, which is exactly the statement (9).

There is a slight difference with the hypotheses of Proposition 1.3. There one assumed that the basis could not be partitioned into two non-empty orthogonal subsets. Here we are in a situation in which the dual graph is not necessarily connected. Namely, as we work in the hyperplane $\mathcal{H}_w$, we drop the component E_w from the exceptional divisor; therefore the dual graph of the remaining components gets decomposed in a finite positive number of connected components. The associated partition of $\mathcal{P}(\pi) \setminus \{w\}$ induces an orthogonal direct sum decomposition of $\mathcal{H}_w$, each term of this sum having a connected dual graph. The dual basis of $(E_a)_{a \in \mathcal{P}(\pi) \setminus \{w\}}$ is the union of the dual bases of the individual terms of this orthogonal direct sum. Apply then Proposition 1.3 to each such term. One easily gets in this way the characterization of the case of equality in (8). ☐

The point (III) in the following reformulation of Proposition 1.18 was already stated by the third author in the summary [41] of the work [19].

Proposition 1.19. *Let X_π be a good model of X, and let E_u, E_v, and E_w be not necessarily distinct exceptional prime divisors of π. Then the following statements hold:*

(I) $\langle u, v \rangle \cdot \langle v, w \rangle \leq \langle v, v \rangle \cdot \langle u, w \rangle$, *with equality if and only if v separates u from w in the dual graph Γ_π.*

(II) *The function ρ is a metric on the finite set $\mathcal{P}(\pi)$, with equality in the triangle inequality $\rho(u, v) + \rho(v, w) \geq \rho(u, w)$ if and only if v separates u from w in Γ_π.*

(III) *Endow the real vector space $\mathcal{E}(\pi)_{\mathbb{R}}$ with the Euclidean structure equal to the opposite of the intersection form. On its unit sphere, consider the pairwise distinct vectors which are positively proportional to $\check{E}_u, \check{E}_v, \check{E}_w$. Join them by shortest geodesics, obtaining a spherical triangle called simply uvw. This triangle has all its angles in the interval $(0, \pi/2]$. Moreover, it is rectangular at v if and only if v separates u from w in Γ_π.*

Proof. The equivalence of the inequality (7) with the inequality (I) and the assertion on the triangle inequality in (II) are a simple consequence of Definitions 1.6 and 1.13 and Proposition 1.14.

The reformulation (III) needs a little more explanation. First, note that inequality (7) may be rewritten as

$$\frac{-\check{E}_u\cdot\check{E}_v}{\sqrt{(-\check{E}_u\cdot\check{E}_u)(-\check{E}_v\cdot\check{E}_v)}}\cdot\frac{-\check{E}_v\cdot\check{E}_w}{\sqrt{(-\check{E}_v\cdot\check{E}_v)(-\check{E}_w\cdot\check{E}_w)}}\leq\frac{-\check{E}_u\cdot\check{E}_w}{\sqrt{(-\check{E}_u\cdot\check{E}_u)(-\check{E}_w\cdot\check{E}_w)}}.$$

Measuring the angles using the opposite of the intersection form (which is indeed a Euclidean metric on the real vector space $\mathcal{E}(\pi)_{\mathbb{R}}$ by Theorem 1.2(1)), the previous inequality may be rewritten as

$$\cos(\angle\check{E}_u\check{E}_v)\cdot\cos(\angle\check{E}_v\check{E}_w)\leq\cos(\angle\check{E}_u\check{E}_w). \tag{10}$$

Recall now the spherical law of cosines for a geodesic triangle on a unit sphere, whose edges have lengths denoted $a, b, c\in(0,\pi)$, the angle opposite to the edge of length a being denoted $A\in(0,\pi)$ (see for instance Prasolov and Tikhomirov [42, section 5.1, p. 87], Ratcliffe [44, Theorem 2.5.3], or Van Brummelen [50, Chapter 6]):

$$\cos a=\cos b\cdot\cos c+\sin b\cdot\sin c\cdot\cos A.$$

Applying it to the spherical triangle uvw, with preferred vertex v, we see that the inequality (10) is equivalent to the fact that the angle at vertex v belongs to the interval $(0,\pi/2]$. The fact that one has equality if and only if the angle is $\pi/2$ is the content of the *spherical Pythagorean theorem*, which may also be obtained as a consequence of the spherical law of cosines. □

Remark 1.20. We may speak about the spherical triangle with vertices at u, v, w without mentioning the model on which we work because, by Proposition 1.5, this triangle is independent of the model up to isometry. Note that a spherical triangle may have 2 or 3 angles $\geq\pi/2$, but that in our case at most one angle is equal to $\pi/2$, the two others being acute. This results from the fact that *if v separates u from w, then neither u separates v from w nor w separates u from v.*

There exist other kinds of extensions of the usual Pythagorean theorem to the three kinds of bidimensional Riemannian geometries of constant curvature (see for instance Maraner [34] and Foote [17]).

For the moment we have no applications of the spherical geometrical viewpoint (III), but we think that it is intriguing and that it is worth formulating as a very vivid way of remembering the inequality of Proposition 1.18.

1.3. **A reformulation of the ultrametric problem.** In this subsection we begin the study of the function u_L introduced by the first three authors in [19], defined whenever L is a fixed branch on the normal surface singularity X. Given a finite set $\mathcal{F}$ of branches, in Corollary 1.25 we reformulate the condition that for every branch $L\in\mathcal{F}$ the function u_L is an ultrametric on $\mathcal{F}\setminus\{L\}$ as the condition that the angular distance on $\mathcal{F}$ is *tree-like*. Then we recall the correspondence between tree-like distances on finite sets $\mathcal{F}$ and metric trees having a subset of vertices labeled by $\mathcal{F}$ (see Proposition 1.29).

Let L be a fixed branch on X. If A, B are two other branches, assumed to be distinct from L, let us define the following (see [19]).

$$\boxed{u_L(A,B)}:=\begin{cases}\dfrac{(L\cdot A)\,(L\cdot B)}{A\cdot B} & \text{if}\quad A\neq B,\\[2ex] 0 & \text{if}\quad A=B.\end{cases} \tag{11}$$

The following vocabulary was introduced in [19].

Definition 1.21. A normal surface singularity is called **arborescent** if the dual graphs of its good models are trees.

In [19, Theorem 4.18], the first three authors proved the following theorem as a generalization of a theorem of Płoski [40] concerning the case where X is *smooth*.

Theorem 1.22. *If X is an arborescent singularity, then for every branch L on X, the function u_L is an ultrametric on the set $\mathcal{B}(X) \setminus \{L\}$ of branches on X which are distinct from L.*

The present paper is an outgrowth of our desire to understand in which measure Theorem 1.22 extends to other normal surface singularities.

Let us begin with a reformulation of the ultrametric inequality for u_L, whose simple proof is left to the reader.

Proposition 1.23. *Let L, A, B, C be four pairwise distinct branches on X. Consider an embedded resolution π of their sum. Denote by l, a, b, c the prime divisorial valuations corresponding to the representing divisors on X_π of L, A, B and, respectively, C (see Definition* 1.9*). Then the following inequalities are equivalent, as well as the corresponding equalities:*

(1) $u_L(A,B) \leq \max\{u_L(A,C), u_L(B,C)\}$.
(2) $(A \cdot B)(L \cdot C) \geq \min\{(A \cdot C)(L \cdot B),\ (B \cdot C)(L \cdot A)\}$.
(3) $\langle a, b\rangle \cdot \langle l, c\rangle \geq \min\{\langle a, c\rangle \cdot \langle l, b\rangle,\ \langle b, c\rangle \cdot \langle l, a\rangle\}$.
(4) $\rho(a,b) + \rho(l,c) \leq \max\{\rho(a,c) + \rho(l,b),\ \rho(b,c) + \rho(l,a)\}$.

The next proposition is subtler.

Proposition 1.24. *Let $\mathcal{F}$ be a set of branches on X. If u_L is an ultrametric on $\mathcal{F} \setminus \{L\}$ for one branch L in $\mathcal{F}$, then the same is true for any branch of $\mathcal{F}$.*

Proof. This proof is inspired by the explanations of Böcker and Dress in [3, Lemma 6, Corollary 7, Remark 5]. Let L and M be two distinct branches on X. We assume that u_L is an ultrametric on $\mathcal{F} \setminus \{L\}$. We want to prove that u_M is an ultrametric on $\mathcal{F} \setminus \{M\}$.

Consider three pairwise distinct branches A, B, C in $\mathcal{F} \setminus \{M\}$ (if this set has less than three elements, then there is nothing to prove). If $L \in \{A, B, C\}$, then the equivalence of (1) and (2) in Proposition 1.23 shows that the ultrametric inequalities of the restriction of u_M to $\{A, B, C\}$ are equivalent to the ultrametric inequalities of the restriction of u_L to $\{M, A, B, C\} \setminus \{L\}$.

Assume now that $L \notin \{A, B, C\}$. Using again the equivalence of (1) and (2) in Proposition 1.23, we see that the fact that u_M is an ultrametric in restriction to $\{A, B, C\}$ is equivalent to the fact that among the products $(B \cdot C)(M \cdot A)$, $(A \cdot C)(M \cdot B)$, $(A \cdot B)(M \cdot C)$, two are equal and the third one is not less than they are. An immediate computation shows that this is equivalent to the fact that

(12)
$$\begin{gathered}\textit{among the products } u_L(B,C) \cdot u_L(M,A), u_L(A,C) \cdot u_L(M,B),\\ u_L(A,B) \cdot u_L(M,C), \textit{two are equal and the third one is not greater than them.}\end{gathered}$$

This is the statement which we will prove. If the six values taken by u_L in restriction to pairs of distinct elements of the set $\{M, A, B, C\}$ are equal, then the assertion (12) is obvious, the three products being equal.

Assume therefore that not all six values are equal. In order to follow the next reasoning, we recommend that the reader draw the edges of a tetrahedron with vertices M, A, B, C and look successively at its faces. The basic fact which will be used many times for various triples is that in an ultrametric space, among the distances between three points, two are equal and the third one is not bigger than them.

Up to permuting the labels M, A, B, C, we may consider that $u_L(M,A) > u_L(A,B)$. As u_L is ultrametric on $\{M, A, B\}$, we get the relations $u_L(M,A) = u_L(M,B) > u_L(A,B)$. Let us now compare $u_L(M,A)$ to $u_L(M,C)$.

• **Suppose that** $u_L(M,C) < u_L(M,A) = u_L(M,B)$**.** As u_L is ultrametric on $\{M, A, C\}$ and on $\{M, B, C\}$, we deduce that $u_L(A,C) = u_L(M,A)$ and $u_L(M,B) = u_L(B,C)$. Therefore $u_L(A,C) = u_L(M,A) = u_L(M,B) = u_L(B,C)$, and this number is strictly bigger than both $u_L(A,B)$ and $u_L(M,C)$. Therefore

$$u_L(B,C) \cdot u_L(M,A) = u_L(A,C) \cdot u_L(M,B) > u_L(A,B) \cdot u_L(M,C).$$

• **Suppose that** $u_L(M,C) = u_L(M,A) = u_L(M,B)$**.** As u_L is ultrametric on $\{A, B, C\}$, we have the relations $u_L(A,B) \leq u_L(B,C) = u_L(C,A)$, up to permutation. Therefore

$$u_L(B,C) \cdot u_L(M,A) = u_L(A,C) \cdot u_L(M,B) \geq u_L(A,B) \cdot u_L(M,C).$$

• **Suppose that** $u_L(M,C) > u_L(M,A) = u_L(M,B)$**.** Using again the fact that u_L is ultrametric on $\{M, A, C\}$ and on $\{M, B, C\}$, we deduce that $u_L(C,A) = u_L(M,C) = u_L(B,C)$. Therefore we get again

$$u_L(B,C) \cdot u_L(M,A) = u_L(A,C) \cdot u_L(M,B) > u_L(A,B) \cdot u_L(M,C).$$

We see that the assertion (12) is true in all cases, which proves the proposition. □

In Proposition 1.23, the branches L, A, B, C were fixed. By applying this proposition to all the quadruples in a finite set of branches $\mathcal{F}$ and by using also Proposition 1.24, we get immediately:

Corollary 1.25. *Let $\mathcal{F} \subset \mathcal{B}(X)$ be a finite set of branches on X. Consider an embedded resolution π of their sum and denote by $\mathcal{F}_\pi \subset \mathcal{P}(\pi)$ the set of prime exceptional divisors representing the elements of $\mathcal{F}$ in X_π according to Definition* 1.9*. Then the following properties are equivalent:*

(1) *For some $L \in \mathcal{F}$, the function u_L is an ultrametric on $\mathcal{F} \setminus \{L\}$.*
(2) *For every $L \in \mathcal{F}$, the function u_L is an ultrametric on $\mathcal{F} \setminus \{L\}$.*
(3) *The bracket $\langle \cdot, \cdot \rangle$ satisfies the inequality*

$$\langle a, b\rangle \cdot \langle l, c\rangle \geq \min\{\langle a, c\rangle \cdot \langle l, b\rangle,\ \langle b, c\rangle \cdot \langle l, a\rangle\} \quad \textit{for all } (a,b,c,l) \in (\mathcal{F}_\pi)^4.$$

(4) *The angular distance ρ satisfies the inequality*

$$\rho(a,b) + \rho(l,c) \leq \max\{\rho(a,c) + \rho(l,b),\ \rho(b,c) + \rho(l,a)\} \quad \textit{for all } (a,b,c,l) \in (\mathcal{F}_\pi)^4.$$

Let us introduce the following vocabulary concerning the metrics which satisfy condition (4) of Corollary 1.25.

Definition 1.26. Let S be a finite set. One says that a distance δ on S is **tree-like** if, for all $(a,b,c,d) \in S^4$, one has the following **4-point condition**:

$$\delta(a,b) + \delta(c,d) \leq \max\{\delta(a,c) + \delta(b,d), \delta(a,d) + \delta(b,c)\}. \tag{13}$$

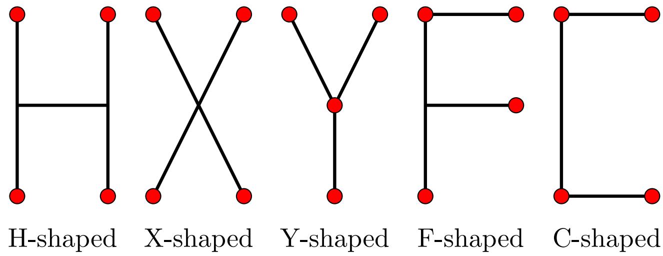

FIGURE 1. The 5 possible S-trees, when S has 4 elements.

This means that, up to a permutation of the three sums, one has

$$\delta(a,b)+\delta(c,d) \leq \delta(a,c)+\delta(b,d) = \delta(a,d)+\delta(b,c). \tag{14}$$

The term 4-*point condition* was introduced by Buneman in [7]. We chose the name tree-like for the previous kind of metrics because such finite metric spaces may be interpreted geometrically as special kinds of trees (see Proposition 1.29 below). Let us introduce first more vocabulary about trees.

Definition 1.27. A **finite tree** is a finite simply connected simplicial complex of dimension 1. The **convex hull** $\boxed{\mathrm{Conv}(\mathcal{F})}$ of a set $\mathcal{F}$ of vertices of a tree is the subtree obtained as the union of the paths joining pairwise the elements of $\mathcal{F}$. If S is a finite set, then an S**-tree** is a finite tree whose set of vertices contains the set S and such that all its vertices of valency 1 or 2 are elements of S. An **isomorphism** of S-trees is an isomorphism of trees which is the identity in restriction to the set S.

Given two S-trees, the fact that all their vertices of valency 1 are elements of S implies that there exists at most one isomorphism between them. When S has 4 elements, there are exactly 5 different S-trees up to isomorphism. They are represented in Figure 1, together with the names we will use for them in what follows.

Definition 1.28. A **metric tree** is a finite tree endowed with a map from its set of edges to the set of positive real numbers. The number associated to an edge is called its **length**. The **induced distance** of a metric S-tree is the distance on S associating to each pair of elements of S the sum of length of the edges lying on the unique path joining them in the tree.

An example of a metric S-tree is shown in Figure 2. Here $S = \{a, \dots, e\}$. Denoting by δ the induced distance on S, one has for instance $\delta(a,d) = 3+2+2$ and $\delta(b,c) = 2+1$.

It is immediate to check that the distance induced by a metric S-tree on the finite set S satisfies the 4-point condition. Therefore, it is tree-like, in the sense of Definition 1.26. Conversely, one has the following proposition (see Buneman's paper [7] and the successive generalizations of Bandelt and Steel [2] and Böcker and Dress [3]).

Proposition 1.29. *Let S be a finite set and let δ be a distance on it. If δ is tree-like, then there exists a unique S-tree T endowed with a length function such that the induced distance on S is equal to δ.*

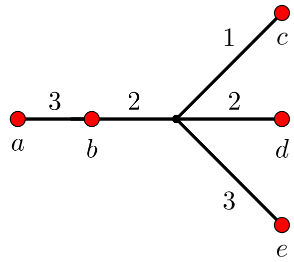

FIGURE 2. An $\{a, b, c, d, e\}$-tree endowed with a length function.

The main idea of the proof of the previous proposition is that an S-tree is determined up to isomorphism by the isomorphism types of the convex hulls of all quadruples of elements of S, which are in turn determined by the inequalities which are equalities in the 4-point condition and in the triangle inequalities concerning them. More precisely, given a quadruple $Q \subset S$ (see Figure 1):

- the H-shaped and X-shaped Q-trees are those Q-trees for which one has only strict triangle inequalities: among them, the H-shaped tree is characterized by the fact that one has a strict inequality in the 4-point condition (14), for a convenient labeling of the elements of Q by the letters a, b, c, d;
- the Y-shaped Q-trees are those Q-trees such that for exactly one triple of points of Q, all the corresponding triangle inequalities are strict;
- the F-shaped Q-trees are those Q-trees such that for exactly two triples of points of Q, one of the corresponding triangle inequalities is an equality;
- the C-shaped Q-trees are those Q-trees such that for all triple of points of Q, one of the corresponding triangle inequalities is an equality.

Proposition 1.29 allows us to define:

Definition 1.30. Let δ be a tree-like metric on a finite set S. Then the unique S-tree endowed with a length function such that the induced distance on S is equal to δ is called the **tree hull** of the metric space (S, δ).

1.4. A theorem about special metrics on the set of vertices of a graph.

Let X_π be a good model of X. Consider the angular distance ρ on the vertex set $\mathcal{V}(\Gamma_\pi) = \mathcal{P}(\pi)$ of the associated dual graph Γ_π. In Proposition 1.19, we saw that the cases of equality in the triangle inequalities associated to the metric space $(\mathcal{V}(\Gamma_\pi), \rho)$ are characterized by separation properties in Γ_π. The aim of this subsection is to prove that if a metric δ on the set of vertices $\mathcal{V}(\Gamma)$ of a connected graph Γ satisfies this kind of constraint, then it becomes tree-like (in the sense of Definition 1.26) in restriction to special types of subsets $\mathcal{F}$ of $\mathcal{V}(\Gamma)$ (see Theorem 1.38). Moreover, the tree hull of $(\mathcal{F}, \delta)$ (according to Definition 1.30) may be described as the convex hull of $\mathcal{F}$ in a tree canonically associated to the graph Γ, its **brick-vertex tree** $\mathcal{BV}(\Gamma)$ (see Definition 1.34).

In what follows, we will use the following notion of graph.

Definition 1.31. A **graph** Γ is a finite cell complex of dimension at most 1. In particular, it may have loops or multiple edges, and it may have connected components which are simply points. We will denote by $\boxed{\mathcal{V}(\Gamma)}$ its set of vertices and by $\boxed{\mathcal{A}(\Gamma)}$ its set of edges. The **valency** of a vertex v of Γ is the number of

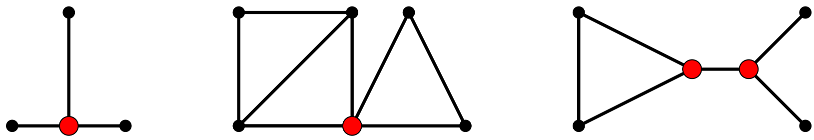

FIGURE 3. A few separable graphs and their cut-vertices marked in red.

germs of edges adjacent to v (a loop based at v counting twice, as it contributes with two germs in this count).

If we want to insist on the graph Γ in which we compute the valency (in situations where we deal with several graphs at the same time), we will speak about the **Γ-valency** of a vertex v.

It will be important for us to look at the edges of a connected graph Γ according to their separation properties.

Definition 1.32. Let Γ be a connected graph. A **cut-vertex** of Γ is a vertex whose removal disconnects Γ. A **bridge** of Γ is an edge such that the removal of its interior disconnects Γ. The graph Γ is called **separable** if it admits at least one cut-vertex (see Figure 3). Otherwise, it is called **non-separable**.

The only non-separable graphs which are trees are the segments. All the other non-separable graphs have the property that any two of their edges are contained in a **circuit**, that is, a union of edges whose underlying topological space is homeomorphic to a circle. The trees may be characterized as the connected graphs all of whose edges are bridges.

Every connected graph contains a distinguished family of non-separable subgraphs, its *blocks*, among which we distinguish the *bricks* and the *bridges*.

Definition 1.33. The **blocks** of a connected graph Γ are its maximal subgraphs which are non-separable (see Figure 4). A block which is equal to an edge of Γ is called a **bridge**; otherwise it is called a **brick**.

The notions of bridge of Definitions 1.32 and 1.33 are equivalent.

The blocks of a connected graph Γ may be characterized as the unions of edges of each equivalence class for the following equivalence relation on the set $\mathcal{A}(\Gamma)$: two edges are equivalent if they are either equal or they are both contained in the same circuit. Trees may be characterized as the connected finite graphs which have no bricks.

It is elementary to check that the following construction leads indeed to a tree.

Definition 1.34. The **brick-vertex tree** $\boxed{\mathcal{BV}(\Gamma)}$ of a connected graph Γ is the tree whose vertex set is the union of the set of bricks of Γ and of the set of its vertices. The set of its edges consists of the bridges of Γ and of new edges connecting a brick of Γ to a vertex of Γ (seen as vertices of $\mathcal{BV}(\Gamma)$) if and only if the brick contains the vertex. A vertex of $\mathcal{BV}(\Gamma)$ associated to a brick of Γ will be called a **brick-vertex**.

If a is a vertex (resp. if B is a brick) of Γ, we will denote by $\overline{a}$ (resp. $\overline{B}$) the vertex of $\mathcal{BV}(\Gamma)$ defined by it. If $e = \{a, b\}$ is a bridge of Γ, then $\overline{e} = \{\overline{a}, \overline{b}\}$ is also a bridge of $\mathcal{BV}(\Gamma)$. Similarly, if $\mathcal{F}$ is a set of vertices of Γ, we denote by $\overline{\mathcal{F}}$ the same set seen as a set of vertices of $\mathcal{BV}(\Gamma)$.

Examples of planar brick-vertex trees are shown in Figures 4 and 8. The bricks are emphasized by shading the plane regions spanned by their vertices and edges.

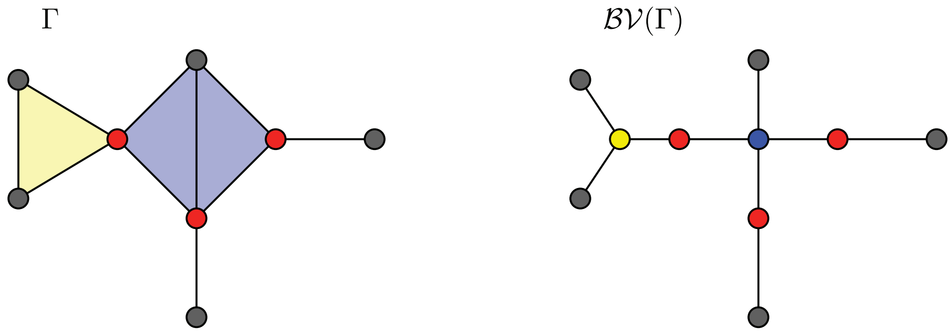

FIGURE 4. The brick-vertex tree of a connected graph.

Remark 1.35. Whitney introduced the blocks of a finite graph in his 1932 paper [53] under the name of *components*. His definition was slightly different: the blocks were the final graphs (necessarily inseparable) of a process which chooses at each step a cut-vertex of the graph and decomposes the connected component which contains it into the connected subgraphs which are joined at that vertex. The term *block* seems to have been introduced for this concept in Harary's 1959 paper [22]. In Tutte's 1966 book [49], the blocks are called *cyclic elements*, a term originating from general topology (see Remark 2.50). The use of the term *brick* for the blocks which are not bridges seems to be new. A construction related to the brick-vertex tree is known under the name of *cut-tree* (see Tutte's book [49, section 9.5]), *block-cut tree* (see Harary's book [23, p. 36]), or *block tree* (see Bondy and Murty's book [4, section 5.2]). In that construction, which was introduced by Gallai [18] and Harary and Prins [24], one considers only the set of cut-vertices of Γ, instead of the full set of vertices, and all the blocks, not only the bricks. Later on, Kulli [30] introduced the *block-point tree* of a connected graph, in which one still considers all the blocks, but also all the vertices, not only the cut-vertices.

The following proposition, which uses the notation explained after Definition 1.34, is the reason why we introduced the notion of brick-vertex tree.

Proposition 1.36. *Let a, b, c be three not necessarily pairwise distinct vertices of the connected graph Γ. Then the following properties are equivalent:*

(1) *a separates b from c in the graph Γ;*
(2) *$\overline{a}$ separates $\overline{b}$ from $\overline{c}$ in the brick-vertex tree $\mathcal{BV}(\Gamma)$.*

Proof. First notice that if $b = c \neq a$, then a does not separate b and c either in Γ or in $\mathcal{BV}(\Gamma)$, while if a coincides with either b or c, then it separates b from c both in Γ and $\mathcal{BV}(\Gamma)$ (see Definition 1.16). Hence, we may suppose that a, b, c are pairwise distinct.

• **Suppose first that a does not separate b from c in Γ.** Therefore, there exists a path γ joining b and c in $\Gamma \setminus \{a\}$. Decompose γ in a finite sequence of concatenating edges e_j with endpoints v_{j-1}, v_j for $j = 1, \dots, n$, with $v_0 = b$,

$v_n = c$, and $v_j \neq a$ for all j. We construct a path $\tilde{\gamma}$ joining $\overline{b}$ and $\overline{c}$ in $\mathcal{BV}(\Gamma) \setminus \{a\}$ as follows:

- If v_{j-1} and v_j belong to a brick B, then we replace the edge e_j with the concatenation of the two edges $\{\overline{v_{j-1}}, \overline{B}\}$, $\{\overline{B}, \overline{v_j}\}$ of $\mathcal{BV}(\Gamma)$.
- If the edge e_j connecting v_{j-1} and v_j is a bridge, then we consider the associated edge $\overline{e_j} = \{\overline{v_{j-1}}, \overline{v_j}\}$ of $\mathcal{BV}(\Gamma)$.

• **Suppose now that $\overline{a}$ does not separate $\overline{b}$ from $\overline{c}$ in $\mathcal{BV}(\Gamma)$.** Therefore, there exists a path $\tilde{\gamma}$ joining $\overline{b}$ and $\overline{c}$ in $\mathcal{BV}(\Gamma) \setminus \{\overline{a}\}$. Denote by $\epsilon_j = \{w_{j-1}, w_j\}$ the sequence of edges of $\tilde{\gamma}$ (notice that as $\mathcal{BV}(\Gamma)$ is a *tree*, the edges are determined by their extremities). Therefore, every vertex w_j of this path corresponds to a vertex or to a brick of Γ. We construct a path γ joining b and c in $\Gamma \setminus \{a\}$ as follows. The endpoints of every edge ϵ_j of $\tilde{\gamma}$ either correspond simultaneously to vertices of Γ or one corresponds to a vertex and the other to a brick of Γ. In the first case, we define e_j to be the unique bridge of Γ which projects to ϵ_j. In the second case, since the vertices $\overline{b}$ and $\overline{c}$ of $\mathcal{BV}(\Gamma)$ correspond to vertices of Γ we can assume, up to replacing j by $j+1$ if necessary, that $w_{j-1} = \overline{v_{j-1}}$ and $w_{j+1} = \overline{v_{j+1}}$ correspond to vertices v_{j-1} and v_{j+1} of Γ and that $w_j = \overline{B}$ corresponds to the unique brick B containing them. Notice that a could be a vertex of B. Since v_{j-1} and v_{j+1} belong to B, there exist two paths of Γ inside the brick B joining v_{j-1} to v_{j+1}, which intersect only at their endpoints. Therefore, at least one of them does not pass through a. We define then $\gamma_{j-1,j+1}$ to be such a path avoiding a and contained inside the brick B of Γ. Finally, the path γ of Γ obtained as the union of all the previous elementary paths e_j and $\gamma_{j-1,j+1}$ joins indeed b and c without passing through a. □

Remark 1.37. Proposition 1.36 holds also if we replace the brick-vertex tree by Kulli's block-point tree (see Remark 1.35 for its definition), the proof being completely analogous. In fact, we could work in this first part of the paper with the block-point tree of Γ. We chose to work with Definition 1.34 since it has the advantage of extending directly to graphs of $\mathbb{R}$-trees (see section 2.6). Notice that for a tree Γ, its brick-vertex tree coincides with Γ, while its block-point tree is isomorphic to the barycentric subdivision of Γ.

By Proposition 1.36, the brick-vertex tree of Γ encodes precisely the way in which the vertices of Γ get separated by the elimination of one of them.

Recall the reformulation of Proposition 1.18 given in Proposition 1.19 (II). It states that if one looks at the angular distance ρ on the vertex set $\mathcal{V}(\Gamma_\pi)$ of the dual graph Γ_π of a good model X_π of X, then one has an equality $\rho(u,v) + \rho(v,w) = \rho(u,w)$ in the triangular inequality associated to the triple (u,v,w) of vertices of Γ_π if and only if v separates u from w in Γ_π. The following theorem, which is the main result of this section, describes special subsets of vertices of the graphs endowed with metrics having the same formal property (recall that the convex hull of a finite set of vertices of a tree was introduced in Definition 1.27).

Theorem 1.38. *Let Γ be a finite connected graph and let $\delta : \mathcal{V}(\Gamma)^2 \to [0,\infty)$ be a metric such that one has the equality*

$$\delta(a,b) + \delta(b,c) = \delta(a,c) \tag{15}$$

if and only if the vertex b separates a from c in Γ. Consider a set $\mathcal{F}$ of vertices of Γ and their convex hull $\mathrm{Conv}(\overline{\mathcal{F}})$ *in the brick-vertex tree $\mathcal{BV}(\Gamma)$ of Γ. If each brick*

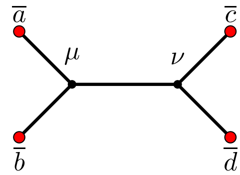

FIGURE 5. The case of an H-shaped tree in the proof of Theorem 1.38.

of Γ *has* $\mathrm{Conv}(\overline{\mathcal{F}})$*-valency at most* 3*, then the restriction of* δ *to* $\mathcal{F}$ *is tree-like, and its tree hull* (*see Definition* 1.30) *is isomorphic as an* $\mathcal{F}$*-tree to* $\mathrm{Conv}(\overline{\mathcal{F}})$*.*

Proof. Assume that $\mathcal{F} \subset \mathcal{V}(\Gamma)$ satisfies the hypotheses of the theorem. Consider four pairwise distinct points $a, b, c, d \in \mathcal{F}$ and the convex hull $\mathrm{Conv}(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ of their images in the brick-vertex tree $\mathcal{BV}(\Gamma)$.

We will consider several cases according to the shape of this convex hull. In every case we will prove that in restriction to $\{a, b, c, d\}$, the metric δ satisfies the 4-point condition and that the shape of $\mathrm{Conv}(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ is determined by the cases of equality in the 4-point conditions and in the triangle inequalities associated to the four triples of points among a, b, c, and d (see the explanations following Proposition 1.29). Then, thanks to Proposition 1.29, we conclude that the tree hull of $(\{a, b, c, d\}, \delta)$ in the sense of Definition 1.30 is indeed isomorphic as an $\{a, b, c, d\}$-tree to the convex hull $\mathrm{Conv}(\overline{a}, \overline{b}, \overline{c}, \overline{d})$, finishing the proof of the proposition.

• **Assume that** $\mathrm{Conv}(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ **is H-shaped.** Denote by μ and ν the two 3-valent vertices of $\mathrm{Conv}(\overline{a}, \overline{b}, \overline{c}, \overline{d})$. We may assume, up to renaming the four points, that μ and ν separate $\overline{a}$ and $\overline{b}$ from $\overline{c}$ and $\overline{d}$, as illustrated in Figure 5. We claim that there exists then a cut-vertex p of Γ with the following properties:

(a) p separates both a and b from both c and d;
(b) either p does not separate a from b or it does not separate c from d.

In order to prove this, let us consider two cases:

(i) *One of the points* μ *and* ν *of* $\mathcal{BV}(\Gamma)$ *is a cut-vertex of* Γ*.* Assume for instance that $\mu = \overline{p}$, where $\overline{p}$ is a cut-vertex of $\mathcal{BV}(\Gamma)$. The convex hull $\mathrm{Conv}(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ having the shape illustrated in Figure 5, we see that p has the announced properties.

(ii) *Both points* μ *and* ν *of* $\mathcal{BV}(\Gamma)$ *are bricks of* Γ*.* By construction, all edges of $\mathcal{BV}(\Gamma)$ join either two vertices coming from Γ or a brick-vertex with a vertex coming from Γ. We deduce that there exists necessarily a cut-vertex $\overline{p}$ in the interior of the geodesic $[\mu\nu]$ of $\mathcal{BV}(\Gamma)$. Again, since the convex hull $\mathrm{Conv}(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ has the shape illustrated in Figure 5, we see that p has the announced properties.

Using the fact that p satisfies properties (a) and (b) above, the hypothesis that δ is a distance on $\mathcal{V}(\Gamma)$, and the characterization of the equality in the triangle inequality, we get

$$\begin{aligned}\delta(a,b) + \delta(c,d) &< (\delta(a,p) + \delta(b,p)) + (\delta(c,p) + \delta(d,p)) \\ &= (\delta(a,p) + \delta(c,p)) + (\delta(b,p) + \delta(d,p)) = \delta(a,c) + \delta(b,d) \\ &= (\delta(a,p) + \delta(d,p)) + (\delta(b,p) + \delta(c,p)) = \delta(a,d) + \delta(b,c).\end{aligned}$$

This shows that δ satisfies the 4-point condition in restriction to $\{a, b, c, d\}$ and that one has a strict inequality in this condition. In addition, one has by Proposition 1.36 and the hypothesis that there is no equality among the 4 triangle inequalities concerning triples of points among a, b, c, d.

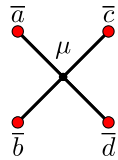

FIGURE 6. The case of an X-shaped tree in the proof of Theorem 1.38.

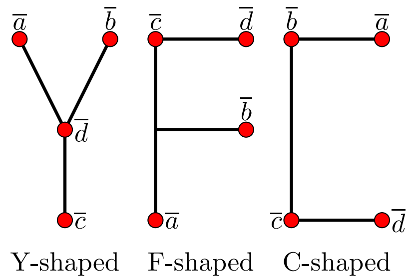

FIGURE 7. The Y-shaped, F-shaped, and C-shaped trees in the proof of Theorem 1.38.

• **Assume that** $\mathrm{Conv}(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ **is X-shaped.** Denote by μ the unique point of this graph that is of valency 4 (see Figure 6). By hypothesis, no brick of $\mathrm{Conv}(\mathcal{F})$ is of valency ≥ 4. Therefore, $\mu = \overline{p}$, where p is a separating vertex of Γ. Moreover, p separates pairwise the points a, b, c, d. Therefore

$$\begin{aligned}\delta(a,b) + \delta(c,d) &= (\delta(a,p) + \delta(b,p)) + (\delta(c,p) + \delta(d,p)) \\ &= (\delta(a,p) + \delta(c,p)) + (\delta(b,p) + \delta(d,p)) = \delta(a,c) + \delta(b,d) \\ &= (\delta(a,p) + \delta(d,p)) + (\delta(b,p) + \delta(c,p)) = \delta(a,d) + \delta(b,c).\end{aligned}$$

This shows again that δ satisfies the 4-point relation in restriction to $\{a, b, c, d\}$. As in the previous case, one has no equality among the 4 triangle inequalities concerning triples of points among a, b, c, d.

In the remaining cases we assume that $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$ are as in Figure 7.

• **Assume that** $\mathrm{Conv}(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ **is Y-shaped**. By Proposition 1.36, the point d separates simultaneously a from b, b from c, and a from c. Using this fact and the hypotheses of the theorem, we get that

$$\delta(a,b) + \delta(c,d) = \delta(a,c) + \delta(b,d) = \delta(a,d) + \delta(b,c) = \delta(a,d) + \delta(b,d) + \delta(c,d).$$

Thus the 4-point condition (14) is verified with equalities in this case. Reasoning as in the previous cases, one gets that the only equalities among the triangle inequalities are of the form $\delta(x,y) = \delta(x,d) + \delta(d,y)$ for $x, y \in \{a, b, c\}$, $x \neq y$.

• **Assume that** $\mathrm{Conv}(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ **is F-shaped.** By Proposition 1.36, we have that neither c nor d separates a from b, but c separates b from d and also c separates a from d. We obtain the following triangle (in)equalities:

$$\begin{aligned}\delta(a,b) &< \delta(a,c) + \delta(b,c), \qquad & \delta(a,b) &< \delta(a,d) + \delta(b,d), \\ \delta(b,d) &= \delta(b,c) + \delta(c,d), \qquad & \delta(a,d) &= \delta(a,c) + \delta(c,d).\end{aligned}$$

It is immediate to see from these relations that the 4-point condition (14) holds with a strict inequality, where the right hand side of (14) is equal to $\delta(a,c)+\delta(b,c)+\delta(c,d)$.

• **Assume that** $\mathrm{Conv}(\overline{a},\overline{b},\overline{c},\overline{d})$ **is C-shaped.** By Proposition 1.36, we have that b separates a from d, that b separates a from c, and that c separates b from d. The triangle inequalities become equalities in this case:

$$\delta(a,d)=\delta(a,b)+\delta(b,d),\quad \delta(a,c)=\delta(a,b)+\delta(b,c),\quad \text{and}\quad \delta(b,d)=\delta(b,c)+\delta(c,d).$$

It follows that the 4-point condition (14) holds with a strict inequality, where the right hand side (14) is equal to $\delta(a,b)+2\delta(b,c)+\delta(c,d)$. □

Example 1.39. Consider Figure 8. In the left picture, we have a graph Γ. Here $\mathcal{F}=\{a_1,\dots,a_{13}\}$ is depicted in light green. In this example, all the vertices in $\mathcal{F}$ are of valency 1 (which is not a hypothesis of Theorem 1.38). The cut-vertices are in red. Shaded areas correspond to bricks. Dark green shaded edges represent some of the bridges (the one whose endpoints are both cut points). In the right picture, we have represented the brick-vertex tree $\mathcal{BV}(\Gamma)$. The light green shaded subgraph is the set $\mathrm{Conv}(\mathcal{F})\subset\mathcal{BV}(\Gamma)$. Notice that there are four brick-vertices of $\mathcal{BV}(\Gamma)$ which have valency at least 4 (three of them have valency 4 and one of them has valency 5). But at those vertices the convex hull $\mathrm{Conv}(\mathcal{BV}(\mathcal{F}))$ has only valency 3. This convex hull also has two points of valency 4, but both of them are cut-vertices. Therefore, we have here a situation in which the hypothesis of Theorem 1.38 that each brick of Γ has $\mathrm{Conv}(\overline{\mathcal{F}})$-valency at most 3 is satisfied.

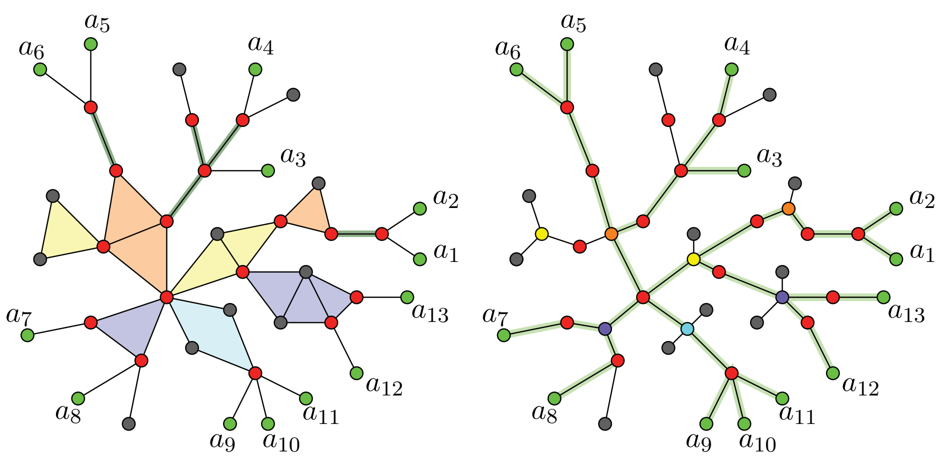

FIGURE 8. Example 1.39, in which the hypothesis of Theorem 1.38 about valencies of bricks is satisfied.

1.5. **Applications to finite sets of branches on normal surface singularities.** The main result of this subsection (Theorem 1.42) is the announced generalization to arbitrary normal surface singularities of the fact that u_L is an ultrametric on arborescent singularities (see Theorem 1.22). This generalization, stating that in general u_L is an ultrametric in restriction to special sets of branches describable topologically on any embedded resolution of their sum, is an immediate corollary of Theorem 1.38 of the previous subsection.

Applying Theorem 1.38 to the angular distance ρ, we get:

Corollary 1.40. *Let X be a normal surface singularity and let π be a good resolution of X. Consider a subset $\mathcal{F}$ of the set of vertices of the dual graph Γ_π and its convex hull* $\mathrm{Conv}(\mathcal{F})$ *in the brick-vertex tree $\mathcal{BV}(\Gamma_\pi)$ of Γ_π. If each brick of Γ_π has* $\mathrm{Conv}(\mathcal{F})$*-valency at most* 3*, then the restriction of ρ to $\mathcal{F}$ is tree-like and the associated tree is isomorphic as an $\mathcal{F}$-tree to* $\mathrm{Conv}(\mathcal{F})$*.*

In order to state the next results, it is convenient to introduce the following vocabulary.

Definition 1.41. If $\mathcal{F} \subset \mathcal{B}(X)$ is a finite set of branches on X, then an **injective resolution** of $\mathcal{F}$ is an embedded resolution of their sum such that different branches in $\mathcal{F}$ have different representing divisors (in the sense of Definition 1.9).

If π is an injective resolution of $\mathcal{F}$, then we have a canonical injection of $\mathcal{F}$ in $\mathcal{P}(\pi)$. We will identify sometimes $\mathcal{F}$ and its image, saying for instance that $\mathcal{F}$ is a subset of the set of vertices of Γ_π.

We deduce immediately from Corollaries 1.40 and 1.25 the following theorem.

Theorem 1.42. *Let X be a normal surface singularity. Consider a finite set $\mathcal{F}$ of branches on it and denote by L one of them. Let π be an injective resolution of the sum of branches in $\mathcal{F}$. Identify $\mathcal{F}$ with the set of prime divisors representing its elements. If each brick of Γ_π has* $\mathrm{Conv}(\mathcal{F})$*-valency at most* 3*, then the function $u_L : (\mathcal{F} \setminus \{L\})^2 \to [0, \infty)$ is an ultrametric and the associated rooted $\mathcal{F}$-tree is isomorphic to* $\mathrm{Conv}(\mathcal{F})$*.*

Note that Theorem 1.22 is indeed a special case of Theorem 1.42. This is a consequence of the fact that for arborescent singularities, Γ_π has no bricks.

Remark 1.43. The rooted tree associated to u_L in Theorem 1.42 is *end-rooted* in the sense of [19, Definition 3.5]; that is, its root is of valency 1. It corresponds to a supplementary element associated to the set of closed balls of the ultrametric, which may be thought of as a ball of infinite radius. The approach of the paper [19] was to work exclusively with rooted trees associated to ultrametrics. By contrast, in the present paper our trees are associated to metrics satisfying the 4-point condition (see Definition 1.26); therefore they are not canonically rooted. One may translate one approach into the other one using Proposition 1.23.

An important aspect of Theorem 1.42 is that *it depends only on the topology of the total transform of the branches on an embedded resolution of their sum* and not on special properties of the values of the intersection numbers of the prime exceptional divisors nor on their genera.

Example 1.44. The condition on the valency of brick-points in Theorem 1.42 (and of analogous theorems like Theorem 2.53) is not necessary in general. For example, consider a singularity X whose minimal good resolution has a tetrahedral dual graph. Denote by E_1, E_2, E_3, E_4 the exceptional primes, and assume that they all have the same self-intersection $-k$, where $k \geq 4$. By symmetry, $\check{E}_i \cdot \check{E}_j$ is constant for any $1 \leq i \neq j \leq 4$. The brick-vertex tree has here a brick-vertex of valency 4, but the 4-point condition is satisfied. See Examples 2.55 and 2.56 for a deeper analysis of this example.

1.6. **An ultrametric characterization of arborescent singularities.** The aim of this subsection is to prove a converse to Theorem 1.22. Namely, we prove that if u_L is an ultrametric for every branch L on X, then X is arborescent (see Theorem 1.46).

In the next proposition we show that if the normal surface singularity is not arborescent, then one may find four branches on it such that for any one of them, called L, the associated function u_L is not an ultrametric on the set of the remaining three branches (even if the proposition is not stated like this, the fact that its conclusion may be formulated in this way is a consequence of Proposition 1.23).

Proposition 1.45. *Let X_π be a good model of X. Assume that a, b, m, p are four pairwise distinct vertices of the dual graph Γ_π such that:*

- *both m and p are adjacent to a;*
- *a does not separate b from either m or p.*

Denote by x_m the intersection point of E_a and E_m and by x_p the intersection point of E_a and E_p. Let A and B be branches on X whose representing divisors on X_π are E_a and E_b, respectively. Then there exist branches C_m and C_p whose strict transforms on X_π pass through x_m and x_p, respectively, such that:

$$(A \cdot B)(C_m \cdot C_p) < (C_m \cdot A)(C_p \cdot B) < (C_m \cdot B)(C_p \cdot A). \tag{16}$$

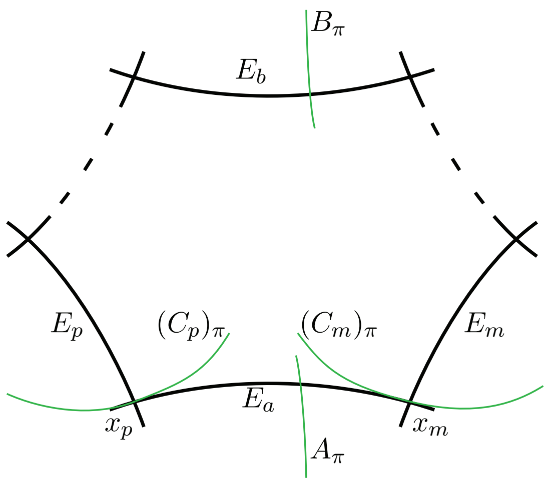

FIGURE 9. Geometric situation of Proposition 1.45.

Proof. Consider a branch C_m whose strict transform $(C_m)_\pi$ passes through the point x_m and is smooth and tangent to the prime exceptional divisor E_a. Denote by $s \in \mathbb{N}^*$ the intersection number $(C_m)_\pi \cdot E_a$. As $(C_m)_\pi \cdot E_m = 1$ and the intersection numbers of $(C_m)_\pi$ with the other irreducible components of the exceptional divisor of π are all 0, we deduce that

$$(C_m)_\pi^{ex} = -\check{E}_m - s\check{E}_a.$$

Consider an analogous branch C_p whose strict transform passes through x_p and such that one has $(C_p)_\pi \cdot E_a = t \in \mathbb{N}^*$. One gets

$$(C_p)_\pi^{ex} = -\check{E}_p - t\check{E}_a.$$

See Figure 9 for the relative positions of prime exceptional divisors and strict transforms of branches.

As the strict transforms $(C_m)_\pi$ and $(C_p)_\pi$ are disjoint, Corollary 1.12 implies that

$$C_m \cdot C_p = -(C_m)^{ex}_\pi \cdot (C_p)^{ex}_\pi.$$

We use the analogous equalities for the other intersection numbers appearing in (16) (in each case, the strict transforms of the corresponding branches by the modification π are again disjoint). As $A^{ex}_\pi = -\check{E}_a$ and $B^{ex}_\pi = -\check{E}_b$, the system of inequalities (16) becomes

$$\begin{aligned} \langle a,b\rangle \cdot (\langle m,p\rangle + t\langle m,a\rangle + s\langle a,p\rangle + ts\langle a,a\rangle) \\ < (\langle m,a\rangle + s\langle a,a\rangle)(\langle p,b\rangle + t\langle a,b\rangle) \\ < (\langle m,b\rangle + s\langle a,b\rangle)(\langle p,a\rangle + t\langle a,a\rangle). \end{aligned} \tag{17}$$

We want to show that we may find pairs $(s,t) \in \mathbb{N}^* \times \mathbb{N}^*$ such that (17) holds. Let us consider in turn both inequalities.

- The left-hand inequality in (17) becomes

$$(\langle a,a\rangle\langle b,p\rangle - \langle a,b\rangle\langle a,p\rangle)s + (\langle a,m\rangle\langle b,p\rangle - \langle a,b\rangle\langle m,p\rangle) > 0. \tag{18}$$

Note that the left-hand side of (18) is a polynomial of degree 1 in the variable s. By Proposition 1.19 and the hypothesis that a does not separate b from p in the dual graph of π, the coefficient $\langle a,a\rangle\langle b,p\rangle - \langle a,b\rangle\langle a,p\rangle$ of s is positive. Therefore, the inequality (18) becomes true for s big enough.

- Similarly, the right-hand inequality of (17) becomes

$$\begin{aligned} (\langle a,a\rangle\langle b,m\rangle - \langle a,b\rangle\langle a,m\rangle)t - (\langle a,a\rangle\langle b,p\rangle - \langle a,b\rangle\langle a,p\rangle)s + \langle a,p\rangle\langle b,m\rangle \\ - \langle a,m\rangle\langle b,p\rangle > 0. \end{aligned} \tag{19}$$

Assume that s was chosen such that (18) holds. The left-hand side of (19) is then a polynomial of degree 1 in the variable t. Its dominating coefficient $\langle a,a\rangle\langle b,m\rangle - \langle a,b\rangle\langle a,m\rangle$ is > 0, because a does not separate b from m. Therefore, the inequality (19) becomes true for t big enough. □

We get the announced characterization of arborescent singularities.

Theorem 1.46. *Let X be a normal surface singularity. Then the following properties are equivalent:*

(1) *For every branch $L \in \mathcal{B}(X)$, the function u_L is an ultrametric on the set $\mathcal{B}(X) \setminus \{L\}$.*
(2) *There exists a branch $L \in \mathcal{B}(X)$ such that the function u_L is an ultrametric on the set $\mathcal{B}(X) \setminus \{L\}$.*
(3) *The bracket $\langle \cdot,\cdot \rangle$ satisfies the following inequality:*

$$\langle a,b\rangle \cdot \langle l,c\rangle \geq \min\{\langle a,c\rangle \cdot \langle l,b\rangle,\ \langle b,c\rangle \cdot \langle l,a\rangle\} \quad \textit{for all } (a,b,c,l) \in (\mathcal{P}(X))^4.$$

(4) *The singularity X is arborescent.*

Proof. The equivalences (1) $\iff$ (2) $\iff$ (3) are direct consequences of Corollary 1.25.

The implication (4) $\implies$ (1) is a direct consequence of Theorem 1.22.

In order to prove the implication (2) $\implies$ (4) we proceed by contradiction and suppose that X is not arborescent. We will show that for every choice of branch L, there exist branches A, C_m, C_p such that the quadruple L, A, C_m, C_p does not satisfy the 4-point condition. Fix a good model X_π of X, which is an embedded resolution of the branch L. Denote by E_l the exceptional prime representing L in X_π, and look at l as a vertex in the dual graph Γ_π of π. By Proposition 1.45, it suffices to find three vertices a, m, p in Γ_π such that m and p are adjacent to a and a does not separate l from either m or p.

As X is not arborescent, the dual graph Γ_π contains a cycle Θ. Replacing perhaps X_π by another model obtained from it by blowing up points of the divisor represented by Θ, we may assume that Θ has at least four vertices. If l is a vertex of Θ we take a, m, p, three other successive vertices of Θ, and apply Proposition 1.45. Otherwise, l does not belong to Θ. As Γ_π is connected, there exists a path Π inside it connecting l to a vertex d of Θ such that d is the only vertex common to Θ and to this path. As Θ has at least four vertices, one may find three successive vertices m, a, p of it which are different from d. Then the vertices a, m, and p satisfy the condition we were looking for. $\square$

2. Ultrametric distances on valuation spaces

In this second part of the paper, we generalize the results of Section 1 to the setting of valuation spaces. We keep denoting by (X, x_0) a normal surface singularity and by $\mathcal{O}_X$ its local ring. We denote by R the completion $\hat{\mathcal{O}}_X$ of its local ring relative to its maximal ideal and by $\mathfrak{m}$ the unique maximal ideal of R.

2.1. Semivaluation spaces of normal surface singularities.

In this section we recall the definitions of *semivaluations* and *valuations* of X, as well as that of *normalized* such objects. Then we recall the classification of semivaluations into *divisorial*, *quasi-monomial* (in particular *irrational*), *curve*, and *infinitely singular*.

Let $[0, +\infty]$ be the union of the set of non-negative real numbers and of the single-element set $\{+\infty\}$, endowed with the usual total order. In this paper we will consider the following notion of semivaluation.

Definition 2.1. A **semivaluation** on X (or on R) is a function $\nu \colon R \to [0, +\infty]$ satisfying the following axioms:

(1) $\nu(0) = +\infty$ and $\nu(1) = 0$;
(2) $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$ for all $\phi, \psi \in R$;
(3) $\nu(\phi + \psi) \geq \min\{\nu(\phi), \nu(\psi)\}$ for all $\phi, \psi \in R$;
(4) $0 < \nu(\mathfrak{m}) < +\infty$,

where $\nu(\mathfrak{m}) := \min\{\nu(\phi) \mid \phi \in \mathfrak{m}\}$. The semivaluation ν is **normalized** if in addition $\nu(\mathfrak{m}) = 1$. The semivaluation ν is a **valuation** if $\nu^{-1}(+\infty) = \{0\}$. The set of semivaluations on X will be denoted by $\boxed{\hat{\mathcal{V}}_X^*}$, while the set of normalized semivaluations will be denoted by $\boxed{\mathcal{V}_X}$.

Remark 2.2. There are more general notions of semivaluations which do not require the condition (4) on Definition 2.1 or which take values on the non-negative part of the additive semigroup $\mathbb{R}^2$, with respect to the lexicographical ordering. In the literature, the semivaluations of Definition 2.1 are usually called *centered* (which makes reference to the condition $\nu(\mathfrak{m}) > 0$), *finite* (meaning that $\nu(\mathfrak{m}) < +\infty$), and of *rank* 1 (since they take values on the non-negative part of $(\mathbb{R}, +)$).

If ν is a semivaluation on X, so is $\lambda\nu$ for any $\lambda \in \mathbb{R}_+^* := (0, +\infty)$. In particular, any semivaluation is proportional to a normalized one.

Remark 2.3. The normalization with respect to the maximal ideal is not the only possible one. It is sometimes useful to normalize with respect to other ideals of R. A typical choice (see [14,15] for the smooth setting) is to normalize with respect to the value taken on a given irreducible element x of R, that is, by considering only semivaluations which satisfy $\nu(x) = 1$. In this case special care must be taken for the curve semivaluation ν_C with $C = \{x = 0\}$, since $\mathrm{int}_C(x) = +\infty$ (see below for the definitions of ν_C and int_C).

If $\mathfrak{a}$ is an ideal of R, we denote $\nu(\mathfrak{a}) := \min\{\nu(\phi) \,|\, \phi \in \mathfrak{a}\}$ for any semivaluation ν. One may define equivalently a semivaluation ν as a function on the set of ideals of R satisfying similar properties as those in Definition 2.1 (see [21]).

Note that for any semivaluation ν, the set $\nu^{-1}(+\infty)$ is a prime ideal of R. Therefore, it defines either the point x_0 or a branch on X.

Definition 2.4. The **support** of a semivaluation of R is the vanishing locus of the prime ideal $\nu^{-1}(+\infty)$.

The spaces $\hat{\mathcal{V}}_X^*$ and $\mathcal{V}_X$ come equipped with natural topologies.

Definition 2.5. The **weak topologies** on the sets $\hat{\mathcal{V}}_X^*$ and $\mathcal{V}_X$ are the weakest ones such that the maps $\nu \mapsto \nu(\phi)$ are continuous for any $\phi \in R$.

In the foundational work [57], Zariski gave a classification of semivaluations according to some algebraic invariants (*rank*, *rational rank*, *transcendence degree*). Those different kinds of semivaluations can also be characterized by their geometric properties. We recall here a few facts about this classification in our setting.

• **Divisorial valuations**. They are the valuations associated to the prime exceptional divisors, as seen in section 1.1. Let X_π be a good model of X, and let $E \in \mathcal{P}(\pi)$ be any irreducible (and reduced) component of the exceptional divisor $\pi^{-1}(x_0)$. Then the map $\boxed{\mathrm{div}_E}$, which associates to a function $\phi \in R$ the order of vanishing of $\phi \circ \pi$ along E, defines a valuation of X. We say that a valuation is **divisorial** if it is of the form $\lambda\, \mathrm{div}_E$, with $\lambda \in \mathbb{R}_+^*$. When $\lambda = 1$, the divisorial valuation is called **prime**, a denomination already used in section 1. For any exceptional prime $E \in \mathcal{P}(\pi)$, we denote by $\boxed{\nu_E} := b_E^{-1}\mathrm{div}_E$ the normalized valuation proportional to div_E, where $\boxed{b_E} := \mathrm{div}_E(\mathfrak{m}) \in \mathbb{N}^*$ is the **generic multiplicity** of ν_E. Finally, for any good model X_π of X, we denote by $\boxed{\mathcal{S}_\pi^*}$ the set of normalized divisorial valuations associated to the primes of π.

• **Quasi-monomial and irrational valuations**. Quasi-monomial valuations of X are constructed as follows. Let X_π be a good model of X, and let $P \in E(\pi)$ be any point in the exceptional divisor $E(\pi)$ of π. Pick local coordinates (x, y) at P adapted to $E(\pi)$ (i.e., so that $E(\pi) \subseteq \{xy = 0\}$ locally at P). For any $(r, s) \in (\mathbb{R}_+^*)^2$, we may consider the monomial valuation $\mu_{r,s}$ on the local ring of X_π at P, defined on the set of monomials in x and y by setting $\mu_{r,s}(x) = r$ and $\mu_{r,s}(y) = s$ and extended to any element ϕ of this ring by taking the minimum of $\mu_{r,s}$ on the set of monomials appearing in ϕ. The valuation $\nu_{r,s}$ defined by $\boxed{\nu_{r,s}} := \pi_*\mu_{r,s} : \phi \mapsto \mu_{r,s}(\phi \circ \pi)$ is an element of $\hat{\mathcal{V}}_X^*$, called a **quasi-monomial valuation**. If r and s are rationally dependent, it turns out that $\nu_{r,s}$ is a divisorial valuation (associated to an exceptional prime obtained after a toric modification of X_π in the coordinates (x, y)). If r and s are rationally independent, we call the valuation $\nu_{r,s}$ an **irrational valuation**. Notice that we can also define $\nu_{r,s}$ when either r or s vanishes. For example, suppose that $E(\pi) = \{x = 0\} = E$ locally at P. Then the valuation $\nu_{1,0}$ coincides with div_E, while $\nu_{0,1}$ is not a centered valuation: it would correspond up to a multiplicative constant to the order of vanishing along the branch determined by the projection of $\{y = 0\}$ to X.

• **Curve semivaluations**. They are the semivaluations associated to branches in $\mathcal{B}(X)$. Given such a branch L, a **curve semivaluation associated to** L is any positive real multiple of $\boxed{\mathrm{int}_L}$, which in turn is defined by $\mathrm{int}_L(\phi) := L \cdot (\phi)$, where $\phi \in R$ and (ϕ) denotes the divisor of ϕ. As for divisorial valuations, we denote by $\boxed{\nu_L} := m(L)^{-1}\mathrm{int}_L$ the normalized semivaluation proportional to int_L, where $m(L) \in \mathbb{N}^*$ is the multiplicity of L. Notice that curve semivaluations are never valuations, since $\mathrm{int}_L(\phi) = +\infty$ for any $\phi \in R$ vanishing on L. In fact, the support of int_L according to Definition 2.4 is exactly L.

• **Infinitely singular valuations**. These are the remaining elements of $\hat{\mathcal{V}}_X^*$. They are characterized by having rank and rational rank equal to 1 and transcendence degree equal to 0. They are also characterized as valuations whose value group is not finitely generated over $\mathbb{Z}$. They can be thought of as curve semivaluations associated to branches of infinite multiplicity (see [14, Chapter 4]).

Definition 2.6. Given a good model X_π, we denote by $\boxed{\mathcal{S}_\pi}$ the set of centered normalized quasi-monomial valuations described above, for all the points $p \in \pi^{-1}(x_0)$, and call it the **skeleton of** X_π.

Notice that $\mathcal{S}_\pi$ admits a structure of finite connected graph, with set of vertices $\mathcal{S}_\pi^*$ and edges between two points ν_E and ν_F for each intersection point between E and F in $\pi^{-1}(x_0)$. This graph is homeomorphic to the dual graph Γ_π of π introduced in Definition 1.17.

Remark 2.7. In section 1, we considered only divisorial valuations. Given such a valuation u, we denoted by E_u the exceptional prime associated to it. Since here we consider other types of valuations not associated to exceptional primes, we prefer to denote by $\nu \in \mathcal{V}_X$ any kind of valuation and write $\nu = \nu_E$ if ν is the divisorial valuation associated to the exceptional prime E.

2.2. **Valuation spaces as projective limits of dual graphs.** The aim of this section is to explain some basic relations between dual graphs, skeleta, and the valuation space.

Let $\pi\colon X_\pi \to X$ be a good resolution of the normal surface singularity X and let $\nu \in \hat{\mathcal{V}}_X^*$ be a semivaluation of X. By the valuative criterion of properness, ν has a unique **center** in X_π, which lies in the exceptional divisor of π. The center is characterized as the unique scheme-theoretic point $\xi \in X_\pi$ so that ν takes nonnegative values on the local ring $\mathcal{O}_{X_\pi,\xi}$ of elements of the fraction field of R whose pull-backs to X_π are regular at ξ, and strictly positive values exactly on its maximal ideal $\mathfrak{m}_\xi$.

Then one can define as follows a retraction r_π from $\mathcal{V}_X$ to the skeleton $\mathcal{S}_\pi$ of the good model X_π (see Definition 2.6). Let $\nu \in \mathcal{V}_X$ be a normalized semivaluation, and let $\xi \in \pi^{-1}(x_0)$ be its center. If ξ is the generic point of an exceptional prime E or if it is a closed point belonging to a unique exceptional prime E of $\mathcal{P}(\pi)$, then we set $r_\pi(\nu) := \nu_E$, the divisorial valuation associated to E. If ξ is a closed point P belonging to the intersection of two exceptional primes E and F, then $\nu = \pi_*\mu$ where μ is a semivaluation centered at P. Pick local coordinates (x, y) at P so that $E = \{x = 0\}$ and $F = \{y = 0\}$. Then we set $r_\pi(\nu)$ to be the quasi-monomial valuation $\pi_*\mu_{r,s}$ at P with weights $r = \mu(x)$ and $s = \mu(y)$. By a result of Thuillier's paper [48], the map $r_\pi : \mathcal{V}_X \to \mathcal{S}_\pi$ is in fact a strong deformation retract.

If $\pi' : X_{\pi'} \to (X, x_0)$ is another good resolution dominating π, then we have $r_\pi = r_\pi \circ r'_\pi$. Hence we get a natural continuous map from the valuation space $\mathcal{V}_X$ to the projective limit $\varprojlim_\pi \mathcal{S}_\pi$ of the skeleta, which turns out to be a homeomorphism (see [51, Theorem 7.5] and [13, p. 399]). This approach can be taken in order to construct the valuation space $\mathcal{V}_X$ directly as the projective limit of the dual graphs of the good resolutions of (X, x_0).

In particular, we can characterize arborescent singularities as the normal surface singularities X for which the valuation space $\mathcal{V}_X$ is contractible. Indeed, if X is arborescent, then the dual graph of each good resolution π is a tree; hence $\mathcal{S}_\pi$ is contractible, and so is $\mathcal{V}_X$ that deformation retracts onto it. Similarly, if X is not arborescent, then we can find a non-trivial loop on the dual graph of a good resolution π, and its image inside $\mathcal{S}_\pi \subset \mathcal{V}_X$ gives a non-trivial loop inside $\mathcal{V}_X$.

2.3. **B-divisors on normal surface singularities.** In the first part of the paper, it was crucial to associate a dual to any prime divisor on a model of X. By looking at the divisor as a prime divisorial valuation and by collecting its associated dual divisors on all the models, one gets a particular b-divisor, in the sense of Definition 2.11. In this section we explain how to extend the previous construction to all semivaluations on X (see Definition 2.10). As an application, we show how to extend to the space of normalized semivaluations the notions of bracket (see Definition 2.12) and of angular distance (see Definition 2.15).

Let $\nu \in \hat{\mathcal{V}}_X^*$. One may define unambiguously the value $\nu(D)$ taken by ν on any divisor $D \in \mathcal{E}(\pi)_\mathbb{R}$ (see for instance [29, section 7.5.2] for the case where R is regular, which extends without changes to our case, or [21, section 2.2]). The idea is to define first $\nu(D)$ when D is prime by evaluating ν on a local defining function of D and to extend it then by linearity. Such local defining functions may be taken as pull-backs of elements of the localization of R at the defining prime ideal $\nu^{-1}(+\infty)$ of the support of ν, to which ν extends canonically.

Any semivaluation on X induces a dual divisor on X_π, according to the next proposition (see [13, p. 400] or [21, Proposition 2.5]):

Proposition 2.8. *For any semivaluation $\nu \in \hat{\mathcal{V}}_X^*$, there exists a unique divisor $Z_\pi(\nu) \in \mathcal{E}(\pi)_\mathbb{R}$ such that $\nu(D) = Z_\pi(\nu) \cdot D$ for each $D \in \mathcal{E}(\pi)_\mathbb{R}$.*

We will use the following name for this divisor.

Definition 2.9. The divisor $\boxed{Z_\pi(\nu)}$ characterized in Proposition 2.8 is called the **dual divisor** of ν in the model X_π.

The name alludes to the fact that for a divisorial valuation div_E, we have $Z_\pi(\mathrm{div}_E) = \check{E}$. Here $\check{E}$ denotes the dual divisor of E, as defined by relations (2).

Definition 2.10. The collection $\boxed{Z(\nu)} = (Z_\pi(\nu))_\pi$, where π varies among all good resolutions of X, is called the **b-divisor associated to** ν.

This name is motivated by the fact that $Z(\nu)$ is a b-divisor in the following sense, due to Shokurov [47] (the letter "b" is the initial of "birational").

Definition 2.11. A collection $(Z_\pi)_\pi$, where π varies among all good resolutions of X and $Z_\pi \in \mathcal{E}(\pi)_\mathbb{R}$, is called a **b-divisor** of X if for any pair of models (π, π') such that π' dominates π, one has $\psi_* Z_{\pi'} = Z_\pi$ if $\pi' = \pi \circ \psi$.

In section 1, we noticed that the intersection of two dual divisors does not depend on the model used to compute it (see Proposition 1.5). This allows us to define the intersection number $Z(\nu) \cdot Z(\mu)$ of two b-divisors associated to divisorial valuations $\nu, \mu \in \hat{\mathcal{V}}_X^*$. In the general case of an arbitrary pair of semivaluations (ν, μ) of X, the intersection number $Z_\pi(\nu) \cdot Z_\pi(\mu)$ *may depend* on the model π. In fact, we always have $Z_{\pi'}(\nu) \cdot Z_{\pi'}(\mu) \leq Z_\pi(\nu) \cdot Z_\pi(\mu)$, for any model π' dominating π. More precisely, the intersection remains constant as far as ν and μ have different centers in X_π (see [21, Proposition 2.13]), while it decreases if the centers coincide (see [21, Proposition 2.17]). This allows us to define

$$\boxed{Z(\nu) \cdot Z(\mu)} := \inf_\pi \big(Z_\pi(\nu) \cdot Z_\pi(\mu)\big) \in [-\infty, 0).$$

We refer to [5,13,21] for further details on b-divisors associated to semivaluations.

Recall that in Definition 1.6 we introduced the *bracket* of two prime divisorial valuations. The next definition extends the bracket to arbitrary pairs of semivaluations.

Definition 2.12. Let $\nu, \mu \in \hat{\mathcal{V}}_X^*$ be two semivaluations of X. Their **bracket** is defined by

$$\boxed{\langle \nu, \mu \rangle} := -Z(\nu) \cdot Z(\mu) \in (0, +\infty].$$

When $\nu = \mu$, the self-bracket $\boxed{\alpha(\nu)} := \langle \nu, \nu \rangle$ is called the **skewness** of ν.

Remark 2.13. The skewness $\alpha(\nu)$ has been analyzed for germs of smooth surfaces in [14], where it was defined as the supremum of the ratio between the values of ν and of the multiplicity function. With this interpretation, the skewness is sometimes called the *Izumi constant* of ν, a denomination which refers to the works [27, 28] of Izumi. Its study has been the focus of several works; see e.g. [6, 10, 37, 45, 46]. The b-divisor interpretation given by Favre and Jonsson is more recent, and it has been used to study several properties of valuation spaces for smooth and singular surfaces (see e.g. [21, 29]).

Let us consider now the restriction of the bracket to the space $\mathcal{V}_X$ of normalized semivaluations. The skewness is always finite for quasi-monomial valuations, while it is always infinite for curve semivaluations. It can be any value in $(0, +\infty]$ for infinitely singular valuations (see [14, Theorem 3.26] for the smooth case and [21, Proposition 2.17] for the singular case). We denote by $\boxed{\mathcal{V}_X^\alpha}$ the set of normalized valuations with finite skewness.

More generally, one can show (see [21, Proposition 2.13]) that $\langle \nu, \mu \rangle$ is determined on a model X_π; i.e., $\langle \nu, \mu \rangle = -Z_\pi(\nu) \cdot Z_\pi(\mu)$ as far as ν and μ have different centers on X_π. Since for two distinct normalized semivaluations, there is always a model on which their centers are disjoint, we deduce that:

Proposition 2.14. *The bracket of two distinct normalized semivaluations is always finite.*

Carrying on the analogies with the divisorial case of Section 1, we define the notion of angular distance of semivaluations, as introduced in [21].

Definition 2.15. The **angular distance** of two normalized semivaluations $\mu, \nu \in \mathcal{V}_X$ is

$$\boxed{\rho(\nu, \mu)} := -\log \frac{\langle \nu, \mu \rangle^2}{\alpha(\nu) \cdot \alpha(\mu)} \in [0, \infty] \tag{20}$$

if $\nu \neq \mu$, and 0 if $\nu = \mu$.

Remark 2.16. The function ρ defines an **extended distance** on $\mathcal{V}_X$ (see [21, Proposition 2.40]), in the sense that it vanishes exactly on the diagonal, it is symmetric, and it satisfies the triangular inequality (like a standard distance), but it may take the value $+\infty$ in some cases. In fact, $\rho(\nu, \mu) = +\infty$ exactly when $\nu \neq \mu$ and at least one of the semivaluations ν and μ has infinite skewness. This locus can be precisely determined by reducing first to the smooth case using [21, Lemma 2.43] and by describing then the skewness of a semivaluation in terms of its *Puiseux parameterization*, as in [14, Chapter 4] (when one works over $\mathbb{C}$) or using Jonsson's approach in [29, section 7] (when one works over an arbitrary field, possibly of positive characteristic). In particular, ρ defines a distance on $\mathcal{V}_X^{\alpha}$, hence on the set of normalized quasi-monomial valuations. The topology induced by ρ on $\mathcal{V}_X$ is usually called the **strong topology** in order to distinguish it from the weak topology introduced in Definition 2.5.

2.4. **Ultrametric distances on semivaluation spaces of arborescent singularities.** In Subsection 1.3 we started the study of the function u_L that culminated with the characterization of arborescent singularities given in Theorem 1.46. This section is devoted to the proof of an analog for semivaluation spaces (see Theorem 2.19). We will study functions u_λ depending on an arbitrary semivaluation $\lambda \in \mathcal{V}_X$, defined on $\mathcal{V}_X \times \mathcal{V}_X$. In the particular case in which λ is the curve semivaluation int_L associated to a branch L on X, we get $u_{\mathrm{int}_L} = u_L$ (see Remark 2.18).

Definition 2.17. Let X be a normal surface singularity, and let $\lambda \in \hat{\mathcal{V}}_X^*$ be any semivaluation. Let $\nu_1, \nu_2 \in \mathcal{V}_X$ be any normalized semivaluations on X. We set

$$\boxed{u_\lambda(\nu_1, \nu_2)} := \begin{cases} \dfrac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle}{\langle \nu_1, \nu_2 \rangle} & \text{if } \nu_1 \neq \nu_2, \\ 0 & \text{if } \nu_1 = \nu_2. \end{cases} \tag{21}$$

Remark 2.18. Since $\langle \nu_1, \nu_2 \rangle < +\infty$ when $\nu_1 \neq \nu_2$ (see Proposition 2.14), the function u_λ is well defined with values in $[0, +\infty]$, and it vanishes if and only if $\nu_1 = \nu_2$. The value $+\infty$ is sometimes achieved. In fact, while the denominator is always strictly positive, if λ is normalized we have $\langle \lambda, \nu \rangle = +\infty$ if and only if $\lambda = \nu$ and $\alpha(\lambda) = +\infty$. In particular, u_λ takes only finite values if $\alpha(\lambda) < +\infty$, while it always takes finite values on $(\mathcal{V}_X \setminus \{\lambda\})^2$.

Notice that if ν_1 and ν_2 tend to the same semivaluation ν in the strong topology, then $\frac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle}{\langle \nu_1, \nu_2 \rangle}$ tends to $\frac{\langle \lambda, \nu \rangle^2}{\alpha(\nu)}$. This value is finite as long as $\nu \neq \lambda$, and it is 0 if and only if $\alpha(\nu) = +\infty$. This always happens when ν is a curve semivaluation and never happens for quasi-monomial valuations.

Notice also that u_λ can be extended to $(\hat{\mathcal{V}}_X^*)^2$, setting $u_\lambda(\nu_1, \nu_2) := \frac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle}{\langle \nu_1, \nu_2 \rangle}$ if ν_1 and ν_2 are non-proportional, and equal to zero otherwise. In fact, by homogeneity of the bracket, we have $u_\lambda(b_1 \nu_1, b_2 \nu_2) = u_\lambda(\nu_1, \nu_2)$ for any $b_1, b_2 \in (0, +\infty)$ and also $u_{b\lambda} = b^2 u_\lambda$ for any $b \in (0, +\infty)$.

Finally, Definition 2.17 clearly generalizes (11). In fact, if L, A, B are branches on X, then $u_L(A, B) = u_{\mathrm{int}_L}(\mathrm{int}_A, \mathrm{int}_B)$, where $\mathrm{int}_L, \mathrm{int}_A, \mathrm{int}_B$ are the curve semivaluations associated to L, A, B, respectively.

The aim of this subsection is to prove the following generalization of Theorem 1.46.

Theorem 2.19. *Let X be a normal surface singularity. Then the following properties are equivalent:*

(1) *For every semivaluation $\lambda \in \hat{\mathcal{V}}_X^*$, the function u_λ is an extended ultrametric distance on $\mathcal{V}_X$.*
(2) *There exists a semivaluation $\lambda \in \hat{\mathcal{V}}_X^*$ such that the function u_λ is an extended ultrametric distance on $\mathcal{V}_X$.*
(3) *The singularity X is arborescent.*

Before starting the proof, let us give some definitions and preliminary results, analogous to those described in section 1.

Definition 2.20. Let X be a normal surface singularity, and let $\mu, \nu_1, \nu_2 \in \mathcal{V}_X$ be three normalized semivaluations. We say that μ **separates** ν_1 **and** ν_2 (or **the couple** (ν_1, ν_2)) if either $\mu \in \{\nu_1, \nu_2\}$ or ν_1 and ν_2 belong to different connected components of $\mathcal{V}_X \setminus \{\mu\}$.

Notice that in the previous definition we can consider $\mathcal{V}_X$ endowed indifferently with either the weak or the strong topology, since the connected components of $\mathcal{V}_X \setminus \{\mu\}$ are the same for the two topologies.

Proposition 2.21 ([21, Proposition 2.15])**.** *Let X be a normal surface singularity and let $\mu, \nu_1, \nu_2 \in \mathcal{V}_X$ be three normalized semivaluations. Then we have*

$$\langle \mu, \nu_1 \rangle \cdot \langle \mu, \nu_2 \rangle \leq \langle \mu, \mu \rangle \cdot \langle \nu_1, \nu_2 \rangle. \tag{22}$$

Moreover, the equality holds if and only if μ separates ν_1 and ν_2.

Notice that, by homogeneity, Proposition 2.21 holds also for non-normalized valuations.

Proposition 2.22. *Let X be a normal surface singularity, and let $\nu_j \in \mathcal{V}_X$, for $j = 1, \dots, 4$, be four normalized semivaluations. Suppose that there exists $\mu \in \mathcal{V}_X$ that separates simultaneously the couple (ν_1, ν_2) and the couple (ν_3, ν_4). Then*

$$\langle \nu_1, \nu_2 \rangle \cdot \langle \nu_3, \nu_4 \rangle \leq \langle \nu_1, \nu_3 \rangle \cdot \langle \nu_2, \nu_4 \rangle. \tag{23}$$

Moreover, the equality in (23) *holds if and only if μ also separates simultaneously the couple (ν_1, ν_3) and the couple (ν_2, ν_4).*

Proof. **Suppose first that $\alpha(\mu) = +\infty$.** In this case, μ is necessarily an end of $\mathcal{V}_X$, i.e., $\mathcal{V}_X \setminus \{\mu\}$ is connected. It follows that, up to permuting the roles of ν_1, ν_2 and of ν_3, ν_4, we have either $\nu_1 = \nu_3 = \mu$ or $\nu_1 = \nu_4 = \mu$.

In the first case, if either ν_2 or ν_4 coincides with μ, then both sides of (23) are $+\infty$, and we have equality, in agreement with the statement. If both ν_2 and ν_4 differ from μ, the left hand side of (23) is finite, while the right hand side is $+\infty$, again in agreement with the statement, since μ does not separate ν_2 and ν_4.

In the second case, the left and right hand sides of (23) coincide, and in fact μ separates also the couple (ν_1, ν_3) and (ν_2, ν_4).

Suppose now that $\alpha(\nu) < +\infty$. By Proposition 2.21, we have

$$\langle \mu, \nu_1 \rangle \cdot \langle \mu, \nu_3 \rangle \leq \langle \mu, \mu \rangle \cdot \langle \nu_1, \nu_3 \rangle, \tag{24}$$

$$\langle \mu, \nu_2 \rangle \cdot \langle \mu, \nu_4 \rangle \leq \langle \mu, \mu \rangle \cdot \langle \nu_2, \nu_4 \rangle. \tag{25}$$

We want to prove the inequality

$$\langle \nu_1, \nu_2 \rangle \cdot \langle \nu_3, \nu_4 \rangle \cdot \langle \mu, \mu \rangle \leq \langle \mu, \nu_2 \rangle \cdot \langle \mu, \nu_4 \rangle \cdot \langle \nu_1, \nu_3 \rangle, \tag{26}$$

which implies the statement (23) by applying (25). Now, again by Proposition 2.21, we have

$$\langle \mu, \nu_1 \rangle \cdot \langle \mu, \nu_2 \rangle = \langle \mu, \mu \rangle \cdot \langle \nu_1, \nu_2 \rangle, \tag{27}$$

$$\langle \mu, \nu_3 \rangle \cdot \langle \mu, \nu_4 \rangle = \langle \mu, \mu \rangle \cdot \langle \nu_3, \nu_4 \rangle, \tag{28}$$

where the equalities are given by the fact that μ separates both couples (ν_1, ν_2) and (ν_3, ν_4). From these equalities, together with (24), we deduce that

$$\begin{aligned} \langle \nu_1, \nu_2 \rangle \cdot \langle \nu_3, \nu_4 \rangle \cdot \langle \mu, \mu \rangle^2 &= \langle \mu, \nu_1 \rangle \cdot \langle \mu, \nu_3 \rangle \cdot \langle \mu, \nu_2 \rangle \cdot \langle \mu, \nu_4 \rangle \\ &\leq \langle \mu, \mu \rangle \cdot \langle \nu_1, \nu_3 \rangle \cdot \langle \mu, \nu_2 \rangle \cdot \langle \mu, \nu_4 \rangle, \end{aligned}$$

which gives the desired inequality (26).

Finally, by Proposition 2.21, the inequalities (24) and (25) are equalities if and only if μ separates both the couple (ν_1, ν_3) and the couple (ν_2, ν_4). This concludes the proof. □

Proof of Theorem 2.19. By homogeneity of the bracket, we can assume that the semivaluation λ is normalized (see Remark 2.18). Clearly, (1) implies (2).

Let us prove that (3) $\Longrightarrow$ (1). Let $\lambda \in \mathcal{V}_X$ be any normalized semivaluation. Since by construction u_λ is symmetric and vanishes only on the diagonal, it is enough to show that the ultrametric triangular inequality holds.

Let $\nu_1, \nu_2, \nu_3 \in \mathcal{V}_X$, and assume that $c := \langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle \cdot \langle \lambda, \nu_3 \rangle \in [0, +\infty]$ is finite. This is guaranteed for example if the three semivaluations are taken in $\mathcal{V}_X \setminus \{\lambda\}$. Let us define I_1, I_2, I_3 by

$$\begin{aligned} u_\lambda(\nu_1, \nu_2) &= \frac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle}{\langle \nu_1, \nu_2 \rangle} = \frac{c}{\langle \nu_1, \nu_2 \rangle \cdot \langle \lambda, \nu_3 \rangle} =: \frac{c}{I_3}, \\ u_\lambda(\nu_1, \nu_3) &= \frac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_3 \rangle}{\langle \nu_1, \nu_3 \rangle} = \frac{c}{\langle \nu_1, \nu_3 \rangle \cdot \langle \lambda, \nu_2 \rangle} =: \frac{c}{I_2}, \\ u_\lambda(\nu_2, \nu_3) &= \frac{\langle \lambda, \nu_2 \rangle \cdot \langle \lambda, \nu_3 \rangle}{\langle \nu_2, \nu_3 \rangle} = \frac{c}{\langle \nu_2, \nu_3 \rangle \cdot \langle \lambda, \nu_1 \rangle} =: \frac{c}{I_1}. \end{aligned}$$

We want to show that if X is arborescent, then among the quantities I_1, I_2, I_3 at least two coincide, and they are smaller than or equal to the third one.

Since X is arborescent, the convex hull $\mathrm{Conv}(\nu_1, \nu_2, \nu_3, \lambda)$ of $\{\nu_1, \nu_2, \nu_3, \lambda\}$ has one of the shapes represented in Figure 1. In this setting, the convex hull of a finite subset $S \subset \mathcal{V}_X$ may be defined as the union of the images of all injective continuous paths $\gamma \colon [0, 1] \to \mathcal{V}_X$ (the latter considered with its weak topology) joining any two (distinct) points of S (see Remark 2.23 below for an explicit description of this convex hull).

Possibly reordering the four semivaluations, we may assume that they are in counter-clockwise order, starting from the top right corner. In the case of the Y-shape, assume that the branch point is λ (in other cases the argument is the same).

We study case by case, according to the shape of $\mathrm{Conv}(\nu_1, \nu_2, \nu_3, \lambda)$:

- H**-shaped.** Let μ be any point in the horizontal segment. It separates all couples, except at least one between ν_1, λ and ν_2, ν_3. By Proposition 2.22 we deduce that $I_3 = I_2 < I_1$.
- X**-shaped.** The branch point μ separates all couples, and $I_1 = I_2 = I_3$.
- Y**-shaped.** The branch point $\mu = \lambda$ separates all couples, and again $I_1 = I_2 = I_3$.
- F**-shaped.** Let μ be the branch point. It separates all couples, except ν_1, ν_2. We get $I_1 = I_2 < I_3$.
- C**-shaped.** Let μ be any point in the vertical segment. It separates all couples, except ν_1, ν_2 and ν_3, λ. We get $I_1 = I_2 < I_3$.

The case when some of the semivaluations $\nu_1, \nu_2, \nu_3, \lambda$ coincide is easier and is left to the reader. We conclude that u_λ defines an ultrametric distance on $\mathcal{V}_X \setminus \{\lambda\}$ (and an extended ultrametric on $\mathcal{V}_X$).

We conclude the proof of Theorem 2.19 by showing that (2) $\Longrightarrow$ (3). We proceed by contradiction and assume that X is not arborescent; i.e., there exists a good model π such that its dual graph Γ_π has a loop. Denote by $E_1, \dots, E_r$ the vertices of such a loop, where $E_j \in \mathcal{P}(\pi)$ are exceptional primes satisfying $E_j \cdot E_{j+1} = 1$ for all $j = 1, \dots, r$ (with cyclic indices). It follows that $\mathcal{V}_X$ has itself a loop S, given by the quasi-monomial valuations which are either the divisorial valuations ν_{E_j} or the quasi-monomial ones at $p_j = E_j \cap E_{j+1}$ for all $j \in \{1, \dots, r\}$. We have fixed a semivaluation λ for which u_λ is an ultrametric distance. We will show that there exist $\nu_1, \nu_2, \nu_3 \in \mathcal{V}_X$ satisfying

$$\langle \nu_3, \lambda \rangle \cdot \langle \nu_1, \nu_2 \rangle < \langle \nu_2, \lambda \rangle \cdot \langle \nu_1, \nu_3 \rangle < \langle \nu_1, \lambda \rangle \cdot \langle \nu_2, \nu_3 \rangle, \tag{29}$$

or $I_3 < I_2 < I_1$ if we use the notation introduced in the previous part of the proof. This would contradict the hypothesis that u_λ is an ultrametric distance.

But this is the valuative counterpart of Proposition 1.45, which can be proved in this more general setting by using Proposition 2.21 instead of Proposition 1.18. The role of a, b, m, p will be played by $\nu_3, \lambda, \nu_1, \nu_2$, respectively. In particular, given b, it suffices to pick ν_3 as any point in S so that λ is in the connected component of $\mathcal{V}_X \setminus \{\nu_3\}$ containing $S \setminus \{\nu_3\}$. We may assume that ν_3 is divisorial, associated to an exceptional prime divisor E_a. Fix a model X_π such that λ and ν_3 have different centers on it. Denote by E_m and E_p the exceptional prime divisors adjacent to E_a, whose associated valuations belong to S. Up to taking a higher model, we may also assume that the center of λ is disjoint from E_m and E_p and that ν_3 does not separate λ from either ν_{E_m} or ν_{E_p}. Proposition 1.45 gives two valuations, ν_1 and ν_2, corresponding respectively to monomial valuations at the points x_m and x_p of Figure 9, which satisfy (29). □

Remark 2.23. The convex hull mentioned in the previous proof can be described in terms of the skeleton of a model. Fix a good resolution π, and for any closed point $P \in \pi^{-1}(x_0)$, denote by $\mathcal{V}_P$ the topological closure of the set of semivaluations in X_π centered at P. This set $\mathcal{V}_P$ can be naturally identified with the valuative tree $\mathcal{V}$ of [14]. If S is contained in $\mathcal{V}_P$ for some P, the convex hull $\mathrm{Conv}(S)$ is taken in $\mathcal{V}_P$ with respect to its tree structure inherited by $\mathcal{V}$. If this is not the case, then there exist finitely many points $P_1, \dots, P_r$ (with $r \geq 2$) such that $S \subset \bigcup_j \mathcal{V}_{P_j}$. In this situation, one has to consider first for each $j \in \{1, \dots, r\}$ the convex hull inside $\mathcal{V}_{P_j}$ of the union of $S \cap \mathcal{V}_{P_j}$ with $r_\pi(S \cap \mathcal{V}_{P_j})$, as defined above, where $r_\pi : \mathcal{V}_X \to \mathcal{S}_\pi$ is the retraction defined in section 2.2. Then the convex hull $\mathrm{Conv}(S)$ is obtained as the union of those convex hulls with the convex hull of $r_\pi(S)$ inside $\mathcal{S}_\pi$ (which is a tree, since X is arborescent by hypothesis). In fact, in this case $\mathcal{V}_X$ itself has a structure of an $\mathbb{R}$-tree (see Proposition 2.45).

2.5. **$\mathbb{R}$-trees and graphs of $\mathbb{R}$-trees.** In section 1.4, we associated to any finite connected graph Γ a tree $\mathcal{BV}(\Gamma)$, called its *brick-vertex tree*. Then we applied this construction to the dual graph of the embedded resolution of the sum of a finite set $\mathcal{F}$ of branches on a normal surface singularity X, and using it we were able to describe a situation in which u_L defines an ultrametric distance on $\mathcal{F} \setminus \{L\}$ (see Theorem 1.42).

In section 2.6 we construct an analog of the brick-vertex tree for the space $\mathcal{V}_X$. With this scope in mind, we first recall the tree structure carried by the space of normalized semivaluations of a smooth surface singularity. Then we introduce the more general concept of *graph of* $\mathbb{R}$*-trees* (see Definition 2.25) and we explain how to associate to such a graph a topological space, called its *realization* (see Definition 2.26). We conclude the section by introducing several operations on graphs of $\mathbb{R}$-trees, *regularizations* (see Definition 2.36) and *refinements* (see Definition 2.38), which will be used in the next subsection in the construction of the *brick-vertex tree* of a graph of $\mathbb{R}$-trees.

When X is smooth, the space of normalized semivaluations $\boxed{\mathcal{V}} := \mathcal{V}_X$ has been deeply studied by Favre and Jonsson in [14] (see also Jonsson's course [29]). It is referred to as the *valuative tree*, since it carries the structure of an $\mathbb{R}$*-tree* in the sense of [29, Definition 2.2]. Let us first recall the definition of this notion.

Definition 2.24. An **interval structure** on a set I is a partial order $\leq$ on I under which I becomes isomorphic as a poset to the real interval $[0,1]$ or to the trivial real interval $\{0\}$ (endowed with the standard total order of the real numbers). A subinterval $J \subseteq I$ is a subset of I that becomes a subinterval of $[0,1]$ under such an isomorphism. If I is a set with an interval structure, we denote by I^- the same set with the opposite interval structure.

An $\mathbb{R}$**-tree** is a set W together with a family $\{\boxed{[x,y]} \subseteq W \mid x,y \in W\}$ of subsets endowed with interval structures and satisfying the following properties:

(T1) $[x,x] = \{x\}$;

(T2) if $x \neq y$, then $[x,y] = [y,x]^-$ as posets; moreover, $x = \min[x,y]$ and $y = \min[y,x]$;

(T3) if $z \in [x,y]$, then $[x,z]$ and $[z,y]$ are subintervals of $[x,y]$ such that $[x,z] \cup [z,y] = [x,y]$ and $[x,z] \cap [z,y] = \{z\}$;

(T4) for any $x,y,z \in W$, there exists a unique element $w = \boxed{x \wedge_z y} \in [x,y]$ such that $[z,x] \cap [y,x] = [w,x]$ and $[z,y] \cap [x,y] = [w,y]$;

(T5) if $x \in W$ and $(y_\alpha)_{\alpha \in A}$ is a net in W such that the segments $[x, y_\alpha]$ increase with α (relative to the inclusion partial order of the subsets of W), then there exists $y \in W$ such that $\bigcup_\alpha [x, y_\alpha) = [x,y)$.

Here we used the notation $\boxed{[x,y)} := [x,y] \setminus \{y\}$. We define analogously $\boxed{(x,y]}$ and $\boxed{(x,y)}$.

Recall that a **net** is a sequence indexed by a directed set, not necessarily countable.

An $\mathbb{R}$-tree structure on the set W induces a natural topology, called **weak topology**. It is constructed as follows. Fix any $z \in W$, and pick any two points $x, y \in W \setminus \{z\}$. We say that $x \sim_z y$ if $z \notin [x, y]$ (a condition equivalent to $(z, x] \cap (z, y] \neq \emptyset$, found sometimes in the literature). An equivalence class is called a **tangent direction** $\boxed{\overrightarrow{v}}$ at z, and the set of all such classes is denoted by $\boxed{T_z W}$ (see Example 2.33). Tangent directions need to be thought of as *branches* at a point z of W and in some way as infinitesimal objects (hence the name *tangent direction*). For this reason we distinguish an element $\overrightarrow{v} \in T_z W$ from the set $\boxed{U_z(\overrightarrow{v})}$ of points $x \in W \setminus \{z\}$ representing $\overrightarrow{v}$, which is seen as a subset of W. We declare $U_z(\overrightarrow{v})$ to be open for any z varying in W and $\overrightarrow{v}$ varying among all tangent directions at z. The weak topology is generated by such open sets (i.e., it is the weakest topology for which all the sets $U_z(\overrightarrow{v})$ are open). When considering the $\mathbb{R}$-tree structure of $\mathcal{V}$, the weak topology defined here coincides with the weak topology defined in Subsection 2.1.

The structure of the space of normalized semivaluations $\mathcal{V}_X$ associated to a normal surface singularity X has been investigated from a viewpoint similar to that of the present paper by Favre [13] and by Gignac and the last-named author in [21]. It has also been investigated from somewhat different perspectives by Fantini [11, 12], Thuillier [48], and de Felipe [8]. Roughly speaking, $\mathcal{V}_X$ is obtained by patching together copies of the valuative tree $\mathcal{V}$ along any skeleton $\mathcal{S}$ associated to a good resolution π (see Proposition 2.51). As the name suggests, the space $\mathcal{V}_X$ admits an $\mathbb{R}$-tree structure if and only if the singularity X is arborescent (see Propositions 2.43 and 2.45).

To cover the general case, we introduce the concept of *graph of* $\mathbb{R}$*-trees*, which combines the concepts of $\mathbb{R}$-trees and finite graphs.

Seen combinatorially, a finite graph is given by a set of vertices V and a set of edges E, both seen abstractly and related by incidence maps. One may then consider a topological realization of it: the edges can be seen as real segments $I_e = [0, 1]$, and the incidences may be realized by maps $i_e \colon \{0, 1\} \to V$, which give the identifications between the ends of the segment I_e and some vertices of V. We may assume that every vertex in V is in the image of one such map i_e. The graph can then be realized topologically as the disjoint union of all segments I_e (and of the set V) quotiented by the identification of the ends to vertices according to the maps i_e. In order to define graphs of $\mathbb{R}$-trees, we replace in this construction the segments with $\mathbb{R}$-trees.

Definition 2.25. A **graph of $\mathbb{R}$-trees of finite type** is defined by the following data:

(G1) Three sets V, E, D, with V and E finite.

(G2) A family $(W_e)_{e \in E}$ of $\mathbb{R}$-trees with two distinct marked points $x_e, y_e \in W_e$, together with a map $i_e \colon V_e := \{x_e, y_e\} \to V$.

(G3) A family $(W_d)_{d \in D}$ of $\mathbb{R}$-trees with a marked point $x_d \in W_d$, together with a map $i_d \colon V_d := \{x_d\} \to V$.

We denote such a structure by $\boxed{(V, W)}$, where $W := (W_a)_{a \in A}$ is a family of $\mathbb{R}$-trees as described above, with $A := E \sqcup D$. An element W_a is called a **tree element** of (V, W). If $a \in E$, W_a is called an **edge element**, while if $a \in D$, W_a is called a **decoration element** of (V, W). The maps i_a are called **identification maps**.

The previous definition has both topological aspects (as we consider $\mathbb{R}$-trees as building blocks) and combinatorial ones (as one has incidence maps). As for finite graphs, this definition allows us to get a topological space.

Definition 2.26. Given a graph of $\mathbb{R}$-trees (V, W), its **realization** Z is the set defined as

$$\boxed{Z(V,W)} := \bigsqcup_{a \in A} W_a \Big/ \sim,$$

where $W_a \ni x \sim x' \in W_{a'}$ if and only if $x \in V_a, x' \in V_{a'}$, and $i_a(x) = i_{a'}(x')$.

Remark 2.27. Notice that we defined the realization Z of a graph of $\mathbb{R}$-trees (V, W) merely as a set and not as a topological space, even though it is endowed naturally with the topology induced by the one on the tree elements through the quotient by the equivalence relation $\sim$. This topology, to which we will refer as the **quotient topology**, is not well adapted to our purposes (see Remark 2.34). We will introduce a second topology, called the **weak topology** (see Definition 2.32), and we will consider a realization of Z as a topological space with respect to the weak topology.

Up to restricting V if necessary, we will always assume that for any $v \in V$, there exists an $a \in A$ such that $v \in i_a(V_a)$. In this case, we can identify v with the class of elements of the form $i_a(x)$ that satisfy $i_a(x) = v$.

Denote by $\boxed{\mathrm{pr}}$ the natural projection from $\bigsqcup_{a \in A} W_a$ to Z. Let $x, y \in Z$ be two points, and suppose that there exists $a \in A$ such that $x, y \in \mathrm{pr}(W_a)$. If W_a is an edge element (i.e., $a \in E$) and $x = y = \mathrm{pr}(v)$ with $v \in V$, we denote by $[x, y]$ the singleton $\{\mathrm{pr}(v)\}$ and by $[x, y]_a$ the projection of the segment $[x_a, y_a]_a \subseteq W_a$ given by the $\mathbb{R}$-tree structure of W_a, where x_a, y_a are the marked points of W_a.

If all other situations, there exist unique $\tilde{x}$ and $\tilde{y}$ in W_a so that $\mathrm{pr}(\tilde{x}) = x$ and $\mathrm{pr}(\tilde{y}) = y$. In this case we denote by $[x, y]_a$ the projection of the unique segment $[\tilde{x}, \tilde{y}]_a$ in W_a.

To ease notation, if clear from the context, we will omit the projection map and denote $\mathrm{pr}(W_a) \subseteq Z$ simply by W_a.

Remark 2.28. We say that the graph in Definition 2.25 is of finite type because we impose both the set of vertices V and the set E parametrizing the edge elements to be finite. One can remove these conditions in (G1) and get more general objects. Since our interest in graphs of $\mathbb{R}$-trees lies solely in the description of valuation spaces, we will only need to work with graphs of $\mathbb{R}$-trees of finite type. We will hence assume all graphs of $\mathbb{R}$-trees to be of finite type, without further mention.

Nevertheless, most of the results in this section will apply for general graphs of $\mathbb{R}$-trees. We will use the finiteness of V and E in the next sections to deduce the finiteness of the number of bricks (see section 2.6).

Moreover, the definition of graphs of $\mathbb{R}$-trees can be easily adapted to other situations, for example to $\mathbb{Q}$-trees or trees of spheres, etc.

From a graph of $\mathbb{R}$-trees, we can easily extract a finite graph (in the sense of Definition 1.31), which encodes its geometric complexity.

Definition 2.29. Let (V, W) be a graph of $\mathbb{R}$-trees, with realization $Z(V, W)$. Its **skeleton** $\boxed{S(V, W)}$ is the subset of $Z(V, W)$ obtained as the union of the projected segments $[x_e, y_e]_e$, while e varies in E.

Example 2.30. The top left part of Figure 10 depicts an example of a graph of $\mathbb{R}$-trees (V, W), where V consists of two points $\{v_r, v_g\}$ (depicted in red and green) and W consists of four tree elements: one decoration element and three edge elements. Marked points are colored red or green according to the identification maps. On the right part, we can see its realization, obtained by gluing together the tree elements along the marked points according to the identification maps. Its skeleton $S(V, W)$, represented by thick lines, consists of the projection to Z of the three segments between the marked points of the three edge elements. The lower left part of Figure 10 depicts the regularization of (V, W), a notion introduced below in Definition 2.36.

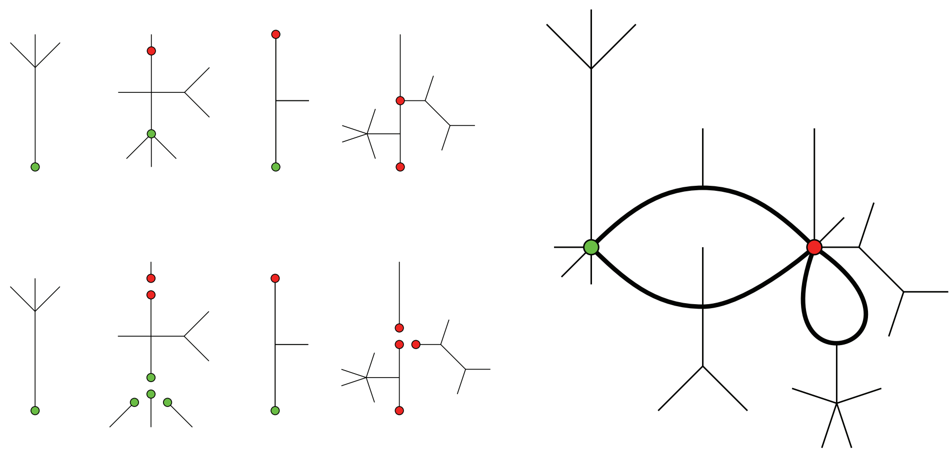

FIGURE 10. A graph of $\mathbb{R}$-trees, its regularization, their realization, and the corresponding skeleton.

As indicated in Remark 2.27, the quotient topology on the realization of a graph of $\mathbb{R}$-trees is not well adapted. Another topology can be introduced, using the notion of *arc* between two points of the realization:

Definition 2.31. Let (V, W) be a graph of $\mathbb{R}$-trees, with realization Z. Let x, y be two points in Z. An **arc** γ between x and y is a subset of Z obtained as a finite concatenation of segments $[s_j, s_{j+1}]_{a_j}$, $j = 0, \dots, n$, where

- $s_0 = x$, $s_{n+1} = y$, and $s_j \in V$ for all $j = 1, \dots, n$;
- $s_j, s_{j+1} \in W_{a_j}$ for all $j = 0, \dots, n$;
- any two segments in the concatenation intersect in at most finitely many points.

Here is the definition of the topology on the realization:

Definition 2.32. Let (V, W) be a graph of $\mathbb{R}$-trees, with realization Z. For any $z \in Z$ and any $x, y \in Z \setminus \{z\}$, we say that $x \sim_z y$ if there exists an arc between x and y which does not contain z. The **weak topology** on Z is the weakest topology for which any subset U of Z representing an equivalence class for $\sim_z$, for any $z \in Z$, is a open set.

Notice that in contrast with the situation for $\mathbb{R}$-trees, the equivalence classes for $\sim_z$ do not correspond directly with tangent vectors at z. In fact, one can define tangent vectors at a point $z \in Z$ as the union of tangent vectors at $z \in W_a$ for all $a \in A$. When Z admits cycles, the spaces associated to two tangent vectors at a point z of the cycle could belong to the same equivalence class with respect to $\sim_z$. See [21, section 2.4] for a description of this phenomenon for normalized semivaluation spaces attached to normal surface singularities.

Example 2.33. Consider again the graph of $\mathbb{R}$-trees (V, W) described in Example 2.30 and its realization Z, depicted on the top left and right part of Figure 10, respectively. The tangent space at the green point v_g consists of six tangent vectors, associated to the $1 + 4 + 1$ tangent vectors appearing on the first three tree elements. By contrast, $Z \setminus \{v_g\}$ has five connected components. The discrepancy is due to the fact that v_g belongs to a cycle of the realization Z of (V, W). Similarly, the red point v_r has seven tangent directions, while $Z \setminus \{v_r\}$ has five connected components.

$\mathbb{R}$-trees and more generally graphs of $\mathbb{R}$-trees should not be thought of only as topological spaces. In fact for applications to semivaluation spaces, one usually needs to go back and forth from the weak topology to the strong topology induced by ρ (see [14–16, 20, 21, 29]). Nevertheless, the weak topology will be very handy, for example in order to be able to talk about connected components of cofinite subsets of $Z(V, W)$ and to define bricks.

Remark 2.34. Let us compare the two topologies introduced for the realization Z of a graph of $\mathbb{R}$-trees: the quotient topology and the weak topology. On the one hand, it is easy to see that the topology induced on W_a by the weak topology on Z does coincide with the weak topology on W_a given by its $\mathbb{R}$-tree structure. On the other hand, the weak topology on Z does not coincide in general with the quotient topology.

Consider for example the graph (V, W) where V consists of just one element $V = \{p\}$, and the family $W = (W_d)_{d \in D}$ is an infinite family of decoration elements (not reduced to a point). In this case, the realization Z admits a structure of an $\mathbb{R}$-tree, and the topology induced by this $\mathbb{R}$-tree structure coincides with the weak topology of its graph of $\mathbb{R}$-tree structure. In particular, an open connected neighborhood of p would contain all decoration elements W_d but for a finite number of $d \in D$. In contrast, an open connected neighborhood of p for the quotient topology is the union of open connected neighborhoods of p in any decoration element W_d, and in particular it need not contain any W_d.

Since it is not the aim of this paper to develop a complete theory of graphs of $\mathbb{R}$-trees, we will not give a definition of morphisms of graphs of $\mathbb{R}$-trees nor of isomorphic graphs of $\mathbb{R}$-trees. Nevertheless, we will consider in this subsection a few operations on graphs of $\mathbb{R}$-trees, which will change the graph structure without changing the underlying realization (seen as a topological space). With this in mind, we will say that two graphs of $\mathbb{R}$-trees are **equivalent** if their realizations are homeomorphic with respect to the weak topologies.

The first operation is related to the choice of the marked points in the tree elements. In fact, following the parallel with classical graphs, we consider the additional condition:

(G4) the marked points V_a of a tree element W_a are **ends** of W_a (i.e., elements that do not disconnect W_a).

Definition 2.35. The graphs of $\mathbb{R}$-trees satisfying the additional condition (G4) are called **regular**.

Given any graph of $\mathbb{R}$-trees (V, W), one can consider the following construction. For any $d \in D$, the tree W_d has a marked point $x = x_d$. For any tangent vector $\overrightarrow{v} \in T_x W_d$, set $W_{d,\overrightarrow{v}} := U_x(\overrightarrow{v}) \cup \{x\}$. The set $W_{d,\overrightarrow{v}}$ is an $\mathbb{R}$-tree, with marked point x. Set $i_{d,\overrightarrow{v}}(x) := i_d(x)$. We replace W_d by the family $(W_{d,\overrightarrow{v}})_{\overrightarrow{v} \in T_x W_d}$.

Analogously, for any $e \in E$, the tree W_e has two marked points, $x = x_e$ and $y = y_e$. Consider the set of connected components of $W_e \setminus V_e$. For any such component U, set $W_{e,U} := \overline{U}$. Notice that there is a unique component U such that $W_{e,U}$ contains V_e, namely, the one containing the open segment (x, y). We set $V_{e,U} := W_{e,U} \cap V_e$ and $i_{e,U} \colon V_{e,U} \to V$ so that it coincides with i_e on its domain of definition. We replace W_e with the family $(W_{e,U})_{e,U}$.

Clearly $(V, (W_{d,\overrightarrow{v}}, W_{e,U})_{d,\overrightarrow{V},e,U})$ defines a graph of $\mathbb{R}$-trees equivalent to (V, W) and satisfying property (G4). Therefore it is regular.

Definition 2.36. The graph of $\mathbb{R}$-trees $(V, (W_{d,\overrightarrow{v}}, W_{e,U})_{d,\overrightarrow{V},e,U})$ constructed above is called the **regularization** of (V, W).

Example 2.37. On the bottom left part of Figure 10, we can see the regularization (V, W') of (V, W) considered in Example 2.30. In this case, W' consists of ten tree elements. Notice that the number of edge elements remains unchanged.

Given a graph of $\mathbb{R}$-trees (V, W), one can define refinements of its structure by adding new vertices. Assume for simplicity that (V, W) is regular (analogous constructions can be done in the non-regular case). Denote by Z the realization of (V, W), and let $p \in Z \setminus V$ be any point. Since p is not a vertex, it belongs to a unique tree element W_a.

If W_a is a decoration element with marked point x, we consider the $\mathbb{R}$-tree $W'_a = W_a$ with marked points x and p. Set $V' = V \cup \{p\}$; then $i'_a(x) = i_a(x)$ and $i'_a(p) = p$. Taking V' as a set of vertices and the family W' obtained from W by replacing W_a with W'_a, we get a new (in general non-regular) graph of $\mathbb{R}$-trees, equivalent to (V, W). Notice that in this case the number of vertices and edges increases by one. Moreover, the skeleton $S(V', W')$ strictly contains $S(V, W)$.

If W_a is an edge element with marked points x and y, set $z = x \wedge_p y$ and $V' = V \cup \{p, z\}$. For any tangent vector $\overrightarrow{v} \in T_z W_a$, define $W'_a(\overrightarrow{v})$ as the closure of $U_x(\overrightarrow{v})$ in W_a. Set $V'_a(\overrightarrow{v}) := W'_a(\overrightarrow{v}) \cap V'$. Notice that $V'_a(\overrightarrow{v})$ always contains z and contains another point in V' in at most three cases (associated to the tangent vectors towards the elements p, x, y). We define $i'_{a,\overrightarrow{v}} \colon V'_a(\overrightarrow{v}) \to V'$ similarly to the previous case. The couple (V', W'), where W' is the family obtained from W by replacing W_a with the family $W'_a(\overrightarrow{v})$, defines again a graph of $\mathbb{R}$-trees equivalent to (V, W). In this case the number of vertices and of edges increase either by 1 or by 2, according to the cases $p \in (x, y)$ or $p \notin (x, y)$. Finally, also in this case $S(V', W') \supseteq S(V, W)$, with equality if and only if $p \in S(V, W)$.

Definition 2.38. Any finite composition of the operation described above and regularizations will be called a **refinement** of the graph structure (V, W).

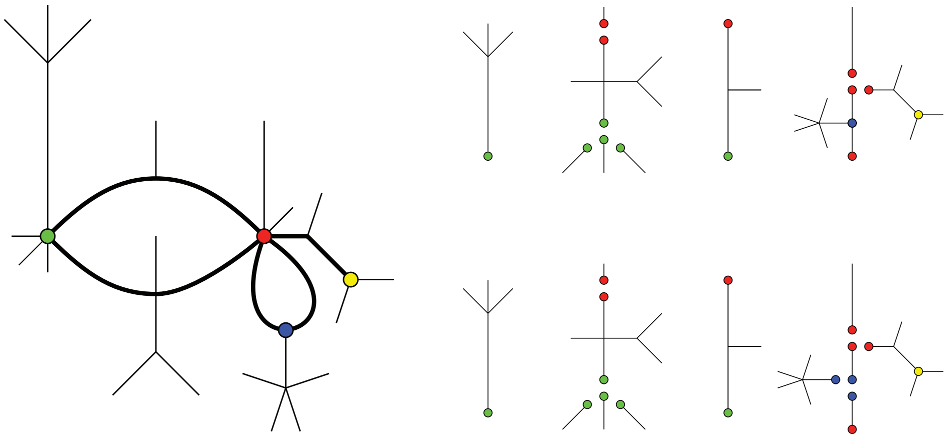

FIGURE 11. Refinement of a graph of $\mathbb{R}$-trees.

Example 2.39. Consider again the regular graph (V, W') described by Example 2.37, with realization Z, depicted in Figure 10. In the left part of Figure 11 we added two vertices, depicted in blue and yellow, obtaining four vertices $V' = \{r_e, r_g, r_b, r_y\}$. The two new vertices belong to unique tree elements that one can see in the top right part of the picture. In the bottom right, we describe the (double) refinement (V', W'') of (V, W') with respect to these two new vertices. The yellow vertex belongs to a decoration element. In this case the new element associated becomes an edge element, and we add a segment to the skeleton (represented by thick segments). The blue vertex belongs to an edge element and to the skeleton $S(V, W')$. In this case, this edge element splits into two edge elements plus a decoration element.

Remark 2.40. Let W be an edge element of some graph of $\mathbb{R}$-trees, with marked points x, y. For any point $z \in [x, y]$, define $\boxed{N_z}$ as $\bigcup_{\vec{v}} U_z(\vec{v}) \cup \{z\}$, where $\vec{v}$ varies among the tangent vectors at z not represented by either x or y. It can also be described as the set of points $w \in W$ such that $[w, z] \cap [x, y] = \{z\}$. The set N_z admits a natural $\mathbb{R}$-tree structure as a subtree of the tree element W. It can also be seen as an $\mathbb{R}$-tree **rooted** at z or again as a graph of $\mathbb{R}$-trees with a single vertex z and a single decoration tree. We will refer to N_z as the **tree at** z **transverse to** $[x, y]$. It will be used below to define *implosions* of graphs of $\mathbb{R}$-trees (see Definition 2.47).

2.6. **Bricks and the brick-vertex tree of a graph of $\mathbb{R}$-trees.** In this section we extend the notions of **brick** and of **brick-vertex tree** to graphs of $\mathbb{R}$-trees (see Definition 2.49). In the next section, we will apply this extended notion of brick-vertex tree to the semivaluation space $\mathcal{V}_X$ of a normal surface singularity X, proving first that it has a structure of graph of $\mathbb{R}$-trees and getting then Theorem 2.53, which is the counterpart of Theorem 1.42 for semivaluation spaces.

The following is an analog of Definition 1.16.

Definition 2.41. Let Z be the realization of a graph of $\mathbb{R}$-trees, and let x, y, z be three points of Z. We say that z **separates** x and y if either $z \in \{x, y\}$ or x and y belong to different connected components of $Z \setminus \{z\}$.

Notice that z separates x and y if and only if all arcs between x and y contain z.

In this section, unless it is specified differently, we will assume that the point z separating x and y never belongs to $\{x, y\}$.

Let us formulate now an analog of Definition 1.33.

Definition 2.42. Let Z be the realization of a graph of $\mathbb{R}$-trees. A subset $C \subseteq Z$ is called **cyclic** if for every couple (x, y) of distinct points of C, no point $z \in C \setminus \{x, y\}$ separates them. A **cyclic element** of Z is a cyclic subset which is maximal with respect to inclusion. A cyclic element is called a **brick** if it does not consist of a single point.

Notice that if $C = \{x\}$, then C is a cyclic element if and only if for all $y \in Z \setminus \{x\}$ there exists $z \in Z \setminus \{x, y\}$ such that z separates x and y in Z.

Proposition 2.43. *Let Z be the realization of a graph (V, W) of $\mathbb{R}$-trees. Then any brick of Z is contained in the skeleton $S(V, W)$.*

Proof. Let x be any point in $Z \setminus S(V, W)$. We want to prove that $\{x\}$ is a cyclic element of Z. This is equivalent to showing that for any point $y \in Z \setminus \{x\}$, there exists a third point z that separates x and y.

Since $x \notin S(V, W)$, there exists a unique $a \in A$ so that $x \in W_a$. We first assume that W_a is a decoration element and denote by z the unique point marked point of W_a. Then z separates x and any point y in $Z \setminus W_a$. Now let y be any point in $W_a \setminus \{x\}$. In this case, any point in (x, y) separates x and y.

Suppose now that W_a is an edge element, say with ends x_a, y_a. By definition we have $W_a \cap S(V, W) = [x_a, y_a]$. Set $z := x_a \wedge_x y_a$. It belongs to $[x_a, y_a]$, and by our assumption it is different from x. In this case, z separates x and any point outside the connected component U of $W_a \setminus [x_a, y_a]$ containing x (i.e., any point representing the tangent vector at z towards x). Finally, let y be any point in $\overline{U} \setminus \{x\}$, where $\overline{U} = U \cup \{z\}$. Then the segment $[x, y]$ is contained in $\overline{U} \subseteq W_a$, and any point in (x, y) separates x and y. $\square$

We deduce that the bricks of Z may be identified with the bricks of the skeleton $S(V, W)$ with respect to its finite graph structure.

As an immediate consequence of Proposition 2.43, we get the following property of graphs of $\mathbb{R}$-trees, assumed as usual to be of finite type.

Corollary 2.44. *Let Z be the realization of a graph of $\mathbb{R}$-trees. Then Z has a finite number of bricks.*

Proof. Pick any graph structure (V, W) whose realization is Z, and denote by $S = S(V, W)$ the skeleton associated to it, with its structure of a finite graph. Let $E = [x, y]$ be an edge of S. Then either E is a bridge of S, in which case every point in (x, y) is a cyclic element, or E is not a bridge, and in this case E belongs to a brick. Since the number of edges is finite, so is the number of bricks. $\square$

The absence of bricks characterizes the graphs of $\mathbb{R}$-trees whose realizations have again a structure of an $\mathbb{R}$-tree.

Proposition 2.45. *Let Z be the realization of a graph of $\mathbb{R}$-trees. Suppose that no cyclic element of Z is a brick. Then Z admits a structure of an $\mathbb{R}$-tree.*

Proof. Let us introduce an $\mathbb{R}$-tree structure on Z satisfying the conditions of Definition 2.24.

Since all cyclic elements of Z are points, we infer that for every couple of points (x, y) in Z, there exists a unique arc $\gamma = \gamma(x, y)$ between x and y. To show this, suppose by contradiction that there are two such arcs that do not coincide. Then in the union of the two we have a cycle, which would be contained in a brick, against the assumption.

Fix any regular structure (V, W) of a graph of $\mathbb{R}$-trees whose realization is Z. Then γ is a finite concatenation of segments $I_j = [s_j, s_{j+1}]$ contained in tree elements W_{a_j}. We set $[x, y] = \gamma$, with the segment structure obtained by taking a concatenation of the orders given by the segment structures on I_j. It is easy to see that (T2) is satisfied for this family of intervals, while property (T3) holds directly by construction.

To verify property (T4), we have to show that for any triple x, y, z of points in Z, there exists a unique element $w = x \wedge_z y$ so that $[z, x] \cap [y, x] = [w, x]$ and $[z, y] \cap [x, y] = [w, y]$. The uniqueness of such w is trivial; hence we only need to show its existence. Consider the set $I = [z, x] \cap [z, y]$, with the partial order induced by the one in $[z, x]$. By uniqueness of arcs between two points, we infer that I is itself a (possibly not closed) interval. Decompose $[z, x] = \bigcup_j [s_j, s_{j+1}]_{a_j}$ where $[s_j, s_{j+1}]_{a_j}$ belongs to W_{a_j}. Let k be the highest index for which $W_{a_k} \cap I \neq \emptyset$. Notice that if $y \notin W_{a_k}$, then $[z, y]$ intersects $W_{a_k} \cap V$ in a point $\tilde{s}$ different from s_k. Set:

- $x_k = x$ if $x \in W_{a_k}$, and $x_k = s_{k+1}$ otherwise;
- $y_k = y$ if $y \in W_{a_k}$, and $y_k = \tilde{s}$ otherwise;
- $z_k = z$ if $z \in W_{a_k}$, and $z_k = s_k$ otherwise.

Now set $w = x_k \wedge_{z_k} y_k$, the wedge being taken with respect to the tree structure on W_{a_k}. Clearly, w satisfies property (T4).

Finally, property (T5) clearly holds for Z. In fact, for any sequence of segments $[x, y_\alpha)$ in Z, there exists $z \in Z$ so that $[z, y_\alpha]$ belongs to a certain tree element W_a for α big enough. Then property (T5) derives directly from the analogous property for W_a. $\square$

We now want to generalize the brick-vertex trees we defined for finite graphs to the case of graphs of $\mathbb{R}$-trees. In order to get such a definition, we need first to introduce a few more constructions.

There is a natural way to associate an $\mathbb{R}$-tree to any non-empty set.

Definition 2.46. Let B be any non-empty set. Let $\sim$ be the equivalence relation on $B \times [0, 1]$ defined by $(x, s) \sim (y, t)$ if and only if $(x, s) = (y, t)$ or $t = s = 0$. The quotient

$$\mathrm{Star}(B) = B \times [0, 1]/\sim$$

is called the **star** over B. We will denote by $\boxed{x_t}$ the class in $\mathrm{Star}(B)$ corresponding to the point (x, t) and by $\boxed{v_B}$ the **apex** of $\mathrm{Star}(B)$, which is represented by $(x, 0)$ for any $x \in B$.

Each star $\mathrm{Star}(B)$ is endowed with a natural structure of an $\mathbb{R}$-tree, whose definition we leave to the reader.

Let (V,W) be a regular graph of $\mathbb{R}$-trees, let Z be its realization, and let B be a brick of Z. For any point $z \in B \setminus V$, there exists a unique edge element $W_{e(z)}$ containing z. We denote by N_z the $\mathbb{R}$-subtree at z transverse to e as defined in Remark 2.40. Then, we consider the graph of $\mathbb{R}$-trees $\boxed{N'_z}$ which has one vertex $\{z\}$ and two decorative elements:

- N_z, with marked point $\{z\}$,
- the segment $[v_B, z_1] \subset \mathrm{Star}(B)$, with marked point $z_1 = (z,1)$,

with natural identification maps. It is easy to see that N'_z has no bricks. In Definition 2.47, N'_z will be considered just as an $\mathbb{R}$-tree, with its structure given by Proposition 2.45.

Given a brick B, let us denote by $\boxed{E(B)}$ the set of indices $e \in E$ such that the edge $[x_e, y_e]$ between the two marked points of an edge element W_e is contained in B.

Definition 2.47. Let (V,W) be a regular graph of $\mathbb{R}$-trees, let Z be its realization, and let B be a brick of Z. For any $z \in B \setminus V$, consider the $\mathbb{R}$-tree N'_z as defined above. Set $V' = V \cup \{v_B\}$, and consider the family W' of $\mathbb{R}$-trees given by:

- the decorative elements W_d, $d \in D$, of W, with same marked point and same identification map;
- the edge elements W_e with $e \in E \setminus E(B)$, with same marked points and same identification map;
- the decorative elements N'_z for $z \in B \setminus V$, with marked point $\{v_B\}$ and natural identification map;
- the edge elements $[v_B, v_1] \subset \mathrm{Star}(B)$, for any $v \in B \cap V$, with marked points v_B and v_1 and identifications $i(v_B) = v_B$ and $i(v_1) = v$.

Then (V', W') is a graph of $\mathbb{R}$-trees, which we call the **implosion** of (V,W) along the brick B. We denote by Z' the realization of the graph (V', W') and by $i_B : Z \to Z'$ the associated natural injection.

Note that the injection $i_B : Z \to Z'$ is not continuous with respect to the weak topologies in Z and Z'. This is due to the fact that the topology induced on $i_B(B)$ by the topology on Z' is the discrete topology, which does not coincide with the topology induced on B by the weak topology of Z (which is the standard topology defined on a graph; see Proposition 2.43). In other terms, we replaced the brick B with its star $\mathrm{Star}(B)$ and not with the *cone* with base B, which corresponds to the analogous construction done by replacing the discrete topology on B with the standard topology of its finite graph structure.

Proposition 2.48. *Let (V,W) be a regular graph of $\mathbb{R}$-trees, and let Z be its realization. Assume that Z has $n \geq 1$ bricks, and let B be any one of them. Let (V', W') be the implosion of (V,W) along the brick B, and let Z' be its realization. Then Z' has exactly $n-1$ bricks, given by the images through the natural injection i_B of the bricks of Z different from B.*

Proof. We only need to check that all points in $\mathrm{Star}(B) \setminus i_B(B)$ form singleton cyclic elements of Z'. By Proposition 2.43, the bricks of Z' are contained in the skeleton $S(V', W')$, which intersects $\mathrm{Star}(B)$ exactly in the edge elements $[v_B, v_1]$ with $v \in V \cap B$ (see Definition 2.47). Let w be any point in $\mathrm{Star}(B) \setminus i_B(B)$, and assume by contradiction that w is contained in a brick B'. Since $\mathrm{Star}(B)$ is a tree, we get that $B' \cap (Z' \setminus \mathrm{Star}(B)) =: C \neq \emptyset$. But then, $B \cup i_B^{-1}(C)$ would be a cyclic subset of Z strictly containing B, which is in contradiction with the maximality of B with respect to inclusion. $\square$

Given any graph of $\mathbb{R}$-trees, we can apply recursively regularizations and brick implosions in order to *kill* all bricks. In fact, by Corollary 2.44, the number of bricks is finite, and by Proposition 2.48, the number of bricks strictly decreases under brick implosion. The final product of this process will be a graph of $\mathbb{R}$-trees (V', W') in which all cyclic elements are singletons. By Proposition 2.45, its realization Z' admits a structure of an $\mathbb{R}$-tree. It is the brick-vertex tree of the starting graph of $\mathbb{R}$-trees.

Definition 2.49. Let Z be the realization of a graph of $\mathbb{R}$-trees (V, W), and let Z' be the $\mathbb{R}$-tree described above, obtained by recursive regularizations and brick implosions of all bricks of Z. Then Z' is called the **brick-vertex tree** of Z and is denoted by $\boxed{\mathcal{BV}(Z)}$. The points of Z' corresponding to apices of bricks of Z are called **brick points** of the brick-vertex tree. We denote by $i_{\mathrm{bv}} \colon Z \to Z'$ the natural injection obtained by the composition of the natural injections i_B described above for brick implosions.

Note that if B, B' are two bricks of a graph of $\mathbb{R}$-trees Z and Z' is the implosion of B, then $i_B(B')$ is a brick in Z'. It follows that the brick-vertex tree of Z does not depend on the order in which we perform the brick implosions.

We end this section with a remark about the notion of cyclic element from a topological perspective.

Remark 2.50. The term *cyclic element* is standard in general topology, while that of *brick* was introduced by us in order to get a common denomination for the graph-theoretic blocks which are not bridges and for the cyclic elements which are not points. Indeed, while the notion of block is combinatorial and that of cyclic element is topological, the underlying topological space of a brick of a finite graph is a brick of its underlying topological space (see Proposition 2.43).

Cyclic elements can be defined for much more general topological spaces than for finite graphs or realization spaces of graphs of $\mathbb{R}$-trees. This notion was introduced by Whyburn in his 1927 paper [54] as a means to describe the overall structure of *Peano continua*, i.e., the compact connected metric spaces which may be obtained as continuous images of the real interval $[0, 1]$ inside some Euclidean space $\mathbb{R}^n$. He defined the *cyclic elements* of such a topological space as its maximal subsets C such that any two distinct points of them are contained in a circle topologically embedded in C. In fact, he initially studied only *plane Peano continua*, and he extended in later papers the theory to arbitrary ones using ingredients from Ayres' 1929 paper [1]. Later on, in the 1930 paper [31], Kuratowski and Whyburn simplified the theory of cyclic elements by defining them as in Definition 2.42 above.

The main point of this theory was to explain that the cyclic elements of a Peano continuum are organized in a tree-like manner. For instance, given any two cyclic elements, there is a unique connected union of cyclic elements which contains them and is minimal for inclusion. This is an analog of the uniqueness of a path joining two points of a tree.

Later, the theory of cyclic elements was extended to more general settings (see e.g. [32, 39, 56] as well as the references in McAllister's surveys [35], [36] of the theory up to 1966 and in the interval 1966–81, respectively). In fact, as pointed out by Rado and Reichelderfer in [43], most of the results of the theory can be obtained in the very general situation of a set endowed with a "cyclic transitive relation" (a cyclic transitive relation $\mathcal{R}$ on a set S is a binary relation which is reflexive, symmetric, and such that if $x_1 \mathcal{R} x_2 \mathcal{R} \dots \mathcal{R} x_n \mathcal{R} x_1$, then $x_i \mathcal{R} x_j$ for all $i, j = 1, \dots, n$). In particular, in this generality one does not need topological spaces in order to talk about cyclic elements. This last aspect is very interesting in our setting, since as already pointed out, valuative spaces carry two natural topologies with quite different properties (the weak topology is non-metrizable, and the space is compact and locally compact, while the strong topology is metrizable, but the space is not locally compact).

Let us mention that the Peano spaces in which all the cyclic elements are points are called *dendrites* (see [55]). Ważewski proved in [52] the existence of a *universal dendrite*, which embeds all other dendrites. Recently, Hrushovski, Loeser, and Poonen found in [26, Corollary 8.2] a representation of it as a special type of valuation space under a countability hypothesis on the base field.

In what concerns the relation between cyclic element theory of topological spaces and block theory of graphs, it is interesting to note that in the paper [53], in which Whitney introduced the notion of *non-separable graph* (see Definition 1.32), he quoted an article of Whyburn on cyclic element theory, but after that date the two fields seem to have evolved quite independently of each other.

2.7. **Valuation spaces as graphs of $\mathbb{R}$-trees.** In this subsection we apply the constructions of the previous section to the space of normalized semivaluations associated to a normal surface singularity. We first prove that it admits space of normalized semivaluations admits a structure of a connected graph of $\mathbb{R}$-trees (see Proposition 2.51). Then we prove the valuative analog of Theorem 1.42, stating that the functions u_λ are ultrametrics on special types of subspaces of the space of normalized semivaluations (see Theorem 2.53). We conclude the paper with several examples which show that the hypotheses of the theorem are not necessary in order to get ultrametrics.

Proposition 2.51. *Let X be a normal surface singularity, and let $\mathcal{V}_X$ be its associated space of normalized semivaluations. Then $\mathcal{V}_X$ admits a structure of a connected graph of $\mathbb{R}$-trees; that is, it is a connected realization space of a graph of $\mathbb{R}$-trees. More precisely, any good resolution defines canonically such a structure.*

Proof. Let $\pi\colon X_\pi \to X$ be any good resolution. We set V as the set of divisorial valuations associated to the primes of π. For any point $p \in \pi^{-1}(x_0)$, we set $W_p = \overline{U_\pi(p)}$, which consists of the set $U_\pi(p)$ of all semivaluations whose center in X_π is p, plus the divisorial valuations of the form ν_E with $E \ni p$ (which belong to V). Since $\pi^{-1}(x_0)$ has simple normal crossings, either p belongs to a unique prime E of π, in which case we declare W_p a decoration element with marked point ν_E, or p belongs to exactly two exceptional primes E and F, in which case we declare W_p an edge element with marked points ν_E and ν_F. Since for any such point p, the germ (X_π, p) is smooth, the set W_p is isomorphic to the valuative tree; hence it is an $\mathbb{R}$-tree. The couple $(V, (W_p)_{p\in\pi^{-1}(x_0)})$ defines a structure of a graph of $\mathbb{R}$-trees on $\mathcal{V}_X$. □

Example 2.52. In Figure 12, we may see on the left the dual graph Γ_π of a good resolution π of some normal surface singularity X. In this example, there are three bricks, depicted in orange, blue and yellow. On the right side, we may see a depiction of the semivaluation space $\mathcal{V}_X$. The structure of a graph of $\mathbb{R}$-trees induced by π in this case has as vertices the vertices of Γ_π under identification with the corresponding valuations (we denoted them as $\mathcal{S}_\pi^*$), edge elements correspond to the trees along the edges of Γ_π, and all other tree elements are decorations. The thick colored segments correspond to bricks of $\mathcal{V}_X$ with respect to its structure of a graph of $\mathbb{R}$-trees.

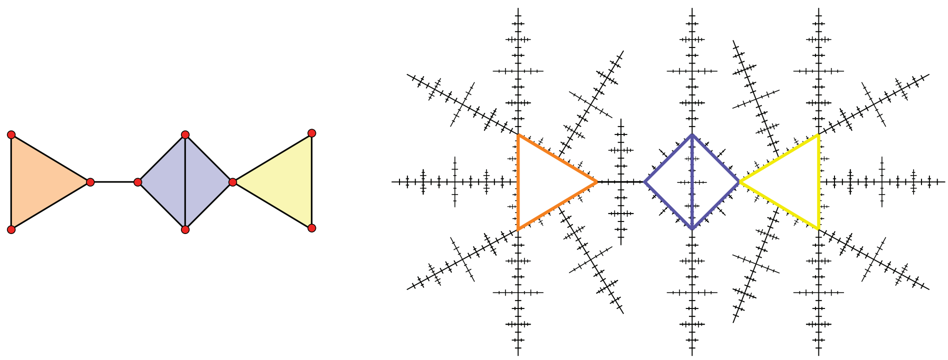

FIGURE 12. The dual graph associated to a good resolution π of a normal surface singularity X, with bricks shaded, and its associated space $\mathcal{V}_X$ of normalized semivaluations.

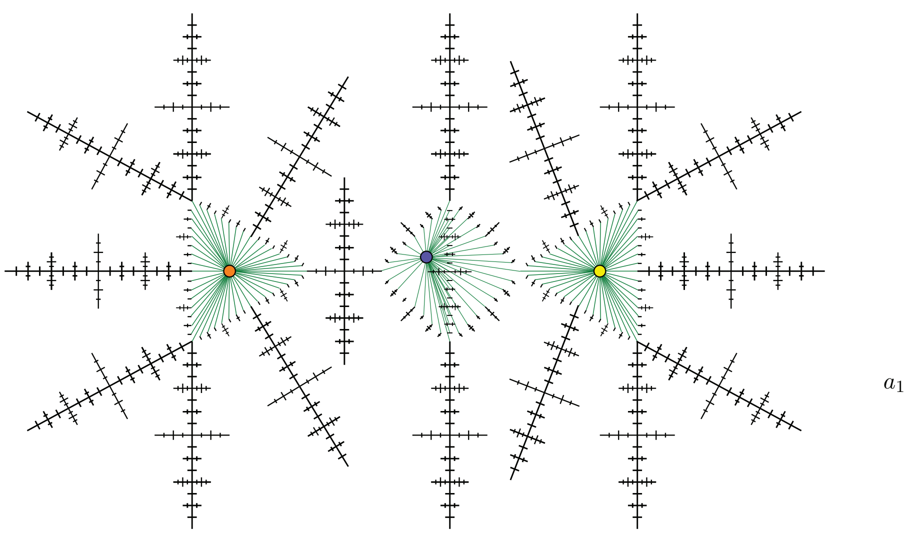

FIGURE 13. The brick-vertex tree $\mathcal{BV}(\mathcal{V}_X)$ for the example of Figure 12.

We are now able to state and prove the following theorem, which is an analog of Theorem 1.42 for valuation spaces.

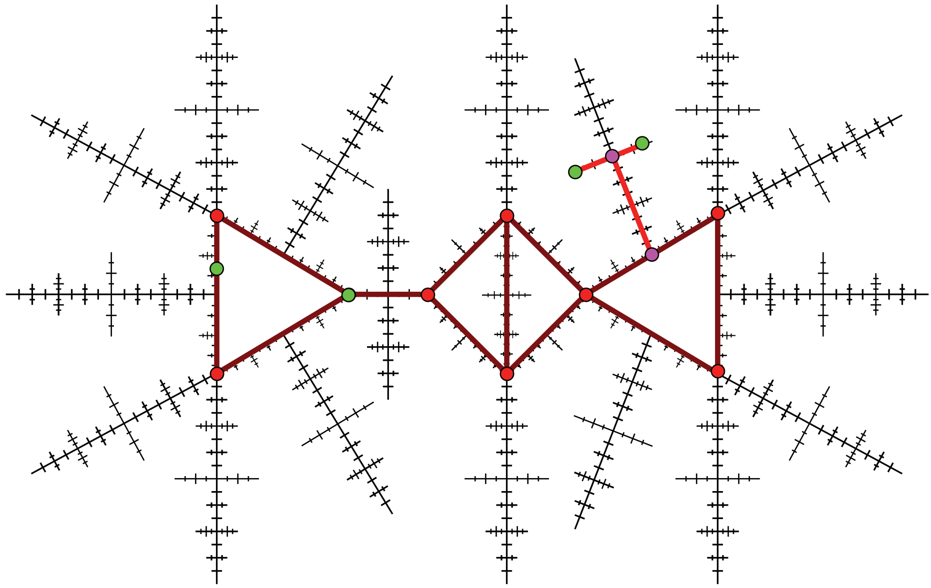

FIGURE 14. Graphs embedded in $\mathcal{V}_X$, illustrating the proof of Theorem 2.53.

Theorem 2.53. *Let X be a normal surface singularity, let $\mathcal{V}_X$ be the associated space of normalized semivaluations, and let $\mathcal{J} \subseteq \mathcal{V}_X$ be any subset of it. Let $\mathcal{BV}(\mathcal{V}_X)$ be the brick-vertex tree of $\mathcal{V}_X$, and consider its subtree $W = \mathrm{Conv}(i_{\mathrm{bv}}(\mathcal{J}))$. If $T_{v_B}W$ consists of at most three points for every brick point $v_B \in W$, then u_λ defines an extended ultrametric distance on $\mathcal{J}$ for any $\lambda \in \mathcal{J}$.*

Proof. Fixing any $\lambda \in \mathcal{J}$, we need to prove that

$$u_\lambda(\nu_1, \nu_3) \leq \max\{u_\lambda(\nu_1, \nu_2), u_\lambda(\nu_2, \nu_3)\} \tag{30}$$

for any triple $\nu_1, \nu_2, \nu_3 \in \mathcal{J}$. Notice that (30) is satisfied if either ν_1, ν_2, or ν_3 coincides with λ. Say for example that $\nu_2 = \lambda$. Then $u_\lambda(\nu_1, \nu_2) = u_\lambda(\nu_2, \nu_3) = \langle \lambda, \lambda \rangle = \alpha(\lambda)$, while $u_\lambda(\nu_1, \nu_3) = \frac{\langle \lambda, \nu_1 \rangle \langle \lambda, \nu_3 \rangle}{\langle \nu_1, \nu_3 \rangle} \leq \langle \lambda, \lambda \rangle$ by Proposition 2.21. We may hence assume that $\lambda \notin \{\nu_1, \nu_2, \nu_3\}$. In particular, the three values in (30) are finite.

By proceeding as in Proposition 1.23 and Corollary 1.25, we get that (30) is equivalent to showing that ρ is tree-like; i.e., it satisfies the 4-point condition (13). Set $J := \{\nu_1, \nu_2, \nu_3, \nu_4\} \subset \mathcal{J}$.

Take any good resolution $\pi \colon X_\pi \to X$. Any semivaluation $\nu \in J$ either belongs to $\mathcal{S}_\pi^*$, or belongs to the weakly open set $U_\pi(p)$ associated to the center $p = p(\nu) \in \pi^{-1}(x_0)$ of ν in X_π. Let $\mathcal{S}_\pi$ denote the skeleton associated to π, and let Γ be the subset of $\mathcal{V}_X$ given by the union of $\mathcal{S}_\pi$ and the segments $[\nu_E, \nu] \subset \overline{U_\pi(p)}$, where $p = p(\nu)$ is as above, E is any exceptional prime of π containing p, and ν varies in J. The set Γ admits a structure of a finite graph. In fact, up to taking higher good resolutions, we may assume that for any distinct $\nu, \nu' \in J$, their centers in X_π are also distinct. We may also assume that any semivaluation in J either belongs to $\mathcal{S}_\pi$ or its center in X_π is a smooth point of $\pi^{-1}(x_0)$. In this case, the structure of a finite graph on Γ has as vertices $\mathcal{S}_\pi^* \cup J$ and as edges all the edges in $\mathcal{S}_\pi$, eventually cut by elements in $J \cap \mathcal{S}_\pi$, plus all the edges associated to the segments $[\nu_E, \nu]$ with $\nu \in J$ as described above (see Figure 14).

The function ρ defines a distance on the set of vertices of Γ, satisfying the condition (15). This is a consequence of Proposition 2.21 applied to the reformulations given in Proposition 1.19.

Consider now the brick-vertex tree $\mathcal{BV}(\Gamma)$ associated to Γ. The embedding of Γ in $\mathcal{V}_X$ induces an embedding of $\mathcal{BV}(\Gamma)$ inside $\mathcal{BV}(\mathcal{V}_X)$. Since the tangent space of W at any brick point consists at most of three points, the Conv(J)-valency of any brick point of Γ is at most 3. We can apply Theorem 1.38 and deduce that ρ is tree-like on the set J, and we are done. $\square$

Notice that, as in the case of finite graphs, we get again the proof of the implication (3) $\Longrightarrow$ (1) of Theorem 2.19 as a direct corollary of Theorem 2.53.

Example 2.54. Figure 13 depicts the brick-vertex tree associated to the semivaluation space $\mathcal{V}_X$ represented in Figure 12. The thick vertices in orange, blue, and yellow denote the brick-vertices of $\mathcal{BV}(\mathcal{V}_X)$, while the dark green segments belong to the stars on them. The image needs to be thought of with the green part not intersecting the rest of the space.

In Figure 14 consider a set J of four semivaluations in $\mathcal{V}_X$ as in the proof of Theorem 2.53, that are depicted in light green. The dark red area denotes the skeleton associated to the minimal good resolution of X, while the light red part corresponds to the part added to $\mathcal{S}_\pi$ to obtain Γ. The thick red dots correspond to the divisorial valuations in $\mathcal{S}_\pi^*$ (not belonging to J), while the pink-purple dots are the rest of the divisorial valuations added for describing the graph structure on Γ.

Example 2.55. As for its counterpart for finite sets of branches formulated in Theorem 1.42, the condition on the valency of brick-points in Theorem 2.53 is not necessary in general. Consider again the singularity studied in Example 1.44, whose minimal good model X_π has four exceptional primes $E_1, \dots, E_4$ of self-intersection -4, which intersect transversely each another. The skeleton associated to it is the 1-skeleton of a tetrahedron. Denote by ν_j the prime divisorial valuation associated to E_j for all $j = 1, \dots, 4$, and denote by μ_t the monomial valuation at the intersection point p between E_1 and E_2, so that $Z_\pi(\mu_t) = (1-t)\check{E}_1 + t\check{E}_2$.

Since all these valuations belong to the skeleton $\mathcal{S}_\pi$, which is included in a unique brick, any choice of four valuations a, b, c, d among $\nu_1, \nu_2, \nu_3, \nu_4, \mu_t$ for $0 < t < 1$ would not satisfy the hypotheses of Theorem 2.53. By computing the bracket between μ_t and ν_j, we get

$$5\langle \nu_1, \mu_t \rangle = 2 - t, \qquad 5\langle \nu_2, \mu_t \rangle = 1 + t, \qquad 5\langle \nu_3, \mu_t \rangle = 5\langle \nu_4, \mu_t \rangle = 1.$$

For any choice of four valuations a, b, c, d, we consider now the values $I_1 = 25\langle a, b\rangle\langle c, d\rangle$, $I_2 = 25\langle a, c\rangle\langle b, d\rangle$, and $I_3 = 25\langle a, d\rangle\langle b, c\rangle$. We recall that a, b, c, d satisfy the 4-point condition if and only if two out of these three values coincide and the third is greater than or equal to the other two. First, pick the quadruple $\nu_1, \mu_t, \nu_3, \nu_4$: we get $I_1 = 2 - t$, $I_2 = I_3 = 1$. In this case the 4-point condition is satisfied. Then, pick the quadruple $\nu_1, \mu_t, \nu_2, \nu_3$: we get $I_1 = 2 - t$, $I_2 = 1$, $I_3 = 1 + t$. In this case the 4-point condition is never satisfied.

Example 2.56. We saw in Example 2.55 how the validity of the 4-point condition may depend on the valuation when it varies inside the same brick. We now investigate how it varies when changing the self-intersections of prime divisors in some model. To this end, consider again the singularity X defined in Example 2.55 and the point p of intersection of E_1 and E_2. Denote by E_5 the exceptional prime divisor corresponding to the blow-up of p. In this new model $X_{\pi'}$, the self-intersections of the strict transforms of E_j, $j = 1, \ldots, 4$, and of E_5 are, respectively, $-5, -5, -4, -4, -1$.

Consider now the normal surface singularity Y whose minimal resolution has the same dual graph as $X_{\pi'}$ but satisfying $E_5^2 = -2$ instead of -1. Denote by ν_j the prime divisorial valuation associated to E_j for all $j = 1, \ldots, 4$ and by ν_5 the one associated to E_5. Let μ'_t be the monomial valuation at the intersection between the strict transform of E_2 and E_5, so that $Z_{\pi'}(\mu'_t) = (1-t)\check{E}_2 + t\check{E}_5$. In this case, we get

$$80\langle \nu_1, \mu'_t \rangle = 7 + 8t, \qquad 80\langle \nu_3, \mu'_t \rangle = 80\langle \nu_4, \mu'_t \rangle = 10.$$

For the choice of valuations a, b, c, d given by $\nu_1, \mu'_t, \nu_3, \nu_4$, we consider $I_1 = 80^2\langle a, b\rangle\langle c, d\rangle$, $I_2 = 80^2\langle a, c\rangle\langle b, d\rangle$, and $I_3 = 80^2\langle a, d\rangle\langle b, c\rangle$. In this case we get $I_2 = I_3 = 100$ and $I_1 = 12(7 + 8t)$.

In particular, we notice that the 4-point condition is satisfied for this quadruple if and only if $t \geq \frac{1}{6}$. Notice also that μ'_t parametrizes the segment $[\nu_2, \nu_5]$, which is contained in the segment $[\nu_2, \nu_1]$. The situation here is quite different from the one described in Example 2.55, where the 4-point condition of the quadruple $\nu_1, \mu_t, \nu_3, \nu_4$ was satisfied for any choice of μ_t. In particular, the valuations $\nu_1, \nu_2, \nu_3, \nu_4$ satisfy the 4-point condition for X, but they do not satisfy the 4-point condition for Y.

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