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Higher Order Polars of Quasi-Ordinary Singularities

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A quasi-ordinary polynomial is a monic polynomial with coefficients in the power series ring such that its discriminant equals a monomial up to unit. In this paper, we study higher derivatives of quasi-ordinary polynomials, also called higher order polars. We find factorizations of these polars. Our research in this paper goes in two directions. We generalize the results of Casas–Alvero and our previous results on higher order polars in the plane to irreducible quasi-ordinary polynomials. We also generalize the factorization of the first polar of a quasi-ordinary polynomial (not necessarily irreducible) given by the first-named author and González-Pérez to higher order polars. This is a new result even in the plane case. Our results remain true when we replace quasi-ordinary polynomials by quasi-ordinary power series.

1 Introduction

In [15], Merle gave a decomposition theorem of a generic polar curve of an irreducible plane curve singularity, according to its topological type. The factors of this decomposition are not necessarily irreducible. Merle's decomposition was generalized to reduced plane curve germs by Kuo and Lu [12], Delgado de la Mata [3], Eggers [4], and García Barroso [5] among others. In [6], García Barroso and González Pérez obtained

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decompositions of the polar hypersurfaces of quasi-ordinary singularities. On the other hand, Casas-Alvero in [2] generalized the results of Merle to higher order polars of an irreducible plane curve. In [8], we improved his results giving a finer decomposition in such a way that we are able to determine the topological type of some irreducible factors of the polar as well as their number.

Our research in this paper goes in two directions. We generalize the results of [2] and [8] on higher order polars to irreducible quasi-ordinary singularities (see Theorem 10.10 and Proposition 10.11). We also generalize the factorization of the first polar of a quasi-ordinary singularity (not necessarily irreducible) from [6] to higher order polars (see Theorem 10.4). This is a new result even in the plane case.

Our approach is based on Kuo–Lu trees, Eggers trees, Newton polyhedra, and resultants. As it was remarked in [18] and [6], the irreducible factors of the polar of a quasi-ordinary singularity are not necessarily quasi-ordinary. For that reason, we measure the relative position of these irreducible factors and those of the quasi-ordinary singularity using a new notion called the *P-contact*, which plays in our situation the role of the *logarithmic distance* introduced by Płoski in [17].

The paper is organized as follows.

In Section 2, we recall the notion of the Newton polyhedron of a Weierstrass polynomial $f \in \mathbf{K}[[\underline{x}]][y]$ and we use it together with the Rond–Schober irreducibility criterium [20], in order to give sufficient conditions for the reducibility of f. The most important result in this section is Corollary 2.6, which allows us to characterize, in Theorem 9.1, the irreducible factors of the higher order polars of the polynomial f.

In Section 3, we present the notion of the Kuo–Lu tree of a quasi-ordinary Weierstrass polynomial. Then, in Section 4, we identify the bars of a Kuo–Lu tree with certain sets of fractional power series called *pseudo-balls* and we introduce the notion of *compatibility* of a Weierstrass polynomial with a pseudo-ball. Every quasi-ordinary Weierstrass polynomial is compatible with every pseudo-ball associated with its Kuo–Lu tree. Moreover, if a Weierstrass polynomial is compatible with a pseudo-ball, then any factor of it is compatible too (see Corollary 4.5). In Lemma 4.6, we prove that, under some conditions, the normalized higher derivatives inherit the compatibility property. In Section 5, we introduce, using Galois automorphisms, an equivalence relation in the set of pseudo-balls, called *conjugacy*, and we explore the compatibility property for conjugate pseudo-balls. We generalize the Kuo–Lu lemma [12, Lemma 3.3] to higher derivatives in Section 6. In Section 7, we introduce our main tool, monomial substitutions, that allows us to reduce several questions to the case of two variables. In particular, if f and g are power series in $d+1$ variables such that after generic monomials substitutions we obtain power series $\bar{f}, \bar{g}$ in two variables with equal Newton polygons, then the Newton polyhedra of f and g are also equal (see Corollary 7.4). In Section 8, we extend the notion of Eggers tree introduced in [4], to quasi-ordinary settings. Remark that the tree we use here is not exactly the Eggers–Wall tree introduced in [18] for the quasi-ordinary situation. The main result of Section 9 is Theorem 9.1, where we characterize the irreducible factors of higher derivatives of quasi-ordinary Weierstrass polynomials. Theorem 9.1 allows us to give factorizations of higher derivatives, in terms of the Eggers tree, in Section 10. Theorem 10.4 generalizes the factorization from [2] on higher order polars to quasi-ordinary singularities (not necessarily irreducible) and also the factorization from [6] to higher order polars. Theorem 10.10 and Proposition 10.11 extend the statements of [8, Theorem 6.2] to irreducible quasi-ordinary Weierstrass polynomials. Finally, in Section 11, we establish that our results also hold for quasi-ordinary power series.

2 Newton Polyhedra

Let $\alpha = \sum \alpha_{\mathbf{i}} \underline{x}^{\mathbf{i}} \in S[[\underline{x}]]$ be a nonzero formal power series with coefficients in a ring S, where $\underline{x} = (x_1, \ldots, x_d)$ and $\underline{x}^{\mathbf{i}} = x_1^{i_1} \cdots x_d^{i_d}$, with $\mathbf{i} = (i_1, \ldots, i_d)$. The *Newton polyhedron* $\Delta(\alpha) \subset \mathbf{R}^d$ of α is the convex hull of the set $\bigcup_{\alpha_{\mathbf{i}} \neq 0} \mathbf{i} + \mathbf{R}^d_{\geq 0}$. By convention, the Newton polyhedron of the zero power series is the empty set.

The Newton polyhedron of a polynomial $f = \sum_{\mathbf{i},j} a_{\mathbf{i},j} \underline{x}^{\mathbf{i}} y^j \in S[[\underline{x}]][y]$ is the polyhedron $\Delta(f) \subset \mathbf{R}^d \times \mathbf{R}$ of f viewed as a power series in $x_1, \ldots, x_d, y$. If Γ is a compact face of $\Delta(f)$, then $f|_\Gamma := \sum_{(\mathbf{i},j) \in \Gamma} a_{\mathbf{i},j} \underline{x}^{\mathbf{i}} y^j \in S[\underline{x}][y]$ is called the *symbolic restriction* of f to Γ.

We say that a subset of $\mathbf{R}^{d+1}$ is a *Newton polyhedron* if it is the Newton polyhedron of some polynomial in $S[[\underline{x}]][y]$.

Let $\mathbf{q} = (q_1, \ldots, q_d) \in \mathbf{Q}^d_{\geq 0}$ and let k be a positive integer. We define the *elementary Newton polyhedron*

$$\left\{ \frac{q}{k} \right\} := \text{convex hull} \left(\{ (q_1, \ldots, q_d, 0), (0, \ldots, 0, k) \} + \mathbf{R}^{d+1}_{\geq 0} \right) .$$

Its *inclination* is, by definition, $\frac{1}{k}\mathbf{q}$. We denote by $\left\{ \frac{\infty}{k} \right\}$ the Newton polyhedron $\Delta(y^k)$, which is the first orthant translated by $(0, \ldots, 0, k)$. By convention, we consider it as an elementary polyhedron.

Example 2.1. The elementary Newton polyhedron $\left\{ \frac{(4,2)}{8} \right\}$ is drawn in Figure 1.

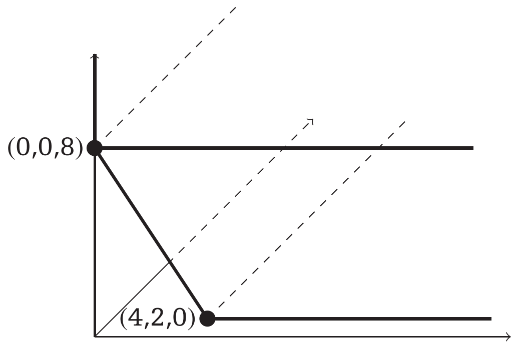

Fig. 1. The elementary Newton polyhedron $\left\{\frac{(4,2)}{8}\right\}$.

A Newton polyhedron is *polygonal* if the maximal dimension of its compact faces is one.

Remember that the Minkowski sum of $A, B \subset \mathbf{R}^{d+1}$ is the set $A + B := \{a + b : a \in A, b \in B\}$. If a Newton polyhedron Δ has a representation of the type

$$\Delta = \sum_{i=1}^{r} \left\{\frac{q_i}{k_i}\right\}, \tag{1}$$

then summing all the elementary Newton polyhedra of the same inclination in (1) we obtain a unique representation, up to the order of the terms, called *canonical representation* of Δ. We introduce in $\mathbf{Q}^d_{\geq 0}$ the partial order: $\mathbf{q} \leq \mathbf{q}'$ if $\mathbf{q}' - \mathbf{q} \in \mathbf{Q}^d_{\geq 0}$. By convention $+\infty$ is bigger than any element of $\mathbf{Q}^d_{\geq 0}$. If the inclinations in (1) can be set in a well-ordered sequence then Δ is polygonal.

2.1 Newton polyhedra and factorizations

Let $\mathbf{K}$ be a field of characteristic zero. We denote by $\mathbf{K}[[x_1^{1/k}, \ldots, x_d^{1/k}]]$ the ring of fractional power series in d variables where all the exponents are nonnegative rational numbers with denominator $k \in \mathbf{N}\backslash\{0\}$. Put $\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] := \bigcup_{k\in\mathbf{N}\backslash\{0\}} \mathbf{K}[[x_1^{1/k}, \ldots, x_d^{1/k}]]$. We will denote by

$$\alpha\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] = \{\alpha w : w \in \mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]\}$$

the ideal of $\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$ generated by $\alpha \in \mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$.

A *Weierstrass polynomial* in $\mathbf{K}[[\underline{x}]][y]$ is a monic polynomial where the coefficients different from the leading coefficient have vanishing constant terms. Notice that, according to this definition, the constant polynomial 1 is a Weierstrass polynomial.

The next lemma gives sufficient conditions for reducibility of Weierstrass polynomials. One of the consequences of this lemma is that a Weierstrass polynomial with a polygonal Newton polyhedron admits a decomposition into coprime factors such that the Newton polyhedron of each factor is elementary (see Theorem 2.4, see also [6, Theorem 3]).

Lemma 2.2. Let $g = y^m + c_1 y^{m-1} + \cdots + c_m \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial. Assume that there exists $\mathbf{q} \in \mathbf{Q}^d$ such that $c_i\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] \subseteq \underline{x}^{i\mathbf{q}}\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$ for all $1 \leq i \leq m$ with equality for some $i = i_0$, $1 \leq i_0 < m$ and strict inclusion for $i = m$. Then, g has at least two coprime factors.

Proof. We will apply [20, Theorem 2.4]. Without lost of generality, we may assume that i_0 is the maximal index $i \in \{1, \ldots, m-1\}$ such that $c_i\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] = \underline{x}^{i\mathbf{q}}\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$. Then, the segment Γ with endpoints $(0, \ldots, 0, m)$ and $(i_0\mathbf{q}, m - i_0)$ is an edge of $\Delta(g)$. The symbolic restriction of g to Γ is the product $g|_\Gamma = y^{m-i_0} \cdot \tilde{g}$, where $\tilde{g} \in \mathbf{K}[\underline{x}][y]$ is coprime with y. The associated polyhedron of g, in the sense of Rond–Schober (see [20, page 4732] is $m\mathbf{q} + \mathbf{R}^d_{\geq 0}$. Hence, the polynomial g verifies the hypothesis of [20, Theorem 2.4] and the lemma follows. ■

Remark 2.3. The assumptions of Lemma 2.2 mean geometrically that the Newton polyhedron $\Delta(g)$ is included in the elementary polyhedron $\left\{\frac{mq}{m}\right\}$, and $\Delta(g)$ has an edge Γ, which endpoints $(0, \ldots, 0, m)$ and $(i_0\mathbf{q}, m - i_0)$, for some $1 \leq i_0 < m$. Figure 2 illustrates the situation.

Theorem 2.4. Let $f \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial. Assume that $\Delta(f)$ is a polygonal Newton polyhedron with canonical representation $\sum_{i=1}^r \left\{\frac{q_i}{k_i}\right\}$.

Then, f admits a factorization $f_1 \cdots f_r$, where $f_i \in \mathbf{K}[[\underline{x}]][y]$ are Weierstrass polynomials, not necessarily irreducible, such that $\Delta(f_i) = \left\{\frac{q_i}{k_i}\right\}$ for $i = 1, \ldots, r$.

Proof. Let $f = g_1 \cdots g_s$ be the factorization of f into irreducible Weierstrass polynomials. Since the Newton polyhedron of a product is the Minkowski sum of the Newton polyhedra of the factors, by hypothesis we get $\Delta(g_j) = \sum_{i=1}^r b_{ij}\left\{\frac{q_i}{k_i}\right\}$ for some $b_{ij} \in \mathbf{Q}_{\geq 0}$. By Remark 2.3 $\Delta(g_j)$ is elementary; hence, for fixed j only one term of the previous sum is nonzero. On the other hand, for fixed i, we get $\sum_j b_{ij} = 1$. Put $f_i := \prod g_j$, where the product runs over all g_j such that $b_{ij} \neq 0$. Then, $f = f_1 \cdots f_r$, where $\Delta(f_i) = \left\{\frac{q_i}{k_i}\right\}$ for $i = 1, \ldots, r$. ■

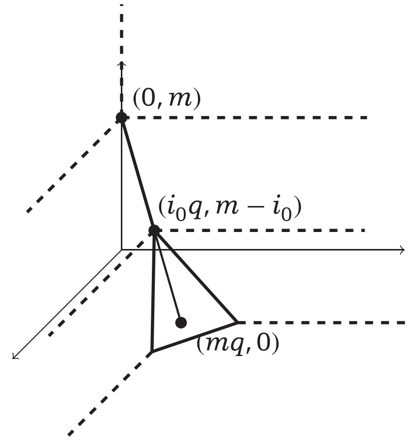

Fig. 2. Illustrating the assumptions of Lemma 2.2.

Theorem 2.5. Let $f(y), g(y) \in \mathbf{L}[y]$ be monic polynomials, where $\mathbf{L}$ is a field of characteristic zero. If $g(y)$ is irreducible in the ring $\mathbf{L}[y]$ then the polynomial $R(T) = \mathrm{Res}_y(T - f(y), g(y))$, where $\mathrm{Res}_y(-,-)$ denotes the resultant, is a power of an irreducible polynomial in $\mathbf{L}[T]$ (this includes the irreducible case).

Proof. Let $y_1, \ldots, y_m$ be the roots of $g(y)$ in the algebraic closure of the field $\mathbf{L}$. Then, $R(T) = \prod_{i=1}^{m}(T - f(y_i))$. Since $\mathbf{L}$ is a field of characteristic zero and $g(y)$ is irreducible, the Galois group of the field extension $\mathbf{L} \hookrightarrow \mathbf{L}(y_1, \ldots, y_m)$ acts transitively on the set $\{y_1, \ldots, y_m\}$. It follows that this group acts transitively on the set $\{f(y_1), \ldots, f(y_m)\}$. Hence, if $R = R_1 \cdots R_s$ is a factorization of $R = R(T)$ into irreducible monic polynomials in the ring $\mathbf{L}[T]$ then $R_i = R_j$ for $i \neq j$. ■

The next corollary will be used in the proof of the main result of the decompositions of higher polars, which is Theorem 9.1.

Corollary 2.6. Let $f(y)$, $g(y) \in \mathbf{K}[[\underline{x}]][y]$ be Weierstrass polynomials. If the resultant $\mathrm{Res}_y(g(y), f(y) - T) \in \mathbf{K}[[\underline{x}]][T]$ satisfies the assumptions of Lemma 2.2, then $g(y)$ is not irreducible in the ring $\mathbf{K}[[\underline{x}]][y]$.

Proof. By Lemma 2.2, the polynomial $R(T)$ has at least two coprime factors. By Theorem 2.5, $g(y)$, considered as a polynomial in $\mathbf{K}((\underline{x}))[y]$, is not irreducible, thus by Gauss Lemma it is not irreducible as a polynomial in $\mathbf{K}[[\underline{x}]][y]$. ■

Remark 2.7. Beata Hejmej in [11] generalizes Theorem 2.5 to polynomials with coefficients in a field of any characteristic. Hence, the results of this section hold for fields of arbitrary characteristic.

3 Kuo–Lu Tree of a Quasi-Ordinary Polynomial

From now on, $\mathbf{K}$ will be an algebraically closed field of characteristic zero. Let $f(y) \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial of degree n. Such a polynomial is *quasi-ordinary* if its y-discriminant equals $\underline{x}^{\mathbf{i}}u(\underline{x})$, where $u(\underline{x})$ is a unit in $\mathbf{K}[[\underline{x}]]$ and $\mathbf{i} \in \mathbf{N}^d$. After Jung–Abhyankar theorem (see [16, Theorem 1.3]) the roots of f are in the ring $\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$ of fractional power series and we may factorize $f(y)$ as $\prod_{i=1}^{n}(y-\alpha_i)$, where α_i is zero or a fractional power series of nonnegative order. Put $\mathrm{Zer} f := \{\alpha_i \,:\, 1 \le i \le n\}$. Since the differences of roots divide the discriminant, for $i \neq j$ we have

$$\alpha_i - \alpha_j = \underline{x}^{\mathbf{q}_{ij}} v_{ij}(\underline{x}), \quad \text{for some } \mathbf{q}_{ij} \in \mathbf{Q}^d \text{ and } v_{ij}(0) \neq 0. \tag{2}$$

The *contact* of α_i and α_j is by definition $O(\alpha_i, \alpha_j) := \mathbf{q}_{ij}$. By convention, $O(\alpha_i, \alpha_i) = +\infty$.

Remember that in $\mathbf{Q}^d_{\ge 0}$ we have the partial order: $\mathbf{q} \le \mathbf{q}'$ if $\mathbf{q}' - \mathbf{q} \in \mathbf{Q}^d_{\ge 0}$ and by convention $+\infty$ is bigger than any element of $\mathbf{Q}^d_{\ge 0}$. As usual, we write $\mathbf{q} < \mathbf{q}'$ when $\mathbf{q} \le \mathbf{q}'$ and $\mathbf{q} \neq \mathbf{q}'$.

After [1, Lemma 4.7], for every $\alpha_i, \alpha_j, \alpha_k \in \mathrm{Zer} f$ one has $O(\alpha_i, \alpha_k) \le O(\alpha_j, \alpha_k)$ or $O(\alpha_i, \alpha_k) \ge O(\alpha_j, \alpha_k)$.

Moreover, we have the *strong triangle inequality:*

$$O(\alpha_i, \alpha_j) \ge \min\{O(\alpha_i, \alpha_k), O(\alpha_j, \alpha_k)\}. \tag{STI}$$

The strong triangle inequality involves three values: $O(\alpha_i, \alpha_j)$, $O(\alpha_i, \alpha_k)$ and $O(\alpha_j, \alpha_k)$. It follows that two of them are equal and the third one is bigger or equal to the other two.

In general, we say that the *contact* between the fractional power series α and β is *well-defined* if and only if $\alpha - \beta = \underline{x}^{\mathbf{q}} w(\underline{x})$, for some $\mathbf{q} \in \mathbf{Q}^d$ and $w \in \mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$ such that $w(0) \neq 0$. In such a case, we put $O(\alpha, \beta) = \mathbf{q}$.

Now, we construct the *Kuo–Lu tree* of a quasi-ordinary Weierstrass polynomial f. Given $\mathbf{q} \in \mathbf{Q}^d_{\ge 0}$ we put $\alpha_i \equiv \alpha_j \bmod \mathbf{q}^+$ if $O(\alpha_i, \alpha_j) > \mathbf{q}$, for $\alpha_i, \alpha_j \in \mathrm{Zer} f$. Let $\mathbf{h}_0$ be the minimal contact between the elements of $\mathrm{Zer} f$. We represent $\mathrm{Zer} f$ as a horizontal bar B_0 and call $h(B_0)$ the *height* of B_0. The equivalence relation $\equiv \bmod h(B_0)^+$ divides $B_0 = \mathrm{Zer} f$

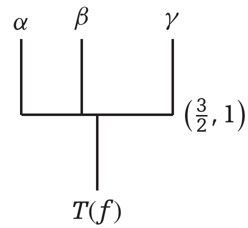

Fig. 3. The Kuo-Lu tree of $f(x, y) = (y^2 - x_1^3 x_2^2)(y - x_1^5 x_2^2)$.

into cosets $B_1, \dots, B_r$. We draw r vertical segments from the bar B_0 and at the end of the jth vertical segment, we draw a horizontal bar that represents B_j. The bar B_j is called a *post-bar* of B_0 and in such a situation we write $B_0 \perp B_j$. We repeat this construction recursively for every B_j with at least two elements. The set of bars ordered by the inclusion relation is a tree. Following [12], we call this tree the *Kuo–Lu tree* of f and denote it $T(f)$. The bar B_0 of minimal height is called the *root* of $T(f)$. For every bar B of $T(f)$, there exists a unique sequence $B_0 \perp B' \perp B'' \perp \cdots \perp B$, starting in B_0 and ending in B.

In the above construction, we do not draw the bars $\{\alpha_i\} \subset \mathrm{Zer} f$. These bars are the *leaves* of $T(f)$ and they are the only bars of infinite height.

Let $B, B' \in T(f)$ be such that $B \perp B'$. All fractional power series belonging to B' have the same term with the exponent $h(B)$. Let c be the coefficient of such term. We say that B' *is supported at* c on B and we denote it by $B \perp_c B'$. Observe that different post-bars of B are supported at different elements of the field $\mathbf{K}$.

This construction is adapted from [12] to quasi-ordinary case.

Example 3.1. Let $f = f_1 f_2 \in \mathbf{C}[[x_1, x_2]][y]$, where $f_1 = y^2 - x_1^3 x_2^2$ and $f_2 = y - x_1^5 x_2^2$. Observe that f is quasi-ordinary since its y-discriminant equals $4x_1^9 x_2^6(-1 + x_1^7 x_2^2)^2$. The roots of f are $\alpha = x_1^{3/2} x_2$, $\beta = -x_1^{3/2} x_2$ and $\gamma = x_1^5 x_2^2$. In Figure 3, we draw the Kuo–Lu tree of f.

In Figure 3, we draw also a vertical segment supporting $T(f)$ called by Kuo and Lu in [12] the *main trunk* of the tree.

4 Compatibility with Pseudo-Balls

Let $\alpha \in \mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$ be a fractional power series and $\mathbf{h} \in \mathbf{Q}^d_{\geq 0}$. The *pseudo-ball* centered in α and of height $\mathbf{h}$ is the set $\alpha + \underline{x}^{\mathbf{h}} \mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$. The *pseudo-ball centered in α of infinite height* is the set $\{\alpha\}$.

Let f be a quasi-ordinary polynomial. Consider the bar $B = \{\alpha_{i_1}, \dots, \alpha_{i_s}\}$ with finite height $\mathbf{h}$ of the Kuo–Lu tree $T(f)$. Set $\tilde{B} := \alpha + \underline{x}^{\mathbf{h}}\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$, where $\alpha \in B$. As $\alpha_{i_k} - \alpha_{i_l} \in \underline{x}^{\mathbf{h}}\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$ for $1 \leq k \leq l \leq s$ the pseudo-ball $\tilde{B}$ is independent of the choice of α. If $B = \{\alpha_i\}$ is a bar of infinite height, then we put $\tilde{B} = B$. The mapping $B \to \tilde{B}$ is a one-to-one correspondence between $T(f)$ and the set of pseudo-balls $\tilde{T}(f) := \{\alpha_i + (\alpha_i - \alpha_j)\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] : \alpha_i, \alpha_j \in \operatorname{Zer} f\}$. For the purposes of this article, it is easier to deal with pseudo-balls; hence from now on, we shall identify the elements of $T(f)$ with corresponding pseudo-balls. Such pseudo-balls will be called *quasi-ordinary pseudo-balls*.

Let $B = \alpha + \underline{x}^{h(B)}\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$ be a quasi-ordinary pseudo-ball of finite height. Every $\gamma \in B$ has a form $\gamma = \lambda_B(\underline{x}) + c_\gamma \underline{x}^{h(B)} + \cdots$, where $\lambda_B(\underline{x})$ is obtained from any $\beta \in B$ by omitting all the terms of order bigger than or equal to $h(B)$ and ellipsis means terms of order bigger than $h(B)$. We call the number c_γ the *leading coefficient of γ with respect to B* and denote it $\operatorname{lc}_B(\gamma)$. Remark that c_γ can be zero.

Let $\mathbf{L}$ be the field of fractions of $\mathbf{K}[[\underline{x}]]$. It follows from [10, Remark 2.3] that any truncation of a root of a quasi-ordinary polynomial is a root of a quasi-ordinary polynomial. Hence, the field extensions $\mathbf{L} \hookrightarrow \mathbf{L}(\lambda_B(\underline{x})) \hookrightarrow \mathbf{L}(\lambda_B(\underline{x}), \underline{x}^{h(B)})$ are algebraic and we can associate with B two numbers:

- the degree of the field extension $\mathbf{L} \hookrightarrow \mathbf{L}(\lambda_B(\underline{x}))$ that we will denote $N(B)$ and
- the degree of the field extension $\mathbf{L}(\lambda_B(\underline{x})) \hookrightarrow \mathbf{L}(\lambda_B(\underline{x}), \underline{x}^{h(B)})$ that we will denote $n(B)$.

In this section, we introduce the notion of *compatibility* of a Weierstass polynomial g with a pseudo-ball B. We define a polynomial $G_B(z)$ that will play an important role in the sequel.

Definition 4.1. Let $g(y) \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial and B be a pseudo-ball of finite height. If

$$g(\lambda_B(\underline{x}) + z\underline{x}^{h(B)}) = G_B(z)\underline{x}^{q(g,B)} + \cdots \tag{3}$$

for some $G_B(z) \in \mathbf{K}[z] \setminus \{0\}$ and some exponent $q(g, B) \in (\mathbb{Q}_{\geq 0})^d$, where ellipsis means terms of order bigger than $q(g, B)$, then we will say that g is compatible with B. The polynomial $G_B(z)$ will be called the B-characteristic polynomial of g.

Example 4.2. Return to Example 3.1. Let $B = \alpha + x_1^{3/2}x_2\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$ be a pseudo-ball of $T(f)$ of height $h(B) = \left(\frac{3}{2}, 1\right)$. Observe that

$$f(\lambda_B + z\underline{x}^{h(B)}) = f(z\underline{x}^{\left(\frac{3}{2},1\right)}) = z(z^2-1)x_1^{9/2}x_2^3 + \cdots$$

Hence, the polynomial f is compatible with the pseudo-ball B and its B-characteristic polynomial is $F_B(z) = z(z^2 - 1)$, but for example the polynomial $g(y) = y - x_1 - x_2$ is not compatible with B.

Our next goal is to prove in Corollary 4.5 that if a Weierstrass polynomial is compatible with a pseudo-ball then any factor of it is also compatible.

Lemma 4.3. Let $g(y) \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial and let B be a pseudo-ball of finite height. Consider $g(\lambda_B(\underline{x}) + z\underline{x}^{h(B)})$ as a fractional power series $\tilde{g}(\underline{x})$ with coefficients in $\mathbf{K}[z]$. Then, $g(y)$ is compatible with B if and only if the Newton polyhedron of $\tilde{g}(\underline{x})$ equals the Newton polyhedron of a monomial.

Proof. If g is compatible with B, then by (3) we get $\Delta\big(\tilde{g}(\underline{x})\big) = \Delta\big(\underline{x}^{q(g,B)}\big)$. Conversely, suppose that the Newton polyhedron of $\tilde{g}(\underline{x})$ equals the Newton polyhedron of the monomial $\underline{x}^{\mathbf{q}}$. Then, $\tilde{g}(\underline{x})$ has a form $\underline{x}^{\mathbf{q}} \sum_{i=0}^{n} a_i(\underline{x}) z^{n-i}$, where at least one of the values $a_i(0)$ is nonzero. Hence, the B-characteristic polynomial of g is $G_B(z) = \sum_{i=0}^{n} a_i(0) z^{n-i}$. ■

Remark 4.4. From the proof of Lemma 4.3, we get that $\tilde{g}(\underline{x})$ has the form $G_B(z)\underline{x}^{q(g,B)} + \sum_{h > q(g,B)} a_h(z)\underline{x}^h$, where $a_h(z) \in \mathbf{K}[z]$.

Corollary 4.5. Let $g \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial compatible with a pseudo-ball B. Then, any factor of g is compatible with B. Moreover, if $g = g_1 g_2$ then $G_B(z) = (G_1)_B(z)(G_2)_B(z)$.

Proof. The Newton polyhedron of the product is the Minkowki sum of Newton polyhedra of the factors. Hence, if $\Delta(\tilde{g}) = \Delta(\underline{x}^{\mathbf{q}})$ and $\tilde{g} = \tilde{g}_1\tilde{g}_2$ then $\Delta(\tilde{g}_i)$ have the form $\Delta(\underline{x}^{\mathbf{q}_i})$ for some $\mathbf{q}_1, \mathbf{q}_2$ such that $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$. The equality $G_B(z) = (G_1)_B(z)(G_2)_B(z)$ follows from Remark 4.4. ■

Next lemma generalizes to d variables [8, Lemma 3.1].

Lemma 4.6. Let $f(y) \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial of degree n compatible with the pseudo-ball B. Then, for every $k \in \{1, \ldots, \deg F_B(z)\}$ the Weierstrass polynomial

$g(y) = \frac{(n-k)!}{n!}\frac{d^k}{dy^k}f(y)$ is also compatible with B and its B-characteristic polynomial is $G_B(z) = \frac{(n-k)!}{n!}\frac{d^k}{dz^k}F_B(z)$.

Proof. Differentiating identity $f(\lambda_B(\underline{x}) + z\underline{x}^{h(B)}) = F_B(z)\underline{x}^{q(f,B)} + \cdots$ with respect to z we get $f'(\lambda_B(\underline{x}) + z^{h(B)})\underline{x}^{h(B)} = F'_B(z)\underline{x}^{q(f,B)} + \cdots$. Hence, $f'(\lambda_B(\underline{x}) + z\underline{x}^{h(B)}) = F'_B(z)\underline{x}^{q(f,B)-h(B)} + \cdots$, which proves the lemma for $k = 1$. The proof for higher derivatives runs by induction on k. ■

Let $f(y) \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial of degree n. The Weierstrass polynomial $\frac{(n-k)!}{n!}\frac{d^k}{dy^k}f(y)$ of Lemma 4.6 will be called the *normalized kth derivative* of the Weierstrass polynomial $f(y) \in \mathbf{K}[[\underline{x}]][y]$ and we will denote it by $f^{(k)}(y)$. The variety of equation $f^{(k)} = 0$ is called the *kth polar* of $f = 0$. Since the normalized nth derivative of f is constant, in the rest of the paper we consider normalized kth derivatives of f for $1 \le k < \deg f$.

Lemma 4.7 Let $f(y) \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial and let B be a pseudo-ball of finite height.

(1) If f is compatible with B, then for any $\gamma \in B$ we have

$$f(\gamma) = F_B(\mathrm{lc}_B\gamma)\underline{x}^{q(f,B)} + \cdots \tag{4}$$

(2) If $f(y) = \prod_{i=1}^n (y - \alpha_i)$ and we assume that one of the following holds: $\underline{x}$ is a single variable and B is arbitrary or f is quasi-ordinary and $B \in \tilde{T}(f)$ then f is compatible with B and we have

$$F_B(z) = \mathrm{const} \prod_{i:\alpha_i \in B} (z - \mathrm{lc}_B\alpha_i) \tag{5}$$

and

$$q(f, B) = \sum_{i=1}^n \min(O(\lambda_B, \alpha_i), h(B)). \tag{6}$$

Proof. Since $\gamma \in B$ we can write $\gamma = \lambda_B(\underline{x}) + \tilde{\gamma}(\underline{x})\underline{x}^{h(B)}$, where $\tilde{\gamma}(\underline{x}) = \mathrm{lc}_B(\gamma) + \cdots$. By Remark 4.4, we have $f(\gamma) = f(\lambda_B(\underline{x}) + \tilde{\gamma}(\underline{x})\underline{x}^{h(B)}) = F_B(\tilde{\gamma}(\underline{x}))\underline{x}^{q(f,B)} + \cdots = F_B(\mathrm{lc}_B\gamma)\underline{x}^{q(f,B)} + \cdots$. This proves (4).

Suppose that $\gamma = \lambda_B(\underline{x}) + e\underline{x}^{h(B)}$, where e is a constant. We have $f(\gamma) = \prod_{i=1}^n (\gamma - \alpha_i)$. In order to prove (5) and (6), it is enough to compute the initial term of every factor $\gamma - \alpha_i$.

If $\alpha_i \in B$, then the initial term of $\gamma - \alpha_i$ equals $(e - \mathrm{lc}_B \alpha_i)\underline{x}^{h(B)}$. Otherwise, the initial terms of $\gamma - \alpha_i$ and $\lambda_B - \alpha_i$ are equal. We finish the proof multiplying the initial terms. ■

Corollary 4.8. Let $f(y) \in \mathbf{K}[[\underline{x}]][y]$ be a quasi-ordinary Weierstrass polynomial. Then, every factor of $f(y)$ is compatible with all pseudo-balls $B \in T(f)$ of finite height.

Lemma 4.9. Let $f(y) \in \mathbf{K}[[\underline{x}]][y]$ be a quasi-ordinary Weierstrass polynomial, $p(y)$ be a factor of $f(y)$ and B, B' be bars of finite heights in $T(f)$ such that $B \perp B'$. Then,

$$q(p,B') - q(p,B) = \sharp(\operatorname{Zer} p \cap B')[h(B') - h(B)].$$

Proof. Consider B and B' as pseudo-balls. Put $p(y) = \prod_{\alpha \in \operatorname{Zer} p}(y - \alpha)$. Let $\gamma \in B, \gamma' \in B'$ be such that $O(\gamma, \alpha) = h(B)$ for all $\alpha \in B \cap \operatorname{Zer} p$ and $O(\gamma', \alpha) = h(B')$ for all $\alpha \in B' \cap \operatorname{Zer} p$. By the STI, we get $O(\gamma', \alpha) = O(\gamma, \alpha)$ for any $\alpha \in \operatorname{Zer} p \backslash B'$. If $\alpha \in \operatorname{Zer} p \cap B'$ then $O(\gamma, \alpha) = h(B)$ and $O(\gamma', \alpha) = h(B')$. Hence,

$$\begin{aligned} q(p,B') - q(p,B) &= \sum_{\alpha \in \operatorname{Zer} p} O(\gamma', \alpha) - \sum_{\alpha \in \operatorname{Zer} p} O(\gamma, \alpha) \\ &= \sharp(\operatorname{Zer} p \cap B')[h(B') - h(B)]. \end{aligned}$$

■

Lemma 4.9 is similar in spirit to [9, Lemma 2.7].

Lemma 4.10. Let B be a quasi-ordinary pseudo-ball and let $g(y) \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial compatible with B. Then,

(1) $G_B(z) = z^k \cdot H(z^{n(B)})$, for some $k \in \mathbf{N}$ and $H(z) \in \mathbf{K}[z]$.
(2) If g is irreducible and quasi-ordinary then $G_B(z) = az^k$ or $G_B(z) = a(z^{n(B)} - c^{n(B)})^l$, for some nonzero $a, c \in \mathbf{K}$ and some $l \in \mathbf{N}$.

Proof. Remember that $\mathbf{L}$ is the field of fractions of $\mathbf{K}[[\underline{x}]]$. By [14, Lemma 5.7] and [10, Remark 2.7] the algebraic extension $\mathbf{L}(\lambda_B(\underline{x})) \hookrightarrow \mathbf{L}(\lambda_B(\underline{x}), \underline{x}^{h(B)})$ is cyclic. Hence, the generator φ of the group $Gal(\mathbf{L}(\lambda_B(\underline{x})) \hookrightarrow \mathbf{L}(\lambda_B(\underline{x}), \underline{x}^{h(B)}))$ acts as follows: $\varphi(\lambda_B(\underline{x})) = \lambda_B(\underline{x})$ and $\varphi(\underline{x}^{h(B)}) = \omega \underline{x}^{h(B)}$, where ω is a primitive $n(B)$th root of the unity. Applying φ to (3) we get

$$g(\lambda_B(\underline{x}) + z\omega x^{h(B)}) = G_B(z)\omega^k x^{q(g,B)} + \cdots \tag{7}$$

for some $0 \leq k < n(B)$. Substituting ωz for z in (3) and comparing with (7) we get $G_B(z)\omega^k = G_B(\omega z)$. Multiplying this equality by $(\omega z)^{n(B)-k}$ and putting $W(z) :=$

$z^{n(B)-k}G_B(z)$ we obtain $W(z) = W(\omega z)$. This implies that $W(z) = \overline{W}(z^{n(B)})$, for some $\overline{W}(z) \in \mathbf{K}[z]$. We finish the proof putting $H(z^{n(B)}) = z^{-n(B)}\overline{W}(z^{n(B)})$. This proves the first part of the lemma.

Suppose now that g is irreducible and quasi-ordinary. Let $\gamma = \lambda_B(\underline{x})+c\underline{x}^{h(B)}+\cdots \in B \cap \operatorname{Zer} g$. Since the extension $\mathbf{L}(\lambda_B(\underline{x})) \hookrightarrow \mathbf{L}(\lambda_B(\underline{x}), \underline{x}^{h(B)}) \hookrightarrow \mathbf{L}(\gamma)$ is Galois, any other root of g belonging to B has the form $\lambda_B(\underline{x}) + \omega^i c\underline{x}^{h(B)} + \cdots$, for some $0 \leq i < n(B)$. Using the first part of the lemma and the equality (5) we complete the proof. ■

5 Conjugate Pseudo-Balls

In this section, we define an equivalence relation between pseudo-balls called *conjugacy relation*. This will allow us to introduce, in Section 8, the notion of the Eggers tree of a quasi-ordinary Weierstrass polynomial.

Let $\mathbf{M}$ be the field of fractions of $\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$. Observe that $\mathbf{M}$ is a field extension of $\mathbf{L}$ (the field of fractions of $\mathbf{K}[[\underline{x}]]$).

Lemma 5.1. Let φ be an $\mathbf{L}$-automorphism of $\mathbf{M}$. Then,

(1) for any $\mathbf{q} \in \mathbf{Q}^d$ there exists a root ω of the unity such that $\varphi(\underline{x}^{\mathbf{q}}) = \omega \cdot \underline{x}^{\mathbf{q}}$,
(2) $\varphi(\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]) = \mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$,
(3) if u is a unit of the ring $\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$ and $\mathbf{q} \in (\mathbf{Q}_{\geq 0})^d$ then $\varphi(u \cdot \underline{x}^{\mathbf{q}}) = \tilde{u} \cdot \underline{x}^{\mathbf{q}}$ for some unit $\tilde{u} \in \mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$.

Proof. Let k be a positive integer. Observe that $x_i = \varphi(x_i) = \varphi\big((x_i^{1/k})^k\big) = \varphi(x_i^{1/k})^k$. Hence, $\varphi(x_i^{1/k}) = c \cdot x_i^{1/k}$ for some $c \in \mathbf{K}\setminus\{0\}$ such that $c^k = 1$. It follows that for any $\mathbf{q} \in \mathbf{Q}^d$ there exists $\omega \in \mathbf{K}$ such that $\varphi(\underline{x}^{\mathbf{q}}) = \omega\underline{x}^{\mathbf{q}}$ and $\omega^m = 1$ for some positive integer m.

Every element of the ring $\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$ can be represented as a finite sum $\sum_{\mathbf{q}} a_{\mathbf{q}}\underline{x}^{\mathbf{q}}$ where $\mathbf{q} = (q_1, \ldots, q_d) \in (\mathbf{Q}_{\geq 0})^d$ $(0 \leq q_i < 1)$ and $a_{\mathbf{q}} \in \mathbf{K}[[\underline{x}]]$. This together with (1) proves items (2) and (3) of the lemma. ■

Let B, B' be pseudo-balls. We say that B and B' are *conjugate* if there exists an $\mathbf{L}$-automorphism φ of $\mathbf{M}$ such that $B' = \varphi(B)$. The conjugacy of pseudo-balls is an equivalence relation. It follows from Lemma 5.1 that conjugate pseudo-balls have the same height. Moreover, two quasi-ordinary pseudo-balls B and B' of the same height are conjugate if any irreducible quasi-ordinary polynomial that has one of its roots in B has another root in B' (in this way conjugate bars were defined in [13, Definition 6.1]). If $B' = \varphi(B)$ then $\lambda_{B'} = \varphi(\lambda_B)$. The converse is also true; if $h(B) = h(B')$ and there exists an $\mathbf{L}$-automorphism φ of $\mathbf{M}$ such that $\lambda_{B'} = \varphi(\lambda_B)$ then B and B' are conjugate. It follows from the above that the number of pseudo-balls conjugate with B is equal to the degree of the minimal polynomial of λ_B, which is the degree $N(B)$ of the field extension $\mathbf{L} \hookrightarrow \mathbf{L}(\lambda_B(\underline{x}))$.

Lemma 5.2. Let B, B' be quasi-ordinary conjugate pseudo-balls. If $p(y) \in \mathbf{K}[[\underline{x}]][y]$ is a Weierstrass polynomial compatible with B then

(1) $p(y)$ is compatible with B'.
(2) $q(p, B) = q(p, B')$.
(3) The characteristic polynomials $P_{B'}(z)$ and $P_B(z)$ of $p(y)$ verify the equality $P_{B'}(z) = \theta P_B(\omega z)$, for some roots of the unity θ and ω.

Proof. Let φ be a $\mathbf{L}$-automorphism of $\mathbf{M}$ such that $\varphi(B) = B'$. Then, $\varphi(\lambda_B) = \lambda_{B'}$. By Lemma 5.1 we have $\varphi(\underline{x}^{h(B)}) = \omega^{-1}\underline{x}^{h(B)}$ and $\varphi(\underline{x}^{q(p,B)}) = \theta\underline{x}^{q(p,B)}$ for some roots of the unity θ and ω. Applying φ to (3), with g replaced by p, we get

$$p(\lambda_{B'} + z\omega^{-1}\underline{x}^{h(B)}) = P_B(z)\theta\underline{x}^{q(p,B)} + \cdots$$

This gives $q(p, B) = q(p, B')$ and $P_{B'}(\omega^{-1}z) = \theta P_B(z)$. ∎

6 Kuo–Lu Lemma for Higher Derivatives

Let $f(y) \in \mathbf{K}[[\underline{x}]][y]$ be a quasi-ordinary Weierstrass polynomial. We begin with combinatorial results concerning the Kuo–Lu tree $T(f)$. Remember that we identify any bar of $T(f)$ with the corresponding quasi-ordinary pseudo-ball. At the end of the section, we apply these results to Newton–Puiseux roots of higher derivatives of $f(y)$.
Take an integer k such that $1 \leq k \leq \deg f$. With every bar B of $T(f)$ we associate the numbers

- $m(B)$ that is the number of roots of $f(y)$ that belong to B,
- $n_k(B) = \max\{m(B) - k, 0\}$, and
- $t_k(B) = n_k(B) - \sum_{B \perp B'} n_k(B')$.

The numbers $m(B)$, $n_k(B)$ and $t_k(B)$ depend not only on B but also on $T(f)$. Moreover, by Lemma 4.7 one has $m(B) = \deg F_B(z)$.

Remark 6.1. For $1 \leq k < m(B)$ we have $n_k(B) > 0$, $t_k(B) > 0$ and for $m(B) \leq k \leq \deg f$ we have $n_k(B) = t_k(B) = 0$.

We denote by $T_k(f)$ the sub-tree of $T(f)$ consisting of the bars $B \in T(f)$ such that $m(B) \geq k$. Let $F \in \mathbf{K}[z]$ be a non-constant polynomial. Let $F^{(k)}$ denotes the kth derivative of F.

Definition 6.2. We will say that F is *k-regular* if one of the following conditions holds:

(1) $F^{(k)}$ is zero or

(2) $F^{(k)}$ is nonzero and there is not a root of F of multiplicity $\leq k$ that is a root of $F^{(k)}$.

Recall that common roots of a polynomial F and its first derivative are multiple roots of F. Hence, any polynomial is 1-regular.

In general, it is not easy to verify the k-regular property. In this paper, polynomials of the form

$$F(z) = (z^n - c)^l \in \mathbf{K}[z] \tag{8}$$

play an important role. Their properties are described in the following lemma, which was proved in [8, Lemma 5.3] for complex polynomials but by Lefschetz Principle it holds true for polynomials over any algebraically closed field of characteristic zero.

Lemma 6.3. Let $\mathbf{K}$ be an algebraically closed field of characteristic zero. If $F(z) = (z^n - c)^e \in \mathbf{K}[z]$ with $c \neq 0$ then for $1 \leq k < \deg F(z)$ one has $\frac{d^k}{dz^k}F(z) = Cz^a(z^n - c)^b \prod_{i=1}^{d}(z^n - c_i)$, where $C \neq 0$ and

(1) $0 \leq a < n$ and $a + k \equiv 0 \pmod{n}$;

(2) $b = \max\{e - k, 0\}$;

(3) $d = \min\{e, k\} - \lceil \frac{k}{n} \rceil$, where $\lceil x \rceil$ denotes the smallest integer bigger than or equal to x;

(4) $c_i \neq c_j$ for $1 \leq i < j \leq d$ and $0 \neq c_i \neq c$ for $1 \leq i \leq d$.

As a consequence we get the following:

Corollary 6.4. The polynomial $F(z)$ as in (8) is k-regular for any k.

Remark 6.5. Let $F(z) = \text{const} \prod_{i=1}^{r}(z - z_i)^{m_i}$, where z_i are pairwise different, $m_i \geq k$ for $1 \leq i \leq s$ and $m_i < k$ for $s < i \leq r$. After differentiating, the multiplicity of any root drops by one. Hence, putting $F^{\lceil k\rceil}(z)=\prod_{i=1}^{s}(z-z_i)^{m_i-k}$ we obtain the decomposition

$$F^{(k)}(z)=F^{\lceil k\rceil}(z)F^{\lfloor k\rfloor}(z) \tag{9}$$

into two coprime polynomials. A polynomial F is k-regular if and only if F and $F^{\lfloor k\rfloor}$ do not have common roots.

Definition 6.6. Let $f\in \mathbf{K}[[\underline{x}]][y]$ be a quasi-ordinary Weierstrass polynomial. We say that f is Kuo–Lu k-regular if for every $B\in T(f)$ of finite height the polynomial $F_B(z)$ is k-regular.

We finish this subsection with some results for Weierstrass polynomials with coefficients in the ring of the formal power series in one variable.

Let $f(y)\in \mathbf{K}[[x]][y]$ be a square-free Weierstrass polynomial. Fix $B\in T_k(f)$ and assume that $\{B_1,\dots,B_s\}$ is the set of post-bars of B in $T_k(f)$. Denote $B^\circ=B\setminus(B_1\cup\dots\cup B_s)$.

Theorem 6.7. Let $f(y)\in \mathbf{K}[[x]][y]$ be a square-free Weierstrass polynomial over the ring of formal power series in one variable. Let $f(y)=\prod_{i=1}^{n}(y-\alpha_i)$ and $f^{(k)}(y)=\prod_{j=1}^{n-k}(y-\beta_j)$ be the Newton–Puiseux factorizations of f and $f^{(k)}$. Then,

(1) for every $B\in T_k(f)$ the set $\{j:\beta_j\in B\}$ has $n_k(B)$ elements;
(2) for every $B\in T_k(f)$ the set $\{j:\beta_j\in B^\circ\}$ has $t_k(B)$ elements;
(3) for every β_j there exists a unique $B\in T_k(f)$ such that $\beta_j\in B^\circ$;
(4) let $B\in T_k(f)$. If the polynomial $F_B(z)$ is k-regular, then for every $\alpha_i\in B$, $\beta_j\in B^\circ$ one has $O(\alpha_i,\beta_j)=h(B)$. Otherwise, there exist $\alpha_i\in B$, $\beta_j\in B^\circ$ such that $O(\alpha_i,\beta_j)>h(B)$.

Proof. *Proof of* (1). Suppose first that $B\in T_k(f)$ has finite height. Then, by Lemma 4.7 $F_B(z)=\text{const}\prod_{i:\alpha_i\in B}(z-\mathrm{lc}_B(\alpha_i))$. By equality (4) of this lemma and Lemma 4.6 we get $F_B^{(k)}(z)=\text{const}\prod_{j:\beta_j\in B}(z-\mathrm{lc}_B(\beta_j))$. Hence, the set $\{j:\beta_j\in B\}$ has $\deg F_B-k=n_k(B)$ elements.

If the height of B is infinite then $B=\{\alpha_i\}$ for exactly one Newton–Puiseux root α_i of $f(y)$. Hence, for $k=1$ $n_1(B)=0$ and $f'(y)$ does not have roots in B since f is square-free, while for $k>1$, $B\notin T_k(f)$.

Proof of (2). It is enough to count the elements of the set $\{j:\beta_j\in B^\circ\}$ using (1).

Proof of (3). Let B_0 be the root of the tree $T(f)$. By (1), $\{\beta_1, \dots, \beta_{n-k}\}$ is a subset of B_0. It is clear that the sets B° for $B \in T_k(f)$ are pairwise disjoint and their union is equal to B_0. This proves (3).

Proof of (4). Assume that $B_1, \dots, B_r$ are the post-bars of B supported at $z_1, \dots, z_r$, respectively, and that $m(B_i) \geq k$ for $i \in \{1, \dots s\}$, $m(B_i) < k$ for $i \in \{s+1, \dots r\}$. Then, by Lemma 4.7 $F_B(z) = \prod_{i=1}^{r} (z - z_i)^{m(B_i)}$.

After Remark 6.5 the kth derivative of $F_B(z)$ is the product of two coprime polynomials

$$F_B^{(k)}(z) = F_B^{\lceil k \rceil}(z) F_B^{\lfloor k \rfloor}(z),$$

where $F_B^{\lceil k \rceil}(z) := \prod_{i=1}^{s} (z - z_i)^{n_k(B_i)}$.

We get $\deg F_B^{\lfloor k \rfloor}(z) = t_k(B)$. Hence, it follows from (2) and (3) that all roots of $F_B^{\lfloor k \rfloor}(z)$ correspond to those Newton–Puiseux roots of $f^{(k)}(y)$ that belong to B°. For $\alpha_i \in B$, $\beta_j \in B^\circ$ one has $O(\alpha_i, \beta_j) > h(B)$ if and only if $\mathrm{lc}_B(\alpha_i) = \mathrm{lc}_B(\beta_j)$, which means that the polynomials $F_B(z)$ and $\deg F_B^{\lfloor k \rfloor}(z)$ have a common root. Since $F_B(z)$ is k-regular if and only if $\deg F_B^{\lfloor k \rfloor}(z)$ and $F_B(z)$ do not have common roots we get (4). ■

Remark 6.8. Let $f(y) \in \mathbf{K}[[x]][y]$ be a square-free Weierstrass polynomial over the ring of formal power series in one variable. Let $B \in T_k(f)$, $\beta_i \in B \cap \mathrm{Zer} f^{(k)}$ and put $c = \mathrm{lc}_B \beta_i$. Then, $F_B^{\lceil k \rceil}(c) \neq 0$ if and only if $\beta_i \in B^\circ$. If $F_B^{\lceil k \rceil}(c) = 0$ then there exists a sequence of post-bars $B \perp_c B_1 \perp \cdots \perp B_l$ such that $\beta_i \in B_l^\circ$ and $B_l \in T_k(f)$.

For Weierstrass polynomials that are Kuo–Lu k-regular, the counterpart of [12, Lemma 3.3] is true:

Corollary 6.9. Let $f(y) \in \mathbf{K}[[x]][y]$ be a square-free Weierstrass polynomial over the ring of formal power series in one variable. Assume that f is Kuo–Lu k-regular. Then, under assumptions and notations of Theorem 6.7; for every $\alpha_i \in \mathrm{Zer} f$, $\beta_s \in \mathrm{Zer} f^{(k)}$ there exists $\alpha_j \in \mathrm{Zer} f$ such that $O(\alpha_i, \beta_s) = O(\alpha_i, \alpha_j)$.

7 Newton Polyhedra of Resultants

In this section, we give a formula for the Newton polyhedron of the resultant $\mathrm{Res}_y(f^{(k)}(y), p(y) - T)$, where $f(y)$ is a Kuo–Lu k-regular quasi-ordinary Weierstrass polynomial, $p(y)$ is a factor of $f(y)$ and T is a new variable. We prove that for irreducible $p(y)$, the Newton polyhedron of the resultant is polygonal. The particular case of this result for $k = 1$ and $p(y) = f(y)$ was proved in [7, Theorem 4.1].

7.1 Monomial substitutions

Let $g(\underline{x}, y) \in \mathbf{K}[[\underline{x}, y]]$. For any monomial substitution $x_1 = u^{r_1}, \dots, x_d = u^{r_d}$, where r_i are positive integers, we put

$$\bar{g}^{[\mathbf{r}]}(u, y) := g(u^{r_1}, \dots, u^{r_d}, y). \tag{10}$$

We will write simply $\bar{g}(u, y)$ when no confusion can arise.

Observe that for $g = \underline{x}^{\mathbf{s}}$ we get $\bar{g}^{[\mathbf{r}]} = u^{\langle \mathbf{r}, \mathbf{s} \rangle}$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product.

Lemma 7.1. Let $f(y) \in \mathbf{K}[[\underline{x}]][y]$ be a quasi-ordinary Weierstrass polynomial. There is a one-to-one correspondence between the bars of $T(f)$ and the bars of $T(\bar{f}^{[\mathbf{r}]})$. If B and $\bar{B}$ are the corresponding bars of $T(f)$ and $T(\bar{f}^{[\mathbf{r}]})$, respectively, then

(1) $h(\bar{B}) = \langle \mathbf{r}, h(B) \rangle$ and $t_k(\bar{B}) = t_k(B)$;

(2) for any factor g of f, the B-characteristic polynomial of g and the $\bar{B}$-characteristic polynomial of $\bar{g}^{[\mathbf{r}]}$ are equal and $q(\bar{g}^{[\mathbf{r}]}, \bar{B}) = \langle \mathbf{r}, q(g, B) \rangle$.

Proof. Set $u^{\mathbf{r}} = (u^{r_1}, \dots, u^{r_d})$. If $\mathrm{Zer} f = \{\alpha_i(\underline{x})\}_{i=1}^n$ then $\mathrm{Zer} \bar{f}^{[\mathbf{r}]} = \{\alpha_i(u^{\mathbf{r}})\}_{i=1}^n$ and $O(\alpha_i(u^{\mathbf{r}}), \alpha_j(u^{\mathbf{r}})) = \langle \mathbf{r}, O(\alpha_i(\underline{x}), \alpha_j(\underline{x})) \rangle$ for $i \neq j$.
Hence, every bar $B = \{\alpha_{i_j}(\underline{x})\}_{j=1}^k$ of $T(f)$ yields the bar $\bar{B} = \{\alpha_{i_j}(u^{\mathbf{r}})\}_{j=1}^k$ of $T(\bar{f}^{[\mathbf{r}]})$ of height $\langle \mathbf{r}, h(B) \rangle$.

Substituting u^{r_i} for x_i in the equation (3) appearing in Definition 4.1, we get

$$\bar{g}^{[\mathbf{r}]}(\lambda_{\bar{B}}(u) + z u^{h(\bar{B})}) = G_B(z) u^{\langle \mathbf{r}, q(g, B) \rangle} + \cdots,$$

hence the second part of the lemma follows. ■

Example 7.2. Let $f = (y^2 - x_1)^2 - x_1^2 x_2 \in \mathbf{K}[[x_1, x_2]][y]$. This polynomial is quasi-ordinary and irreducible. Its roots live in the ring $\mathbf{K}[[x_1^{1/2}, x_2^{1/2}]]$. After any monomial substitution $x_1 = u^{r_1}$, $x_2 = u^{r_2}$, the roots of $\bar{f}$ are in the ring $\mathbf{K}[[u^{1/2}]]$. The degree of the fields extension of the fractions fields of $\mathbf{K}[[u^{1/2}]]$ over the field of fractions of $\mathbf{K}[[u]]$ is 2, which is strictly less than the degree of $\bar{f}$. Hence, $\bar{f}$ is not irreducible over the field of fractions of $\mathbf{K}[[u]]$. Consequently, in general, the irreducibility is not preserved by monomial substitutions.

The proof of the next lemma is similar in spirit to the proof of [7, Theorem 4.1] and the proof of [7, Theorem 9.2]. The same arguments were used there in special situation. Here, we repeat the proof for the convenience of the reader.

Lemma 7.3. Let $g(\underline{x}, y) \in \mathbf{K}[[\underline{x}, y]]$ and $\Delta \subseteq \mathbf{R}^{d+1}$ be a Newton polyhedron. For any $\mathbf{r} \in (\mathbf{R}_{>0})^d$ let $\bar{\Delta}^{[\mathbf{r}]}$ be the image of Δ by the linear mapping $\pi_{\mathbf{r}} : \mathbf{R}^d \times \mathbf{R} \longrightarrow \mathbf{R}^2$ given by $(\mathbf{a}, b) \mapsto (\langle \mathbf{r}, \mathbf{a} \rangle, b)$. If $\Delta(\bar{g}^{[\mathbf{r}]}) = \bar{\Delta}^{[\mathbf{r}]}$ for every $\mathbf{r} \in (\mathbf{N} \backslash \{0\})^d$, then $\Delta(g) = \Delta$.

Proof. For every Newton polyhedron $\Delta \subseteq (\mathbf{R}_{\geq 0})^{d+1}$ and every $v \in (\mathbf{R}_{\geq 0})^{d+1}$ we define the *support function* $l(v, \Delta) = \min\{\langle v, \alpha \rangle \ : \ \alpha \in \Delta\}$. To prove the lemma, it is enough to show that the support functions $l(\cdot, \Delta(g))$ and $l(\cdot, \Delta)$ are equal. As these functions are continuous, it suffices to show the equality on a dense subset of $\mathbf{R}^{d+1}_{\geq 0}$.

Let $\vec{r} = (r_1, \dots, r_{d+1}) = (\mathbf{r}, r_{d+1}) \in \mathbf{R}^{d+1}_{\geq 0}$, where $\mathbf{r} = (r_1, \dots, r_d)$.

Perturbing $\vec{r}$ a little we may assume that the hyperplane $\{\alpha \in \mathbf{R}^{d+1} : \langle \vec{r}, \alpha \rangle = l(\vec{r}, \Delta(g)\}$ supports $\Delta(g)$ at exactly one point $\check{\alpha} = (\underline{\check{\alpha}}, \check{\alpha}_{d+1})$. Since after a small change of $\vec{r}$ the support point remains the same, we can assume, perturbing $\vec{r}$ again if necessary, that all r_i are positive rational numbers.

We will show that

$$l(\vec{r}, \Delta) = l(\vec{r}, \Delta(g)). \tag{11}$$

Multiplying $\vec{r}$ by the common denominator of $r_1, \dots, r_{d+1}$ we may assume that all r_i are positive integers. At this point of the proof, we fix $\vec{r}$. We claim that $l(\vec{r}, \Delta) = l\big((1, r_{d+1}), \bar{\Delta}^{[\mathbf{r}]}\big)$ and $l(\vec{r}, \Delta(g)) = l\big((1, r_{d+1}), \Delta\left(\bar{g}^{[\mathbf{r}]}\right)\big)$.

The first equality follows from the definition of $\pi_{\mathbf{r}}$ and the identity

$$\langle \vec{r}, \alpha \rangle = \langle (1, r_{d+1}), \pi_{\mathbf{r}}(\alpha) \rangle$$

for $\alpha \in \mathbf{R}^{d+1}$.

Write $\alpha = (\underline{\alpha}, \alpha_{d+1}) \in \mathbf{R}^{d+1}$ and $g(\underline{x}, y) = \sum_\alpha d_\alpha \underline{x}^{\underline{\alpha}} y^{\alpha_{d+1}} \in \mathbf{K}[[\underline{x}, y]]$. Since the hyperplane $\{\alpha \in \mathbf{R}^{d+1} : \langle \vec{r}, \alpha \rangle = l(\vec{r}, \Delta(g)\}$ supports $\Delta(g)$ at $\check{\alpha}$, the term $d_{\check{\alpha}} u^{\langle \mathbf{r}, \underline{\check{\alpha}} \rangle} y^{\check{\alpha}_{d+1}}$ of $\bar{g}^{[\mathbf{r}]}$, satisfies the equality $\langle \mathbf{r}, \underline{\check{\alpha}} \rangle + r_{d+1}\check{\alpha}_{d+1} = l(\vec{r}, \Delta(g))$, while for all other terms $d_\alpha u^{\langle \mathbf{r}, \underline{\alpha} \rangle} y^{\alpha_{d+1}}$ with $d_\alpha \neq 0$ appearing in $\bar{g}^{[\mathbf{r}]}$, we have $\langle \mathbf{r}, \underline{\alpha} \rangle + r_{d+1}\alpha_{d+1} > l(\vec{r}, \Delta(g))$.

Hence, $l\big((1, r_{d+1}), \Delta(g)\big) = \langle \mathbf{r}, \underline{\check{\alpha}} \rangle + r_{d+1}\check{\alpha}_{d+1} = l(\vec{r}, \Delta(g))$, so we get (11). ■

Corollary 7.4. Let $g_1(\underline{x}, y), g_2(\underline{x}, y) \in \mathbf{K}[[\underline{x}, y]]$. Suppose that $\Delta(\bar{g}_1^{[\mathbf{r}]}) = \Delta(\bar{g}_2^{[\mathbf{r}]})$ for every $\mathbf{r} \in (\mathbf{N} \backslash \{0\})^d$. Then, $\Delta(g_1) = \Delta(g_2)$.

Theorem 7.5. Assume that $f \in \mathbf{K}[[\underline{x}]][y]$ is a Kuo–Lu k-regular quasi-ordinary Weierstrass polynomial and p is a Weierstrass polynomial that is a factor of f in $\mathbf{K}[[\underline{x}]][y]$.

Then, the Newton polyhedron of $R(T) := \mathrm{Res}_{\,y}(f^{(k)}(y), p(y) - T) \in \mathbf{K}[[\underline{x}]][T]$ is equal to

$$\sum_{\substack{B\in T(f)\\ t_k(B)\neq 0}} \left\{ \frac{t_k(B)q(p,B)}{t_k(B)} \right\}. \tag{12}$$

Proof. First, we will prove the theorem for $d = 1$. We use the notation of Theorem 6.7. Let $\prod_{j=1}^{n-k}(y - \beta_j)$ be the Newton–Puiseux factorization of $f^{(k)}(y)$. By the well-known properties of the resultants we have

$$\mathrm{Res}_{\,y}(f^{(k)}(y), p(y) - T) = \pm \prod_{j=1}^{n-k}(p(\beta_j) - T). \tag{13}$$

By Theorem 6.7, for every β_j there exists a unique bar $B \in T(f)$ such that $\beta_j \in B^\circ$. For such a bar, $h(B)$ is finite and $t_k(B) \neq 0$. By Corollary 4.5, the polynomial p is compatible with B and by (5) of Lemma 4.7 $P_B(z)$ is a factor of $F_B(z)$. By Theorem 6.7 (4), we get that $O(\alpha_i, \beta_j) = h(B)$ for any $\alpha_i \in B$. Hence, $\mathrm{lc}_B\beta_j$ does not belong to the set $\{\mathrm{lc}_B\alpha_i \ : \ \alpha_i \in B\}$. So by the equality (5) in Lemma 4.7, we have $F_B(\mathrm{lc}_B\beta_j) \neq 0$ and consequently $P_B(\mathrm{lc}_B\beta_j) \neq 0$. Now, using equality (4) of Lemma 4.7 we conclude that the Newton polyhedron of $p(\beta_j) - T$ is equal to $\left\{\frac{q(p,B)}{1}\right\}$. Using the property that the Newton polyhedron of a product is the Minkowski sum of the Newton polyhedra of its factors, and (2) of Theorem 6.7 we finish the proof for $d = 1$.

Assume now that $d > 1$.

Let $x_1 = u^{r_1}, \ldots, x_d = u^{r_d}$ be a monomial substitution, where r_i are positive integers. By Lemma 7.1 $f^{[\mathbf{r}]}$ is Kuo–Lu k-regular; hence, by the first part of the proof ($d = 1$)

$$\Delta(\bar{R}^{[\mathbf{r}]}) = \sum_{\substack{B\in T(f)\\ t_k(B)\neq 0}} \left\{ \frac{t_k(\bar{B})q(\bar{p}^{[r]}, \bar{B})}{t_k(\bar{B})} \right\}.$$

For any elementary polyhedron of the above sum, Lemma 7.1 gives

$$\left\{ \frac{t_k(\bar{B})q(\bar{p}^{[r]}, \bar{B})}{t_k(\bar{B})} \right\} = \left\{ \frac{t_k(B)\langle r, q(p,B)\rangle}{t_k(B)} \right\} = \pi_r\left(\left\{ \frac{t_k(B)q(p,B)}{t_k(B)} \right\}\right).$$

Since the image of the Minkowski sum of Newton polyhedra is the Minkowski sum of the images, we get $\Delta(\bar{R}^{[\mathbf{r}]}) = \pi_{\mathbf{r}}(\Delta)$, where Δ denotes the Newton polyhedron given in (12). By Lemma 7.3, we get $\Delta(R) = \Delta$. ■

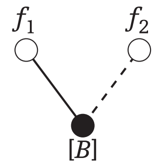

Fig. 4. The Eggers tree of $f = f_1 f_2$ from Example 3.1.

8 Eggers Tree of a Quasi-Ordinary Weierstrass Polynomial

In this section, we introduce the *Eggers tree* of a quasi-ordinary Weierstrass polynomial f, after the conjugacy relation defined in Section 5. Denote by $[B]$ the conjugacy class of the pseudo-ball B of the Kuo–Lu $T(f)$. By definition, the *Eggers tree* of f, denoted by $E(f)$, is the set of conjugacy classes with the natural order induced by the Kuo–Lu tree. This is the natural generalization of the Eggers tree associated with plane curves in [4]. The notion of Eggers tree, for quasi-ordinary singularities, was introduced by Popescu-Pampu in [18] and [19]. He defined a slightly different notion of the Eggers tree, since he generalized to quasi-ordinary singularities the version of Eggers tree defined for curves in [21].

The leaves of $E(f)$ correspond with irreducible factors of f. Following Eggers, we draw them in white color. By definition, the *root* of $E(f)$ is its vertex of minimum height. The *branches* of $E(f)$ are the smallest sub-trees of $E(f)$ containing the root and one of its leaves. Let $[B]$ be a vertex in the branch of $E(f)$ corresponding with the irreducible component f_i of f. Eggers draws in a dashed way the edge leaving from the vertex $[B]$ in this branch if there are not two roots of f_i with contact $h(B)$.

Recall that the number of pseudo-balls conjugate with a quasi-ordinary pseudo-ball B is $N(B)$ (see page 14).

Let $[B]$ be a vertex of the Eggers tree of a quasi-ordinary polynomial f. By Lemma 5.2, for any $k \in \{1, \dots, \deg f\}$, the numbers $n_k(B)$ and $t_k(B)$ do not depend on the representative of $[B]$. Moreover, if $p(y) \in \mathbf{K}[[\underline{x}]][y]$ is a Weierstrass polynomial compatible with B then the number $q(p, B)$ and the degree of its B-characteristic polynomial are also independent of the representative of $[B]$.

The Eggers tree of the quasi-ordinary polynomial $f = f_1 f_2$ from Example 3.1 is drawn in Figure 4.

Remark 8.1. Let p be an irreducible factor of f. Then, following Lemma 4.9, the sequence $\{q(p, B)\}_{[B]}$ is increasing along the branch P of the Eggers tree of f containing the leave representing p. If $[B]$ does not belong to P then, by Lemma 4.9, $q(p,B) = q(p,B_0)$, where $[B_0]$ is the last common vertex of P and the branches of the Eggers tree containing $[B]$. Hence, the set $\{q(p,B)\}_{[B]}$ is well ordered with respect to the coordinate-wise order.

After Remark 8.1 we get

Corollary 8.2. Let $f \in \mathbf{K}[[\underline{x}]][y]$ be a Kuo–Lu k-regular quasi-ordinary Weierstrass polynomial and p a Weierstrass polynomial that is an irreducible factor of f in $\mathbf{K}[[\underline{x}]][y]$. Then, the Newton polyhedron in (12) is polygonal.

Remark 8.3. Remember that we denote by $\bar{f}$ the polynomial $f \in \mathbf{K}[[\underline{x}]][y]$ after monomial substitution. Example 7.2 shows that even though the Kuo–Lu tree of any quasi-ordinary polynomial f yields the Kuo–Lu tree of $\bar{f}$, this is not the case of the Eggers trees of polynomials f and $\bar{f}$.

9 Irreducible Factors of Higher Derivatives

Let f be a quasi-ordinary Weierstrass polynomial. In this section, we study irreducible factors of normalized higher derivatives $f^{(k)}$. We show that every such an irreducible factor can be associated with a certain vertex $[B]$ of the Eggers tree of f. By definition an *Eggers factor* will be the product of all irreducible factors associated with the same vertex of $E(f)$. The *Eggers factorization* of a higher derivative is the product of all its Eggers factors. It generalizes to higher derivatives the factorization of the first polar given in [4] and [5] for plane curves and in [6] for quasi-ordinary polynomials.

Let $F_B(z)$ be the B-characteristic polynomial of f. After Remark 6.5, the polynomial $F_B^{(k)}(z)$ is the product of two coprime polynomials $F_B^{\lceil k\rceil}(z)$ and $F_B^{\lfloor k\rfloor}(z)$, where

$$F_B^{\lceil k\rceil}(z) = \prod_{B\perp_{z_i}B_i} (z - z_i)^{n_k(B_i)}.$$

Theorem 9.1. Let $f(y)$ be a quasi-ordinary Weierstrass polynomial and let $g(y) \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial that is an irreducible factor of $f^{(k)}(y)$. Then, there exists $[B] \in E(f)$, with $B \in T_k(f)$, such that

(1) If $B' \in T_k(f)\backslash[B]$ then every root of $G_{B'}(z)$ is a root of $F_{B'}^{\lceil k\rceil}(z)$.

(2) If $B' \in T_k(f) \cap [B]$ then $G_{B'}(z)$ and $F_{B'}^{\lceil k \rceil}(z)$ do not have common roots. Moreover,

$$G_B(z) = az^l \text{ or } G_B(z) = a(z^{n(B)} - c)^l \tag{14}$$

for some $l \geq 1$ and $a, c \in \mathbf{K} \setminus \{0\}$. If $l = 1$ then $g(y)$ is quasi-ordinary.

Proof. Let $\overline{\beta}$ be a root of $\overline{g}$. Then, by Theorem 6.7, there exists $B \in T_k(f)$ such that $\overline{\beta} \in \overline{B}^o$. Hence, $G_B(z)$ has a root that is not a root of $F_B^{\lceil k \rceil}(z)$, so

$$\mathcal{T} = \{ B \in T_k(f) : G_B(z) \text{ has a root which is not a root of } F_B^{\lceil k \rceil}(z) \}$$

is a nonempty set. By Remark 6.8, $B \in \mathcal{T}$ if and only if for any monomial substitution $\bar{g}$ has a Newton–Puiseux root that belongs to $\bar{B}^\circ$.

Let $\mathcal{E} = \{ [B] \in E(f) : B \in \mathcal{T} \}$. We will show that $\mathcal{E}$ has only one element. Suppose that this is not the case, and let $[B_0]$ be the infimum of $\mathcal{E}$ in the ordered set $E(f)$ (the infimum exists because $E(f)$ has the structure of a tree).

Let $[B']$ be an element of $\mathcal{E}$ different from $[B_0]$ and α be a Newton–Puiseux root of f belonging to B'. By definition of the Eggers tree, there exists $B_1 \in [B_0]$ such that $B' \subsetneq B_1$. Since $B' \in T_k(f)$, the multiplicity of $c := \mathrm{lc}_{B_1}(\alpha)$ as a root of $F_{B_1}(z)$ is bigger than or equal to k. Let p be the irreducible factor of f for which $p(\alpha) = 0$. By the second statement of Lemma 4.10, the polynomial $P_{B_1}(z)$ is up to multiplication by a constant a power of $z^{n(B_1)} - c^{n(B_1)}$ if $c \neq 0$ or a power of z if $c = 0$. Hence, by the first statement of Lemma 4.10 and Remark 6.5 the polynomials $P_{B_1}(z)$ and $F_{B_1}^{\lfloor k \rfloor}(z)$ are coprime.

Let $\bar{g} = \prod_{i=1}^m (y - \bar{\beta}_i)$ be the Newton–Puiseux factorization of g after some monomial substitution. Fix $B \in [B_0]$. By Lemmas 5.2 and 7.1 we get $q(\bar{p}, \bar{B}) = q(\bar{p}, \bar{B}_0)$. Let us define two sets of indexes associated with $\bar{B}$:

$$I_{\bar{B}} = \{ i : \bar{\beta}_i \in \bar{B}, P_B(\mathrm{lc}_{\bar{B}} \bar{\beta}_i) \neq 0 \},$$

$$J_{\bar{B}} = \{ i : \bar{\beta}_i \in \bar{B}, P_B(\mathrm{lc}_{\bar{B}} \bar{\beta}_i) = 0 \}.$$

Directly from the definition of P_B we have the following: if $i \in I_{\bar{B}}$ then $\operatorname{ord} \bar{p}(\bar{\beta}_i) = q(\bar{p}, \bar{B}_0)$, and if $i \in J_{\bar{B}}$ then $\operatorname{ord} \bar{p}(\bar{\beta}_i) > q(\bar{p}, \bar{B}_0)$.

The cardinality of $I_{\bar{B}}$ is equal to the number of roots of $G_B(z)$ counted with multiplicities that are not the roots of $P_B(z)$. Similarly, the cardinality of $J_{\bar{B}}$ is equal to the number of roots of $G_B(z)$ counted with multiplicities that are the roots of $P_B(z)$. Hence, the cardinality of these sets does not depend on the choice of the monomial substitution. Let $I := \bigcup_{B\in[B_0]} I_{\bar{B}}$ and $J := \bigcup_{B\in[B_0]} J_{\bar{B}}$. Observe that $\operatorname{ord}\bar{p}(\bar{\beta}_i) = q(\bar{p},\bar{B}_0)$ for $i \in I$ and $\operatorname{ord}\bar{p}(\bar{\beta}_i) > q(\bar{p},\bar{B}_0)$ for $i \in J$.

The sets I and J depend on the choice of the monomial substitution but their cardinality does not. We will show that the set J is nonempty. Since $B' \in \mathcal{T}$, there exists $\bar{\beta}_i \in \bar{B}'$. Any root of $\bar{p}$ that belongs to $\bar{B}'$ has the same leading coefficient with respect to $\bar{B}_1$ as $\bar{\beta}_i$. Hence, $P_{B_1}(\mathrm{lc}_{\bar{B}_1}\bar{\beta}_i) = 0$, which gives $i \in J_{\bar{B}_1} \subset J$.

Now, we will prove that the set I is empty. Suppose that it is not the case. Put $R(T) := \operatorname{Res}_Y(g, p-T)$ and $\bar{R}(T) := \operatorname{Res}_Y(\bar{g}, \bar{p}-T)$. We can write

$$\begin{aligned} R(T) &= \pm T^m + c_1 T^{m-1} + \cdots + c_m, \\ \bar{R}(T) &= \pm T^m + \bar{c}_1 T^{m-1} + \cdots + \bar{c}_m, \end{aligned}$$

for some $c_i \in \mathbf{K}[[\underline{x}]]$. By a well-known formula for the resultant, we have $\bar{R}(T) = \pm\prod_{i=1}^m(\bar{p}(\bar{\beta}_i) - T)$. Since the Newton polygon of a product is the Minkowski sum of the Newton polygons of its factors, $\Delta(\bar{R}(T))$ has an edge of inclination $q(\bar{p},\bar{B}_0)$ starting in the point $(0,m)$. The projection of this edge to the vertical axis has length $\sharp I$. This gives

$$\begin{aligned} \operatorname{ord}\bar{c}_i &\geq iq(\bar{p},\bar{B}_0) \ \text{ for } 1 \leq i < \sharp I, \\ \operatorname{ord}\bar{c}_i &= iq(\bar{p},\bar{B}_0) \ \text{ for } i = \sharp I, \\ \operatorname{ord}\bar{c}_i &> iq(\bar{p},\bar{B}_0) \ \text{ for } \sharp I < i \leq m. \end{aligned}$$

Since the monomial substitution was arbitrary, we have

$$\begin{aligned} c_i\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] &\subseteq \underline{x}^{iq(p,B_0)}\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] && \text{for} 1 \leq i < \sharp I, \\ c_i\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] &= \underline{x}^{iq(p,B_0)}\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] && \text{for } i = \sharp I, \\ c_i\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] &\subsetneq \underline{x}^{iq(p,B_0)}\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]] && \text{for } \sharp I < i \leq m. \end{aligned}$$

By Corollary 2.6 g is not irreducible and we get a contradiction.

We conclude that $I = \emptyset$. This means that for every $\bar{\beta}_i$ there exists $B \in [B_0]$ such that $\bar{\beta}_i \in \bar{B}$ and $\operatorname{ord}\bar{p}(\bar{\beta}_i) > q(\bar{p},\bar{B})$. By Remark 6.8, $\bar{\beta}_i$ belongs to a post-bar of $\bar{B}$, which has a nonempty intersection with $\operatorname{Zer}\bar{p}$. All post-bars of $B \in [B_0]$ that have nonempty intersection with $\operatorname{Zer} p$ conjugate. They form the vertex of $E(f)$, bigger than $[B_0]$, which is smaller or equal (with the natural order in $E(f)$) than any element of $\mathcal{E}$. Hence, $[B_0]$ cannot be the infimum of $\mathcal{E}$ and we arrive again at a contradiction.

We have shown that $\mathcal{E}$ has only one element. Denote it by $[B_0]$. Hence, for any monomial substitution we have $\mathrm{Zer}\,\bar{g} \subset \bigcup_{B\in[B_0]} \bar{B}^{\circ}$. By Remark 6.8 we get (1) and the first part of (2).

Now, we will find the form of $G_B(z)$, for any $B \in [B_0]$. If for every $\bar{\beta}_i \in B$ the leading coefficient $\mathrm{lc}_{\bar{B}}\bar{\beta}_i$ is 0, then obviously $G_B(z) = az^l$. Otherwise, by Lemma 4.10 there exist $c \neq 0$ and a polynomial $G_1(z)$ coprime with $z^{n(B)} - c^{n(B)}$ such that $G_B(z) = G_1(z)(z^{n(B)} - c^{n(B)})^l$. Let $p(y)$ be the minimal Weierstrass polynomial of $\lambda_B(\underline{x}) + c\underline{x}^{h(B)}$. Then, $P_B(z) = \mathrm{const} \cdot (z^{n(B)} - c^{n(B)})$.

Proceeding as in the first part of the proof we define again the sets I, J of indexes. By the choice of $p(y)$ the set J is nonempty. If the polynomial $G_1(z)$ has positive degree, then the set I is nonempty and we arrive at a contradiction. Hence, $G_1(z)$ is a constant that proves the second part of the theorem.

Now, we prove that if $l = 1$ in (14) then $g(y)$ is quasi-ordinary. Let $p(y) \in \mathbf{K}[[\underline{x}]][y]$ be the minimal polynomial of $\lambda_B(\underline{x})$ if $G_B(z) = az$ or the minimal polynomial of $\lambda_B(\underline{x}) + c\underline{x}^{h(B)}$ if $G_B(z) = a(z^{n(B)} - c^{n(B)})$. Then, $G_B(z)$ is equal to $P_B(z)$ up to multiplication by a constant. By Lemma 5.2 for any $B' \in [B_0] = [B]$ the characteristic polynomials $G_{B'}(z)$ and $P_{B'}(z)$ have the same form, in particular have the same number of roots and all their roots are simple. Take any monomial substitution and let $\bar{\beta}'$, $\bar{\beta}''$ be different roots of $\bar{g}(y)$. Since $\mathrm{Zer}\,\bar{g} \subset \bigcup_{B\in[B_0]} \bar{B}^{\circ}$ there exist $B', B'' \in [B_0]$ such that $\bar{\beta}' \in \bar{B}'$ and $\bar{\beta}'' \in \bar{B}''$. If $B' = B''$ then $O(\bar{\beta}', \bar{\beta}'') = h(\bar{B}')$ because $\bar{\beta}'$ and $\bar{\beta}''$ have different leading coefficients with respect to $\bar{B}'$. If $B' \neq B''$ then $O(\bar{\beta}', \bar{\beta}'') = O(\lambda_{\bar{B}'}, \lambda_{\bar{B}''})$. In both cases, the contact $O(\bar{\beta}', \bar{\beta}'')$ depends only on B' and B''. The same argument applies to the roots of $\bar{p}(y)$. As a consequence any bijection $\Phi : \mathrm{Zer}\,\bar{g} \to \mathrm{Zer}\,\bar{p}$ such that $\Phi(\bar{B}' \cap \mathrm{Zer}\,\bar{g}) = \bar{B}' \cap \mathrm{Zer}\,\bar{p}$ for $B' \in [B_0]$ preserves contacts.

Since the discriminant of a monic polynomial is the product of differences of its roots, the discriminant of $\bar{g}$ and the discriminant of $\bar{p}$ have the same order. Then, by Corollary 7.4, the Newton polyhedra of the discriminants of $g(y)$ and $p(y)$ are equal and we conclude that $g(y)$ is quasi-ordinary. ■

For k-regular quasi-ordinary Weierstrass polynomials we can say more.

Corollary 9.2. Let $f(y)$ be a Kuo–Lu k-regular quasi-ordinary Weierstrass polynomial and let $g(y) \in \mathbf{K}[[\underline{x}]][y]$ be a Weierstrass polynomial that is an irreducible factor of $f^{(k)}(y)$. Then, there exists $[B] \in E(f)$ with $B \in T_k(f)$ such that

(1) if $B' \in T(f) \cap [B]$, then $G_{B'}(z)$ and $F_{B'}(z)$ do not have common roots;
(2) if $B' \in T_k(f) \backslash [B]$, then every root of $G_{B'}(z)$ is a root of $F_{B'}^{\lceil k \rceil}(z)$; and
(3) if $B' \in T(f) \setminus T_k(f)$, then $G_{B'}(z)$ is a nonzero constant polynomial.

Proof. Take $B' \in T_k(f)$. Then, by Lemma 4.6 and the definition of k-regularity $G_{B'}(z)$ and $F_{B'}^{\lfloor k \rfloor}(z)$ do not have common roots. Hence, for $B' \in T_k(f)$ it is enough to use Theorem 9.1. This proves (1). The second statement is the first item of Theorem 9.1.

Now, let $B' \in T(f) \setminus T_k(f)$. Consider the chain of bars $B_0 \perp_c B_1 \perp \cdots \perp B_s = B'$ of $T(f)$ such that $B_0 \in T_k(f)$ and $B_i \notin T_k(f)$ for $1 \leq i \leq s$. Then, the multiplicity of c as a root of $F_{B_0}(z)$ is less than k. Hence, by the k-regularity of $F_{B_0}(z)$, c is not a root of $F_{B_0}^{(k)}(z)$ and consequently $G_{B_0}(c) \neq 0$. Since g is compatible with B_0, after (4) of Lemma 4.7, we have

$$g(\lambda_{B'}(\underline{x}) + c\underline{x}^{h(B')}) = g(\lambda_{B_0}(\underline{x}) + c\underline{x}^{h(B_0)} + \cdots) = G_{B_0}(c)\underline{x}^{q(g,B_0)} + \cdots,$$

which shows that g is also compatible with B' and its B'-characteristic polynomial $G_{B'}(z)$ equals $G_{B_0}(c)$. ■

10 Eggers Factorizations of Higher Derivatives

Let f be a quasi-ordinary Weierstrass polynomial. In this section, we propose a factorization of the normalized derivative $f^{(k)}$ into factors associated with points of Eggers tree $E(f)$.

Definition 10.1 Let $g, p \in \mathbf{K}[[\underline{x}]][y]$ be Weierstrass polynomials. The P-contact between g and p is

$$\mathrm{cont}_{\mathrm{P}}(g,p) := \frac{1}{\deg g \deg p} \Delta(\mathrm{Res}_y(g,p)).$$

The notion of P-contact has its counterpart in the theory of plane analytic curves: for y-regular plane branches, it is related with the *logarithmic distance* studied by Płoski in [17], since in such case $\Delta(\mathrm{Res}_y(g,p))$ equals the Newton polygon of a monomial x^m, where m is the *intersection multiplicity* of the branches $g = 0$ and $p = 0$.

If g is compatible with a pseudo-ball B then we put

$$\mathrm{cont}_{\mathrm{P}}(g,B) := \frac{1}{\deg g} \Delta(\underline{x}^{q(g,B)}).$$

Proposition 10.2. Let B be a quasi-ordinary pseudo-ball of finite height, and let f be an irreducible quasi-ordinary Weierstrass polynomial compatible with B such that $\mathrm{Zer} f \cap B \neq \emptyset$ or equivalently such that $F_B(z)$ has positive degree. Then, $\mathrm{cont}_{\mathrm{P}}(f,B)$ does not depend on f.

Proof. Take any f_1, f_2 satisfying the assumptions of the proposition, and let $\alpha_1 \in \mathrm{Zer} f_1 \cap B$, $\alpha_2 \in \mathrm{Zer} f_2 \cap B$. Choose a constant $c \in \mathbf{K}$ such that $(F_i)_B(c) \neq 0$, for $i = 1, 2$ and let $\gamma = \lambda_B + c\underline{x}^{h(B)}$. Then, $O(\gamma, \alpha_1) = O(\gamma, \alpha_2) = h(B)$ and for any $\xi \in \mathrm{Zer} f_1 \cup \mathrm{Zer} f_2$ we have $O(\xi, \gamma) \leq h(B)$.

Let G be a finite subgroup of $\mathbf{L}$-automorphisms of $\mathbf{M}$ that acts transitively on the sets $\mathrm{Zer} f_1$ and $\mathrm{Zer} f_2$. By the orbit stabilizer theorem, for $i \in \{1, 2\}$ we get

$$\frac{1}{|G|}\sum_{\sigma\in G} O(\gamma, \sigma(\alpha_i)) = \frac{1}{\deg f_i}\sum_{\alpha\in \mathrm{Zer} f_i} O(\gamma, \alpha) = \frac{1}{\deg f_i} q(f_i, B).$$

By STI, we have $O(\gamma, \sigma(\alpha_1)) = O(\gamma, \sigma(\alpha_2))$ for all $\sigma \in G$. Thus, $\frac{1}{\deg f_1} q(f_1, B) = \frac{1}{\deg f_2} q(f_2, B)$. ■

After Proposition 10.2 we define the *self-contact* of a pseudo-ball B of finite height as

$$\text{self-contact}(B) := \mathrm{cont}_{\mathrm{P}}(f, B),$$

for any f satisfying the assumptions of this proposition.

By Lemma 5.2, conjugate pseudo-balls have the same self-contact; hence, the self-contact of $[B]$ is well defined for any vertex $[B]$ of $E(f)$, where B is of finite height.

In the set of Newton polyhedra, we define the next partial order: $\Delta_1 \succeq \Delta_2$ if and only if $\Delta_1 \subseteq \Delta_2$. Observe that $\Delta(\underline{x}^{\mathbf{q}_1}) \succeq \Delta(\underline{x}^{\mathbf{q}_2})$ if and only if $\mathbf{q}_1 \geq \mathbf{q}_2$. Now, we show how the self-contacts of $[B] \in E(f)$ determine the P-contacts between irreducible factors of f.

Proposition 10.3. Let f be a quasi-ordinary Weierstrass polynomial. Then, the self-contacts of vertices of finite height increase along the branches of $E(f)$. Moreover, for any different irreducible factors f_1, f_2 of f

$$\mathrm{cont}_{\mathrm{P}}(f_1, f_2) = \max\{\text{self-contact}([B])\}, \tag{15}$$

where the maximum is taken over all $[B] \in E(f)$ such that $\mathrm{Zer} f_i \cap B \neq \emptyset$ for $i = 1, 2$.

Proof. Let B, B' be pseudo-balls of $T(f)$ of finite height such that $B' \subsetneq B$. Choose an irreducible factor f_i of f such that $\mathrm{Zer} f_i \cap B' \neq \emptyset$. By Lemma 4.9, we get $q(f_i, B) < q(f_i, B')$; hence self-contact$(B) \prec$ self-contact(B').

Let $[B] \in E(f)$ be the maximum (with the order defined in $E(f)$) of the set of all vertices $[B'] \in E(f)$ such that $\mathrm{Zer} f_i \cap B' \neq \emptyset$ for $i = 1, 2$. The pseudo-ball B has the form

$\gamma + (\gamma - \delta)\mathbf{K}[[\underline{x}^{1/\mathbf{N}}]]$, for some $\gamma \in \mathrm{Zer} f_1$ and $\delta \in \mathrm{Zer} f_2$ with maximal possible contact. By the choice of γ and δ, we have $O(\gamma, \delta') \leq h(B)$ for all $\delta' \in \mathrm{Zer} f_2 \cap B$, consequently $(F_2)_B(\mathrm{lc}_B \gamma) \neq 0$. Then, $f_2(\gamma) = (F_2)_B(\mathrm{lc}_B \gamma)\underline{x}^{q(f_2,B)} + \cdots$.

Applying the Galois action associated with the irreducible polynomial f_2 we get $\Delta(f_2(\gamma)) = \Delta(f_2(\gamma'))$, for any $\gamma, \gamma' \in \mathrm{Zer} f_1$. Hence, by the definition of the self-contact and the identity $\Delta(\mathrm{Res}_y(f_1, f_2)) = \sum_{\gamma \in \mathrm{Zer} f_1} \Delta(f_2(\gamma))$ we have

$$\begin{aligned}
\text{self-contact}(B) &= \mathrm{cont}_{\mathrm{P}}(f_2, B) = \frac{1}{\deg f_2}\Delta(\underline{x}^{q(f_2,B)}) \\
&= \frac{1}{\deg f_1 \deg f_2} \deg f_1 \ \Delta(f_2(\gamma)) \\
&= \frac{1}{\deg f_1 \deg f_2}\Delta(\mathrm{Res}_y(f_1, f_2)) = \mathrm{cont}_{\mathrm{P}}(f_1, f_2).
\end{aligned}$$

■

Theorem 10.4. Let $f \in \mathbf{K}[[\underline{x}]][y]$ be a quasi-ordinary Weierstrass polynomial. Then,

$$f^{(k)} = \prod_{[B] \in E(f)} p_{[B]},$$

where $p_{[B]}$ are Weierstrass polynomials such that

(1) the B-characteristic polynomial of $p_{[B]}$ equals $F_B^{\lfloor k \rfloor}$ up to multiplication by constants and $\deg p_{[B]} = N(B)t_k(B)$;

(2) for every irreducible factor g of $p_{[B]}$ and every irreducible factor f_i of f, we get

  (a) $\mathrm{cont}_{\mathrm{P}}(g, B) = \text{self-contact}(B)$,

  (b) if $\mathrm{cont}_{\mathrm{P}}(f_i, B) \prec \text{self-contact}(B)$ then $\mathrm{cont}_{\mathrm{P}}(f_i, g) = \mathrm{cont}_{\mathrm{P}}(f_i, B)$, and

  (c) if $\mathrm{cont}_{\mathrm{P}}(f_i, B) = \text{self-contact}(B)$ then $\mathrm{cont}_{\mathrm{P}}(f_i, g) \succeq \mathrm{cont}_{\mathrm{P}}(f_i, B)$;

(3) if f is k-regular, then the inequalities $\succeq$ in (c) become equalities; and

(4) for every irreducible factor g of $p_{[B]}$ there is an irreducible factor f_i of f such that $\mathrm{cont}_{\mathrm{P}}(f_i, g) = \mathrm{cont}_{\mathrm{P}}(f_i, B) = \text{self-contact}(B)$.

Proof. We define $p_{[B]}$ as the product of all irreducible factors of $f^{(k)}$ having the same $[B]$ in Theorem 9.1 (by convention the product of an empty family is 1). Then, after some monomial substitution, all the roots of $\bar{p}_{[B]}$ belong to $\bigcup_{B' \in [B]} \bar{B'}^0$. Since for every $B' \in [B]$, $\overline{p}_{[B]}$ has $t_k(B)$ roots belonging to B' and $[B]$ has $N(B)$ elements, we get $\deg \overline{p}_{[B]} = N(B)t_k(B)$. Consequently $\deg p_{[B]} = N(B)t_k(B)$.

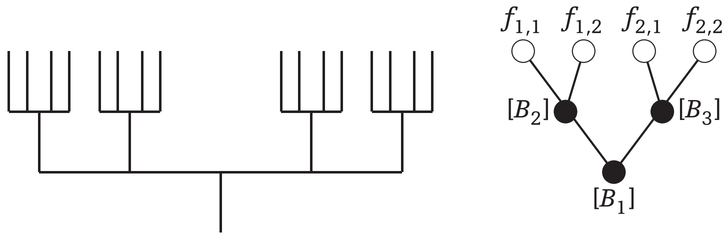

Fig. 5. The Kuo–Lu and the Eggers tree of f from Example 10.5.

Now, we will prove the second statement. Since $p_{[B]}$ has positive degree, we may assume that $B \in T_k(f)$. Let f_i be an irreducible factor of f. If $\mathrm{cont}_{\mathrm{P}}(f_i, B) \prec$ self-contact(B) then by Proposition 10.2 $(F_i)_B(z)$ is a nonzero constant polynomial. Hence, for any $\bar{\gamma} \in \mathrm{Zer}\,\bar{p}_{[B]}$ we have $\mathrm{ord}\,\bar{f}_i(\bar{\gamma}) = q(\bar{f}_i, \bar{B})$, which proves (2)(b). Suppose now that $\mathrm{cont}_{\mathrm{P}}(f_i, B) =$ self-contact(B). For every root $\bar{\gamma} \in \mathrm{Zer}\,\bar{p}_{[B]}$ we have $\mathrm{ord}\,\bar{f}_i(\bar{\gamma}) \geq q(\bar{f}_i, \bar{B})$ with equality in the k-regular case. Hence, if g is an irreducible factor of $p_{[B]}$ then $\mathrm{ord}\,\mathrm{Res}_y(\bar{f}_i, \bar{g}) \geq (\deg g) \cdot q(\bar{f}_i, \bar{B})$ with equality in the k-regular case. This gives (2)(c) and (3).

If the polynomial $F_B(z)$ is as in (8) then, by Corollary 6.4, it is k-regular. In this case, for any irreducible factor f_i of f, with $(F_i)_B(z)$ of positive degree, the polynomials $(F_i)_B(z)$ and $G_B(z)$ do not have common factors.

If $F_B(z)$ is not as in (8), then by Lemma 4.10, there is an irreducible factor f_i of f such that the polynomials $(F_i)_B(z)$ and $G_B(z)$ do not have common factors and $(F_i)_B(z)$ has positive degree. After any monomial substitution, we have $\mathrm{ord}\,\bar{f}_i(\bar{\gamma}) = q(\bar{f}_i, \bar{B})$, for every $\bar{\gamma} \in \mathrm{Zer}\,\bar{g}$. This gives $\mathrm{ord}\,\mathrm{Res}\,(\bar{f}_i, \bar{g}) = \deg g \cdot q(\bar{f}_i, \bar{B})$. Since the monomial substitution was arbitrary, the fourth statement of the theorem holds true in all cases.

It rests to prove (2)(a). Choose f_i as in the proof of the fourth statement. Then, $\Delta(g(\alpha)) = \Delta(\underline{x}^{q(g,B)})$ for any $\alpha \in B \cap \mathrm{Zer}\, f_i$.

Applying the same argument as in the end of the proof of Proposition 10.3, we get $\Delta(\mathrm{Res}_y(f_i, g)) = \deg f_i \Delta(g(\alpha)) = \deg f_i \cdot \Delta(\underline{x}^{q(g,B)})$. After the fourth statement and the definition of the P-contact: self-contact$(B) = \mathrm{cont}_{\mathrm{P}}(f_i, g) = \frac{1}{\deg g \deg f_i} \Delta(\mathrm{Res}_y(f_i, g)) = \frac{1}{\deg g} \Delta(\underline{x}^{q(g,B)}) = \mathrm{cont}_{\mathrm{P}}(g, B)$. ■

Example 10.5. We consider the example in [6, Section 10]: let $f = f_{1,1} f_{1,2} f_{2,1} f_{2,2}$, where $f_{i,j} = (y^2 - i x_1^3 x_2^2)^2 - j x_1^5 x_2^4 y$ are irreducible quasi-ordinary polynomials for $i, j \in \{1, 2\}$. The Kuo–Lu and the Eggers tree of f are drawn in Figure 5.

The heights of the vertices of the Eggers tree are as follows: $h[B_1] = \left(\frac{3}{2}, 1\right)$, $h([B_2]) = h([B_3]) = \left(\frac{7}{4}, \frac{3}{2}\right)$; the self-contacts are self-contact$([B_1]) = \frac{1}{4}\Delta(\underline{x}^{(6,4)})$; and self-contact$([B_2])$ = self-contact$([B_3]) = \frac{1}{4}\Delta\left(\underline{x}^{(13,10)}\right)$.

For any $1 \leq k \leq 16$, the degrees of polynomials $p_{[B_i]}$ are

	$\deg p_{[B_1]}$	$\deg p_{[B_2]}$	$\deg p_{[B_3]}$
$f^{(1)}$	3	6	6
$f^{(2)}$	6	4	4
$f^{(3)}$	9	2	2
$f^{(k)}$	16-k	0	0

The characteristic polynomials are $F_{B_1}(z) = (z^2-1)^4(z^2-2)^4$, $F_{B_2}(z) = (4z^2 - 1)(4z^2-2)$, and $F_{B_3}(Z) = (8z^2 - \sqrt{2})(8z^2 - 2\sqrt{2})$. We can verify that these polynomials are k-regular for any k.

Theorem 10.4 allows us to compute the P-contact between the irreducible factors of f and the irreducible factors of its higher order polars. For any k and any irreducible factor g of $p_{[B_1]}$, we have $\mathrm{cont}_{\mathrm{P}}(f_{i,j}, g)$ = self-contact$([B_1])$. For any k and any irreducible factor g of $p_{[B_2]}$, we have $\mathrm{cont}_{\mathrm{P}}(f_{1,j}, g)$ = self-contact$([B_2])$ and $\mathrm{cont}_{\mathrm{P}}(f_{2,j}, g)$ = self-contact$([B_1])$, for any $j = 1, 2$. We have the symmetric situation for the irreducible factors of $p_{[B_3]}$.

Example 10.6. The second polar of the quasi-ordinary polynomial f from the Example 3.1 (see Figure 4 for its Eggers tree) has only one Eggers factor $p_{[B]} = y$, with $\mathrm{cont}_{\mathrm{P}}(f_2, y) = \Delta(\underline{x}^{(5,2)}) \succ \mathrm{cont}_{\mathrm{P}}(f_1, y) = \Delta(\underline{x}^{(3/2,1)})$ = self-contact(B); hence, in item (2)(c) of Theorem 10.4 we have equality for f_1 and strict inequality for f_2.

Now, we study the examples of [2].

Example 10.7. ([2, Example 5.1]) Let $f = y^3 + x^2y$. The Eggers tree of f has only one vertex $[B]$ of finite height, where $B = x\mathbf{K}[[x^{1/\mathbf{N}}]]$. The B-characteristic polynomial $F_B(z) = z^3 + z$ is not 2-regular. We get $f^{(2)} = p_{[B]} = y$. If $f_1 = y - x$, $f_2 = y + x$ and $f_3 = y$ then $\emptyset = \mathrm{cont}_{\mathrm{P}}(f_3, y) \succeq \mathrm{cont}_{\mathrm{P}}(f_i, y) = \mathrm{cont}_{\mathrm{P}}(f_i, B) = \Delta(x)$ = self-contact(B) for $i = 1, 2$. This illustrates the fourth statement of Theorem 10.4.

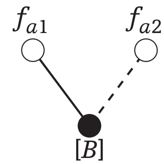

Fig. 6. The Eggers tree of f_a from Example 10.8.

Example 10.8. ([2, Example 5.2]) Let $f_a = y^4 + ax^2y^2 + x^2y + x^{10}$. We get $f_a = f_{a1}f_{a2}$, where f_{a1} is irreducible and the contact of any two different roots of it is $\frac{2}{3}$, and $f_{a2} = 0$ is a smooth curve tangent to $y = 0$. The Eggers tree of f_a is drawn in Figure 6.

The characteristic polynomial $F_B(z)$ equals $z^4 + z$. Hence, f_a is not 2-regular. For any irreducible factor g of $f_a^{(2)}$ we get $\mathrm{cont_P}(f_{a1}, g) = \Delta(x^{2/3}) =$ self-contact(B). For f_{a2} the P-contact depends on a:

$$\mathrm{cont_P}(f_{a2}, g) = \begin{cases} \Delta(x) & \text{for } a \neq 0 \\ \Delta(x^8) & \text{for } a = 0. \end{cases}$$

10.1 Irreducible case

Assume that $f(y) \in \mathbf{K}[[\underline{x}]][y]$ is an irreducible quasi-ordinary Weierstrass polynomial of degree $n > 1$ and $\mathrm{Zer} f = \{\alpha_i\}_{i=1}^n$. By [14], the set $\{O(\alpha_i, \alpha_j) \,:\, i \neq j\} := \{\mathbf{h}_1, \ldots, \mathbf{h}_s\}$ is well ordered, so we may assume that $\mathbf{h}_1 \leq \mathbf{h}_2 \leq \cdots \leq \mathbf{h}_s$. These values are the finite heights of the bars of $T(f)$. The sequence $\mathbf{h}_1, \ldots, \mathbf{h}_s$ is called the sequence of *characteristic exponents* of $f(y)$. Let B_i be any bar in $T(f)$ of height $\mathbf{h}_i$. By [10, Remark 2.7] the degree $n(B_i)$ of the field extension $\mathbf{L}(\lambda_{B_i}(\underline{x})) \hookrightarrow \mathbf{L}(\lambda_{B_i}(\underline{x}), \underline{x}^{\mathbf{h}_i})$ does not depend on the choice of B_i and will be denoted by n_i. Put $e_i := n_{i+1} \cdots n_s$ for $0 \leq i \leq s$ (by convention the empty product is one). Observe that $T(f)$ has a special structure: all bars of the same height are conjugate and there are $n_1 \cdots n_{i-1}$ conjugate bars of height $\mathbf{h}_i$ (see [7, Theorem 6.2]).

By (12) we get

$$\Delta((\mathrm{Res}_y(f^{(k)}, f - T)) = \sum_{i=1}^{s} n_1 \cdots n_{i-1} t_k(B_i) \left\{ \frac{q(f, B_i)}{1} \right\}, \tag{16}$$

where B_i is any ball of $T(f)$ of height $\mathbf{h}_i$ and

$$t_k(B_i) = \begin{cases} (n_i - 1)k & \text{for } 1 \le k \le e_i, \\ e_{i-1} - k & \text{for } e_i \le k \le e_{i-1}, \\ 0 & \text{for } e_{i-1} \le k < n. \end{cases}$$

Let $i_k \in \{1, \ldots, s\}$ be such that $e_{i_k} \le k < e_{i_k - 1}$. Then, $t_k(B_i)$ is positive if and only if $1 \le i \le i_k$.

Newton polyhedron of (16) is polygonal (see Corollary 8.2) and has i_k edges of different inclinations. After Theorem 2.4, we decompose $\mathrm{Res}_y(f^{(k)}, f - T) = \prod_{i=1}^{i_k} R_i$, where $\deg_T R_i = (n_1 \cdots n_{i-1}) t_k(B_i)$ and any R_i has an elementary Newton polyhedron of inclination $q(f, B_i)$.

Such a decomposition of the resultant can be also obtained from Eggers factorization of $f^{(k)}$. By Lemma 4.10 the B_i-characteristic polynomial of f has the form

$$F_{B_i}(z) = \text{constant}(z^{n_i} - c_{B_i})^{e_i}, \tag{17}$$

for some $c_{B_i} \in \mathbf{K} \setminus \{0\}$. The properties of such polynomials were described in Lemma 6.3.

Corollary 10.9. Every irreducible quasi-ordinary Weierstrass polynomial is Kuo–Lu k-regular for any positive integer k.

Theorem 10.10. Let $f(y) \in \mathbf{K}[[\underline{x}]][y]$ be an irreducible quasi-ordinary Weierstrass polynomial of degree $n > 1$ and characteristic exponents $\mathbf{h}_1, \ldots, \mathbf{h}_s$. Let $i_k \in \{1, \ldots, s\}$ be such that $e_{i_k} \le k < e_{i_k - 1}$. Then,

$$f^{(k)}(y) = \prod_{i=1}^{i_k} p_i, \tag{18}$$

where

(1) p_i is a Weierstrass polynomial in $\mathbf{K}[[\underline{x}]][y]$ of degree $n_1 \cdots n_{i-1} t_k(B_i)$;
(2) any irreducible factor g of p_i verifies

$$\mathrm{cont}_{\mathrm{P}}(g, f) = \text{self-contact}(B_i);$$

(3) the B_i-characteristic polynomial of p_i is $(P_i)_{B_i} = const F_{B_i}^{\lfloor k \rfloor}$.

Proof. The theorem follows from Corollary 10.9 and the first, second, and third parts of Theorem 10.4. ■

Proposition 10.11. Let $f(y) \in \mathbf{K}[[\underline{x}]][y]$ be an irreducible quasi-ordinary Weierstrass polynomial with characteristic exponents $\mathbf{h}_1, \ldots, \mathbf{h}_s$. Let a, d be integers such that $0 \leq a < n_i$, $a + k \equiv 0 \pmod{n_i}$, and $d = \min\{e_i, k\} - \lceil \frac{k}{n_i} \rceil$. Then, every p_i of (18) admits a factorization of the form $p_i = p_{i0} p_{i1} \cdots p_{id}$, where

(1) the corresponding B_i-characteristic polynomials are $P_{i0}(z) = \text{const} \cdot z^a$, $P_{ij}(z) = \text{const} \cdot (z^{n_i} - c_j)$ with $c_j \neq c_l$ for $1 \leq j < l \leq d$ and $c_j \neq 0$;

(2) p_{i0} is a Weierstrass polynomial of degree $a \cdot n_1 \cdots n_{i-1}$ not necessarily quasi-ordinary; and

(3) every p_{ij} for $1 \leq j \leq d$ is a quasi-ordinary irreducible Weierstrass polynomial of degree $n_1 \cdots n_i$ and characteristic exponents $\mathbf{h}_1, \ldots, \mathbf{h}_i$.

Proof. After (17) $F_{B_i}(z)$ has the form $a(z^{n_i} - c)^{e_i}$ for some nonzero a and c.

By the first part of Theorem 10.4 and Lemma 6.3 the polynomial $P_{i,B_i}(z) = \text{const} \cdot z^a \prod_{j=1}^{d} (z^{n_i} - c_j)$. This polynomial is the product of the B_i-characteristic polynomials of the irreducible factors of p_i. From the second part of Theorem 9.1, we know that p_i has d irreducible factors $\{p_{ij}\}_{j=1}^{d}$ such that $P_{ij}(z) = \text{const} \cdot (z^{n_i} - c_j)$. If p_i has other irreducible factors, then p_{i0} is their product. It also follows from Theorem 9.1 that p_{ij} are quasi-ordinary for $1 \leq j \leq d$.

By a similar argument as in the first part of the proof of Theorem 10.4 we get $\deg p_{ij} = N(B_i) \deg P_{i,jB_i}(z)$. Since $N(B_i) = n_1 \cdots n_{i-1}$, we obtain the statements about the degrees of p_{ij}.

Fix p_{ij} for $j \in \{1, \ldots, d\}$. The pseudo-ball B_i has $n_1 \cdots n_{i-1}$ conjugate pseudo-balls. Each of these pseudo-balls contains n_i roots of p_{ij}. Since the roots of $P_{i,j}(z)$ are simple, any two roots of p_{ij} belonging to the same pseudo-ball have different leading coefficients with respect to B_i, so their contact equals $\mathbf{h}_i$. Now, if we consider two roots of p_{ij} belonging to different conjugate pseudo-balls, then their contact depends only on these two pseudo-balls; hence, it is equal to $\mathbf{h}_l$ for some $l \in \{1, \ldots, i-1\}$. We conclude that the characteristic exponents of p_{ij} are $\mathbf{h}_1, \ldots, \mathbf{h}_i$. ■

In Proposition 10.11, the integer a can be 0, in such a case $p_{i0} = 1$. If $a = 1$ then p_{i0} is quasi-ordinary with characteristic exponents $\mathbf{h}_1, \ldots, \mathbf{h}_{i-1}$. Moreover, d can be zero and in such a case $p_i = p_{i0}$.

11 Eggers Decomposition for Power Series

In this section, we deal with power series in variables $\underline{x}$ and y. A power series will be called *quasi-ordinary* if it is a product of a unity and a quasi-ordinary Weierstrass polynomial. We outline how to generalize the results of previous sections to quasi-ordinary power series. For that, we need the next generalization of Lemma 4.6.

Lemma 11.1. Let $f = uf^*$ and $\frac{\partial^k}{\partial y^k}f = wg^*$, where u, $w \in \mathbf{K}[[\underline{x}, y]]$ are unities, f^*, $g^* \in \mathbf{K}[[\underline{x}]][y]$ are Weierstrass polynomials and $1 \le k \le n = \deg f^*$. Assume that f^* is compatible with a pseudo-ball B. Then, g^* is compatible with B and $G^*_B(z) = \frac{(n-k)!}{n!}\frac{d^k}{dz^k}F^*_B(z)$.

Proof. Substituting $\underline{x} = 0$ we get $f(0, y) = u(0, 0)y^n + \cdots$. Hence, $\frac{\partial^k f}{\partial y^k}(0, y) = \frac{n!}{(n-k)!}u(0, 0)y^{n-k} + \cdots$. On the other hand, $\frac{\partial^k f}{\partial y^k}(0, y) = w(0, y)g^*(0, y)$, which implies that

$$w(0, 0) = \frac{n!}{(n-k)!}u(0, 0). \tag{19}$$

By the assumption of compatibility of f^* we have

$$f^*(\underline{x}, \lambda_B(\underline{x}) + z\underline{x}^{h(B)}) = F^*_B(z)\underline{x}^{q(f^*,B)} + \cdots .$$

Hence, $f_1(\underline{x}, z) := \underline{x}^{-q(f^*,B)}f(\underline{x}, \lambda_B(\underline{x}) + z\underline{x}^{h(B)})$ is a fractional power series such that

$$f_1(0, z) = u(0, 0)F^*_B(z). \tag{20}$$

By the chain rule of differentiation

$$\frac{\partial^k f_1}{\partial z^k}(\underline{x}, z) = \frac{\partial^k f}{\partial y^k}(\underline{x}, \lambda_B(\underline{x}) + z\underline{x}^{h(B)}) \cdot \underline{x}^{kh(B)-q(f^*,B)}. \tag{21}$$

Differentiating (20) yields $\frac{\partial^k f_1}{\partial z^k}(0, z) = u(0, 0)\frac{d^k}{dz^k}F^*_B(z)$. Thus,

$$\frac{\partial^k f_1}{\partial z^k}(\underline{x}, z) = u(0, 0)\frac{d^k}{dz^k}F^*_B(z) + \text{terms of positive degree in } \underline{x}. \tag{22}$$

Comparing (21) and (22) we get

$$\frac{\partial^k f}{\partial y^k}(\underline{x}, \lambda_B(\underline{x}) + z\underline{x}^{h(B)}) = u(0, 0)\frac{d^k}{dz^k}F^*_B(z) \cdot \underline{x}^{q(f^*,B)-kh(B)} + \cdots$$

By the definition of g^*, the left-hand side of the above equality can be written as

$$w(0,0)\, g^*(\underline{x}, \lambda_B(\underline{x}) + z\underline{x}^{h(B)}) + \cdots ,$$

which gives, after (19)

$$\frac{n!}{(n-k)!} u(0,0) g^*(\underline{x}, \lambda_B(\underline{x}) + z\underline{x}^{h(B)}) = u(0,0) \frac{d^k}{dz^k} F^*_B(z) \cdot \underline{x}^{q(f^*,B)-kh(B)} + \cdots$$

and finishes the proof. ■

Theorem 6.7, Corollary 6.9, Theorem 7.5, Theorem 9.1, Corollary 9.2, Theorem 10.4, Theorem 10.10, and Proposition 10.11, where $f^{(k)}$ stands for the Weierstrass polynomial of kth derivative, remain true for quasi-ordinary power series. For the proofs, it is enough to replace the power series by their Weierstrass polynomials and use Lemma 11.1 instead of Lemma 4.6 when required.

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