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Weierstrass 1-forms and nondicritical generalized curve foliations

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In this paper, we introduce a distinguished expression for a given 1-form with respect to a polynomial $f \in \mathbb{C}\{x\}[y]$, called *Weierstrass form*. We will use this form and the properties of plane analytical curves to give new characterizations of nondicritical generalized curve foliations.

Keywords: Generalized curve foliation; Weierstrass form; plane singular analytic curve; second type foliation.

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1. Introduction

The Weierstrass Division Theorem (WDT) is a powerful tool in the singularity theory, particularly for the study of analytical plane curves. In this case, the WDT allows us to define any plane curve by a polynomial in $\mathbb{C}\{x\}[y]$. In this paper, using the WDT, we introduce, in Definition 4.2, the notion of *Weierstrass form* of

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any 1-form $W \in \Omega^1_{\mathbb{C}^2,0}$, or shortly *Weierstrass* 1-*form* of W, with respect to any polynomial $f \in \mathbb{C}\{x\}[y]$ with a unit as principal coefficient.

Given any element $\overline{\omega}$ in the Kähler differential module $\frac{\Omega^1_{\mathbb{C}^2,0}}{\mathbb{C}\{x,y\}df + f\Omega^1_{\mathbb{C}^2,0}}$ where $f \in \mathbb{C}\{x\}[y]$, we can obtain a polynomial representative ω of $\overline{\omega}$, that is, $\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y \in \mathbb{C}\{x\}[y]\mathrm{d}x + \mathbb{C}\{x\}[y]\mathrm{d}y$ satisfying $\deg_y A < \deg_y f$ and $\deg_y B < \deg_y f - 1$.

Since a foliation in $(\mathbb{C}^2, 0)$ can be defined by 1-forms, we consider that the notion of *Weierstrass* 1-*form* may be interesting in the future for foliation studies. Example of this is Proposition 4.5, where given $f \in \mathbb{C}\{x\}[y]$ with $\mathrm{ord} f = \deg_y(f)$, principal coefficient a unit in $\mathbb{C}\{x\}$ and considering the Weierstrass form of $W \in \Omega^1_{\mathbb{C}^2,0}$ (defining a non-dicritical foliation) with respect to f, we give sufficient conditions when f is the equation of the union of separatrices of the foliation determined by W.

In the study of holomorphic foliations to determine topological information is an important and a non trivial task. A special class of foliations gives some information about this problem: the *generalized curve foliations*. A foliation is said generalized curve if no saddle-nodes appear in its desingularization process. Moreover, a nondicrital generalized curve foliation and the union of its separatrices have the same process of reduction of singularities (see [3, Theorem 2]).

Firstly we characterize, in Theorem 3.4, the nondicrital generalized curved foliations with monomial separatrix using toric resolution, the prenormal form given by Loray (see [10, p. 157]) and the notion of *weighted order* associated with a foliation and a curve (see Definition 3.1). This allows us to give, in Corollary 3.6, the condition in order that a foliation of second type becomes a generalized curve foliation following [6, Theorem 1.2(b)].

In [2], Brunella stabilized relations among the Baum-Bott, Camacho-Sad and Gómez-Mont-Seade-Verjovsky indices associated to one foliation. He proved that if the foliation is generalized curve then the Baum-Bott and Camacho-Sad indices are equal and the Gómez-Mont-Seade-Verjovsky index is zero.

In Proposition 5.3, we characterize the nondicritical generalized curve foliations by means of the polars of the foliation and its union of separatrices.

Using this result and the Weierstrass form of a 1-form W, in Theorem 5.4, we obtain two conditions: one necessary and one sufficient in order to W defines a generalized curve foliation.

In Theorem 5.11, we will give a characterization for nondicritical generalized curve foliations with a single separatrix by means of the Weierstrass forms. Moreover, in Corollary 5.12, when the separatrix has genus one we present a characterization in terms of the weighted order.

2. Preliminaries

Let $f(x,y)$ be a non unit of the ring of convergent power series $\mathbb{C}\{x,y\}$. A *plane curve* $C : \{f(x,y) = 0\}$ is by definition the zero set determined by $f(x,y) \in \mathbb{C}\{x,y\}$.

The curve C is irreducible (respectively, reduced) if f is irreducible in $\mathbb{C}\{x,y\}$ (respectively, f has no multiple factors). An irreducible plane curve is called *branch*. The *multiplicity* of C, denoted by $\operatorname{mult} C$, is by definition the order of the power series $f(x,y)$, that is $\operatorname{mult} C = \operatorname{ord} f$. Remember that the order of f is the minimum of degrees of terms of f.

Consider a branch $C : \{f(x,y) = 0\}$ of multiplicity n. After a change of coordinates we can suppose that $x = 0$ is transversal (not tangent) to C at 0. From Newton's theorem, C admits an expansion with rational exponents $y(x^{1/n})$, such that $f(x, y(x^{1/n})) = 0$. According to Puiseux, the branch C admits n different expansions $\{y_i(x^{1/n})\}_{1\le i\le n}$, where $y_i(x^{1/n}) = y(\varepsilon_i x^{1/n})$ and $\{\varepsilon_i\}_{1\le i\le n}$ are the nth roots of unity in $\mathbb{C}$. We can write f, up to product by a unit in $\mathbb{C}\{x,y\}$, as the product

$$f(x,y) = \prod_{i=1}^{n} (y - y_i(x^{1/n})).$$

The expansions $\{y_i(x^{1/n})\}_{1\le i\le n}$ are called *Newton–Puiseux roots* of the branch C (or equivalently of f). Any Newton–Puiseux root $y_i(x^{1/n})$ is of the form

$$y_i(x^{1/n}) = \sum_{j\ge n} a_j^{(i)} x^{j/n},$$

where $j \in \mathbb{N}$, $a_j^{(i)} \in \mathbb{C}$ and it determines a *parametrization* of C as follows:

$$(x_i(t), y_i(t)) = \left(t^n, \sum_{j\ge n} a_j^{(i)} t^j \right).$$

It is well known that if the multiplicity of a branch is bigger than one, then there exist $g \in \mathbb{N}\backslash\{0\}$ and positive integers $\beta_0 = n$ and

$$\beta_k = \min\{k \,:\, a_k^{(i)} \neq 0 \text{ and } \gcd(n, \beta_1, \beta_2, \ldots, \beta_{k-1}) \text{ does not divide } k\},$$

for $1 \le k \le g$. In the sequel we put $e_k = \gcd(\beta_0, \beta_1, \beta_2, \ldots, \beta_k)$ for $0 \le k \le g$. We get $e_0 = n > e_1 > \cdots > e_g = 1$. Set $n_k := \frac{e_{k-1}}{e_k}$. In particular, $n = \beta_0 = n_1 \cdots n_g$.

The sequence $(\beta_0, \beta_1, \beta_2, \ldots, \beta_g)$ is called the *Puiseux characteristic exponents* of the branch C and the number g is called the *genus* of the branch C. We denote by $K(\beta_0, \beta_1, \beta_2, \ldots, \beta_g)$ the set of plane branches with Puiseux characteristic exponents $(\beta_0, \beta_1, \beta_2, \ldots, \beta_g)$. It is well known, after Brauner-Zariski, that the characteristic exponents and the topological class of the branch C are equivalent data.

Let $h_1(x,y), h_2(x,y) \in \mathbb{C}\{x,y\}$ be two power series and $I = (h_1, h_2)$ the ideal generated by $h_1, h_2 \in \mathbb{C}\{x,y\}$. The *intersection number* of the curves $C_i : \{h_i(x,y) = 0\}$, $1 \le i \le 2$, is $i_0(h_1, h_2) := \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/I$.

Given $f \in K(\beta_0, \beta_1, \beta_2, \ldots, \beta_g)$ the semigroup $\Gamma(f)$ associated to f is

$$\Gamma(f) = \{i_0(f,h) \colon\ h \in \mathbb{C}\{x,y\}\backslash(f)\}.$$

The semigroup $\Gamma(f)$ admits a unique minimal system of generators, given by $\{v_0, v_1, v_2, \ldots, v_g\}$, that is, $\Gamma(f) = \langle v_0, v_1, v_2, \ldots, v_g\rangle = \{\sum_{i=0}^{g} \alpha_i v_i : \alpha_i \in \mathbb{N}\}$. It is

a well-known fact that the semigroup $\Gamma(f)$ and the characteristic exponents are mutually determined. Moreover, we can obtain the minimal system of generators by the Puiseux characteristic exponents using the relations (see [19, Theorem 3.9]):

$$v_0 = n = \beta_0, \quad v_1 = m = \beta_1, \quad v_{i+1} = n_i v_i + \beta_{i+1} - \beta_i \quad \text{for } i = 1, \ldots, g-1. \tag{1}$$

Observe that $e_i = \gcd(v_0, \ldots, v_i)$ for $0 \leq i \leq g$.

For any reduced plane curve (not necessary irreducible) $C : \{f(x,y) = 0\}$, an important topological invariant, useful in this paper, is the *Milnor number*, that is the intersection number $\mu(f) := i_0(f_x, f_y)$, where f_y (respectively, f_x) denotes the partial derivative of f with respect to y (respectively, x).

By Teissier's lemma (see [17, Chap. II, Proposition 1.2]) we have

$$i_0(f, f_y) = \mu(f) + i_0(f, x) - 1. \tag{2}$$

We also have

$$i_0(f, f_x) = \mu(f) + i_0(f, y) - 1. \tag{3}$$

For more details on plane curves see for example [7] or [18].

Let $\Omega^1_{\mathbb{C}^2,0} := \mathbb{C}\{x,y\}\mathrm{d}x + \mathbb{C}\{x,y\}\mathrm{d}y$ be the $\mathbb{C}\{x,y\}$-module of holomorphic 1-form.

A *holomorphic foliation singular at the origin* is defined, in a neighbourhood of the origin, by an equation $\mathcal{F}_W : \{W = 0\}$, where W is a 1-form $W = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y$, with $R(x,y), S(x,y) \in \mathbb{C}\{x,y\}$ without common factors and such that $R(0,0) = S(0,0) = 0$. The *polar* of $\mathcal{F}_W : \{W = 0\}$ with respect the direction $(b : -a) \in \mathbb{P}^1$ is the curve $aR(x,y) + bS(x,y) = 0$.

The *multiplicity* of W is $\operatorname{mult}(W) := \min\{\operatorname{ord}(R(x,y)), \operatorname{ord}(S(x,y))\}$. More precisely if $\operatorname{mult}(W) = n_0$ then we can write $W = \sum_{i+j \geq n_0} a_{i,j} x^i y^j \mathrm{d}x + \sum_{i+j \geq n_0} b_{i,j} x^i y^j \mathrm{d}y$, and for some i, j such that $i + j = n_0$ we get $a_{i,j} \neq 0$ or $b_{i,j} \neq 0$. The *multiplicity* of the foliation $\mathcal{F}_W : \{W = 0\}$ is by definition the multiplicity of the 1-form W.

Let $\mathcal{F}_W$ be a singular foliation at the origin. The *Jacobian matrix* of the linear part of W is the matrix

$$\begin{pmatrix} -S_x(0,0) & -S_y(0,0) \\ R_x(0,0) & R_y(0,0) \end{pmatrix}.$$

Denote by λ_1, λ_2 its complex eigenvalues. We say that the origin is an *irreducible singularity* of $\mathcal{F}_W$ if one of the following conditions is satisfied:

(1) $\lambda_1 \lambda_2 \neq 0$ and $\frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}^+$,
(2) $\lambda_1 = 0$ and $\lambda_2 \neq 0$; or $\lambda_2 = 0$ and $\lambda_1 \neq 0$.

If the first condition holds we say that the origin is a *simple* or *non-degenerate singularity* of $\mathcal{F}_W$. Nevertheless if the second condition holds then we say that the origin is a *saddle-node singularity* of $\mathcal{F}_W$.

Example 2.1. If $W = y^2\mathrm{d}x + x\mathrm{d}y$ then the Jacobian matrix of the linear part of W is $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Its eigenvalues are -1 y 0, so the origin is a saddle-node singularity of $\mathcal{F}_W$.

Let $\Pi : (M, D) \to (\mathbb{C}^2, 0)$ the *process of reduction of singularities* of the foliation $\mathcal{F}_W$ (see [15]), given by a finite composition of quadratic transformations (blow-ups) having $D = \Pi^{-1}(0)$ as the *exceptional divisor*, which is a finite union of projective lines with normal crossings; and where M is a non-singular analytic manifold. There is a 1-form W' such that $\Pi^{-1}(0) \cdot W' = \Pi^*(W) := \Pi^*(\mathcal{F}_W)$. The foliation $\mathcal{F}_{W'}$ is called the *strict transform* of $\mathcal{F}_W$.

A foliation is *generalized curve* if no saddle-nodes appear in its desingularization.

A branch $C : \{f(x, y) = 0\}$ is a *separatrix* of the foliation $\mathcal{F}_W : \{W = 0\}$ if f divides $W \wedge \mathrm{d}f$. This is equivalent to the existence of a parametrization $(x(t), y(t))$ of the branch C such that $R(x(t), y(t))\mathrm{d}x(t) + S(x(t), y(t))\mathrm{d}y(t) = 0$. In [4], Camacho and Sad prove that any singular foliation in $(\mathbb{C}^2, 0)$ admits a separatrix. Their proof is based on the desingularization process of the foliation. In Example 2.1, the branches $x = 0$ and $y = 0$ are separatrices of $\mathcal{F}_W$. According to the number of separatrices that pass through the singular point of the foliation we can classify the foliations in *dicritical* in the case that we have infinite separatrices or *nondicritical* when we have a finite number of separatrices. As a consequence of [3, Theorem 1] we get the following theorem.

Theorem 2.2. *Let $\mathcal{F}_W$ be a nondicritical foliation with union of separatrices $\{f(x, y) = 0\}$. Then* $\mathrm{mult}(W) \geq \mathrm{ord}(f) - 1$.

Example 2.3. If $W_1 = nx\mathrm{d}y - my\mathrm{d}x$ with n and m coprime positive integers, then the branches $\{x = 0\}, \{y = 0\}$ and $\{y^n - cx^m = 0\}$, $c \neq 0$ are separatrices of $W_1 = 0$, so the foliation $\mathcal{F}_{W_1}$ is dicritical. Nevertheless if $W_2 = x\mathrm{d}y + y\mathrm{d}x$ then the foliation $\mathcal{F}_{W_2}$ is nondicritical since its only separatrices are $\{x = 0\}$ and $\{y = 0\}$.

Let $C_i : \{f_i(x, y) = 0\}$, $1 \leq i \leq r$, be the set of different separatrices of a nondicritical foliation $\mathcal{F}_W : \{W = 0\}$ and put $f(x, y) := f_1(x, y) \cdots f_r(x, y)$. The reduced curve $C : \{f(x, y) = 0\}$ will be called the *union of the separatrices* of the nondicritical foliation $\mathcal{F}_W$.

Other important notion we will use is the Newton polygon that we introduce in the sequel. Let $T \subset \mathbb{N}^2$. We consider the convex hull $\mathtt{conv}(T)$ of the Minkowski sum $T + \mathbb{R}^2_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ denotes the non-negative real numbers. By definition, the *Newton polygon* of T, denoted by $\mathcal{NP}(T)$, is the union of the compact edges of the boundary of $\mathtt{conv}(T)$.

The *support of a power series* $f(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ is $\mathrm{supp}\, f := \{(i, j) \in \mathbb{N}^2 : a_{i,j} \neq 0\}$. The *support of a foliation* $\mathcal{F}_W : \{W = 0\}$, where $W = R(x, y)\mathrm{d}x +$

$S(x,y)\mathrm{d}y$, is the union of the supports of $x \cdot R(x,y)$ and $y \cdot S(x,y)$, and we will denote it $\operatorname{supp} W$, that is $\operatorname{supp} W = \operatorname{supp}(x \cdot R(x,y)) \cup \operatorname{supp}(y \cdot S(x,y))$. The *Newton polygon of a power series* $f(x,y) \in \mathbb{C}\{x,y\}$ is by definition the Newton polygon of $\operatorname{supp} f$ and we will denote it $\mathcal{NP}(f)$. The *Newton polygon of a foliation* $\mathcal{F}_W$ is by definition the Newton polygon of $\operatorname{supp} W$, and we will denote it $\mathcal{NP}(W)$. Observe that the Newton polygon depends on coordinates. Remark that $\mathcal{NP}(u \cdot f) = \mathcal{NP}(f)$ for any $u, f \in \mathbb{C}\{x,y\}$, where u is a unit. Hence, we can define the *Newton polygon* of the curve $C : \{f(x,y) = 0\}$ as the Newton polygon of any of its equations.

Proposition 2.4 ([13, Proposition 3.8]). *Let $W = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y$ be a 1-form. If $\mathcal{F}_W$ is a nondicritical generalized curve foliation and $C : \{f(x,y) = 0\}$ is its union of separatrices then the Newton polygons of $\mathcal{F}_W$ and C are the same.*

Saravia, in her PhD thesis [14] (see also [6, Example 3.3]), shows that the foliation $\mathcal{F}_W$, where $W = ((b-1)xy - y^3)\mathrm{d}x + (xy - bx^2 + xy^2)\mathrm{d}y$, with $b \notin \mathbb{Q}$, is not a generalized curve foliation but its Newton polygon equals to the Newton polygon of its union of separatrices $f(x,y) = xy(x-y)$.

The Milnor number can be also defined in the foliation context. Let $R(x,y), S(x,y) \in \mathbb{C}\{x,y\}$ coprime with $R(0,0) = S(0,0) = 0$. Let $W = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y$ be a 1-form and consider the singular foliation at the origin $\mathcal{F}_W$. The *Milnor number at the origin of the foliation* $\mathcal{F}_W$ is $\mu(\mathcal{F}_W) := i_0(R(x,y), S(x,y))$.

By [3, Theorem 4] we have the following characterization of nondicritical generalized curve foliations using Milnor numbers.

Theorem 2.5. *Let $W = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y$ be a 1-form. Suppose that $\mathcal{F}_W$ is a nondicritical generalized curve foliation and $C : \{f(x,y) = 0\}$ is its union of separatrices. Then $\mu(\mathcal{F}_W) \geq \mu(f)$. The equality holds if and only if $\mathcal{F}_W$ is a generalized curve foliation.*

Let $C : \{f(x,y) = 0\}$ be a reduced plane curve. In what follows we consider:

- $[\mathrm{Fol}(f)]$ the set of all 1-forms W, defining a nondicritical foliation $\mathcal{F}_W$, such that f divides $W \wedge \mathrm{d}f$.
- $\mathrm{Fol}(f)$ the set of nondicrital foliations defined by elements of $[\mathrm{Fol}(f)]$ which union of separatrices is $C : \{f(x,y) = 0\}$.

In this paper, we will present new characterizations for generalized curve foliations.

3. Characterization of Generalized Curve Foliations with Monomial Separatrix

In this section, we characterize the nondicritical generalized curve foliations with monomial separatrix, by means of the weighted order defined as following

Definition 3.1. Let $p, q \in \mathbb{N}\backslash\{0\}$. The *weighted order* $\upsilon_{p,q}$ for power series and 1-forms is:

$$\upsilon_{p,q}\left(\sum_{i,j} a_{i,j}x^i y^j\right) = \min\{ip + jq \ : \ a_{i,j} \neq 0\}$$

and

$$\upsilon_{p,q}\left(\sum_{i,j} R_{i,j}x^{i-1}y^j \mathrm{d}x + \sum_{i,j} S_{i,j}x^i y^{j-1}\mathrm{d}y\right) = \min\{ip + jq \ : \ R_{i,j} \neq 0 \text{ or } S_{i,j} \neq 0\}.$$

Let $f \in \mathbb{C}\{x, y\}$ and n, m positive integers not necessarily coprime. For $f = y^n - x^m$, Frank Loray obtained a *prenormal form* of an arbitrary element $\mathcal{F}_W \in \mathrm{Fol}(f)$.

Theorem 3.2 ([10, page 157]). *If $f = y^n - x^m$ and $\mathcal{F}_W \in \mathrm{Fol}(f)$ then*

$$W = \mathrm{d}f + \sum_{\substack{0\leq i\leq m-2\\ 0\leq j\leq n-2}} P_{i,j}(f)x^i y^j(nx\mathrm{d}y - my\mathrm{d}x), \tag{4}$$

for some $P_{i,j}(z) \in \mathbb{C}\{z\}$.

For $f(x, y) = y^n - x^m$ with $0 < n \leq m$ not necessarily coprime, we will give a characterization when the foliation $\mathcal{F}_W \in \mathrm{Fol}(f)$ is generalized curve.

The 1-form W in (4) can be rewritten as

$$W = \mathrm{d}f + (\Delta(x, y) + f(x, y)g(x, y))(nx\mathrm{d}y - my\mathrm{d}x), \tag{5}$$

where $g \in \mathbb{C}\{x, y\}$ and $\Delta(x, y) = \sum_{0\leq i\leq m-2, 0\leq j\leq n-2.} a_{i,j}x^i y^j$.

Since $m \geq n$, if $d := \gcd(n, m)$ then there are integers $p, q \in \mathbb{N} \setminus \{0\}$ with

$$mp - nq = d. \tag{6}$$

Lemma 3.3. *Let $\mathcal{F}_W \in \mathrm{Fol}(f)$ be a foliation defined by W as in (5). If $\upsilon_{n,m}(\Delta(x, y)) \geq mn - n - m$ then the toric morphism*

$$\Pi(u, v) = (u^p v^{\frac{n}{d}}, u^q v^{\frac{m}{d}}),$$

desingularizes $\mathcal{F}_W$ and the singular points of its strict transform are $(0, 0)$ and $(\xi, 0)$, where $\xi^d = 1$. Moreover the Jacobian matrix at the point $(0, 0)$ is

$$\begin{pmatrix} -\dfrac{mn}{d} & 0 \\ 0 & nq \end{pmatrix} \tag{7}$$

and the Jacobian matrix at the point $(\xi, 0)$ is

$$\begin{pmatrix} mn & 0 \\ * & -d\left(1 + \sum_{r,s} a_{r,s}\xi^{p(r+1)+q(s+1)-qn}\right) \end{pmatrix}, \tag{8}$$

where r, s verify $m(r + 1) + n(s + 1) = nm$.

Proof. Note that $\Pi^* W = u^{qn-1} v^{\frac{mm}{d}-1} \cdot W'$, where

$$W' = [(nq - mpu^d) + (nq - mp)G + h(u,v)(nqu^{p+q} v^{\frac{n+m}{d}} - pmu^{d+p+q} v^{\frac{n+m}{d}})]vdu$$
$$+ \left[\frac{mn}{d}(1-u^d)u + \frac{mnh}{d}(u^{p+q+1} v^{\frac{m+n}{d}} - u^{p+q+d+2} v^{\frac{m+n}{d}})\right] dv, \tag{9}$$

with $G = \sum a_{i,j} u^{p(i+1)+q(j+1)-qn} v^{\frac{m(j+1)+n(i+1)-mn}{d}}$ and $h(u,v) \in \mathbb{C}\{u,v\}$. By hypothesis, if $(i,j) \in \operatorname{supp} \Delta$ then $ni + jm \geq mn - m - n$, and we have $\operatorname{ord}_v(G) \geq 0$. On the other hand we claim that $\operatorname{ord}_u(G) > 0$, that is $p(i+1) + q(j+1) > qn$. In effect, from $ni + jm \geq mn - n - m$ and $mp - nq = d$ we get $p(i+1) + q(j+1) = \frac{pn(i+1)+pm(j+1)}{n} - \frac{d(j+1)}{n} \geq \frac{pmn-d(j+1)}{n}$. Since $0 \leq j \leq n-2$ we have $\frac{pmn-d(j+1)}{n} > \frac{pnm-dn}{n}$, so $p(i+1) + q(j+1) > pm - d = nq$.

Therefore the singular points of $\mathcal{F}_{W'}$ are $(0,0)$ and $(\xi, 0)$, with $\xi^d = 1$. The point $(0,0)$ corresponds to the intersection of the exceptional divisors, the other points correspond to the irreducible components of $y^n - x^m$.

Rewrite (9) as $W' = R(u,v)\mathrm{d}u + S(u,v)\mathrm{d}v$. We get $S_u(0,0) = \frac{mn}{d}$, $S_v(0,0) = R_u(0,0) = 0$ and $R_v(0,0) = nq$, so the Jacobian matrix at $(0,0)$ is (7) and the origin is not a saddle-node. On the other hand $S_u(\xi,0) = \frac{mn}{d}$, $S_v(\xi,0) = R_u(\xi,0) = 0$ and $R_v(\xi,0) = 1 + \sum_{m(r+1)+n(s+1)=mn} a_{r,s}\xi^{p(r+1)+q(s+1)-qn}$. Hence the Jacobian matrix at $(\xi,0)$ is (8). □

Using Lemma 3.3 we get the following theorem.

Theorem 3.4. *Set $f = y^n - x^m$ with $1 \leq n \leq m$. Let $\mathcal{F}_W \in \operatorname{Fol}(f)$ be a foliation defined by W as in (5). Then $\mathcal{F}_W$ is a generalized curve foliation if and only if $v_{n,m}(\Delta(x,y)) \geq mn - n - m$, and $(1 + \sum_{m(r+1)+n(s+1)=mn} a_{r,s}\xi^{p(r+1)+q(s+1)-qn}) \notin \mathbb{Q}^- \cup \{0\}$, where $\xi^d = 1$.*

Proof. Suppose that the foliation $\mathcal{F}_W$ is generalized curve. Since its union of separatrices is $\{f(x,y) = 0\}$ and by Proposition 2.4, we get the equality $\mathcal{NP}(W) = \mathcal{NP}(\mathrm{d}f)$ and we have $v_{n,m}(W) \geq mn$. In addition, using the representation given in (5) we have $v_{n,m}(y \sum a_{i,j} x^i y^j \mathrm{d}x) \geq mn$ and $v_{n,m}(x \sum a_{i,j} x^i y^j \mathrm{d}y) \geq mn$; so $v_{n,m}(\Delta(x,y)) + n + m = n(i+1) + m(j+1) \geq mn$. Moreover the points $(\xi, 0)$ are not saddle-nodes, hence by (8) we get $(1 + \sum_{r,s} a_{r,s}\xi^{p(r+1)+q(s+1)-qn}) \notin \mathbb{Q}^- \cup \{0\}$.

On the other hand, the hypothesis $v_{n,m}(\Delta(x,y)) \geq mn - n - m$ allows us to apply Lemma 3.3. Since $(1 + \sum_{r,s} a_{r,s}\xi^{p(r+1)+q(s+1)-qn}) \notin \mathbb{Q}^- \cup \{0\}$ we conclude that $\mathcal{F}_W \in \operatorname{Fol}(f)$ is a generalized curve foliation. □

Example 3.5. Suppose that $W = \mathrm{d}(y^3 - x^6) + axy(3x\mathrm{d}y - 6y\mathrm{d}x)$, where $a \in \mathbb{C}$. In this case $p = q = 1$ (see (6)), and the toric morphism is $x = uv$, $y = uv^2$. The

total transform of W is

$$\Pi^*W : u^2v^5[((3-6u^3)-3au)v\mathrm{d}u + 6(1-u^3)u\mathrm{d}v] = u^2v^5(R(u,v)\mathrm{d}u + S(u,v)\mathrm{d}v),$$

where $R(u,v) = ((3-6u^3)-3au)v$ and $S(u,v) = 6(1-u^3)u$. Then the singularities of the strict transform of W are $(0,0)$, $(\xi,0)$ with $\xi^3 = 1$. Now $6(r+1)+3(s+1) = 18$ if and only if $(r,s) = (1,1)$. Hence, if $(1+a\xi) \in \mathbb{Q}^- \cup \{0\}$, for some ξ with $xi^3 = 1$, then the foliation $\mathcal{F}_W$ is not generalized curve.

Mattei and Salem [11] consider a family of foliations, more general than the generalized curved foliations, called foliations of the *second type*, where saddle nodes are admitted in the reduction process, provided that they lie in the regular part of the divisor, with their *weak separatrices* (the separatrices associated with the zero eingenvalue) transversal to the divisor.

In the following corollary, we give the condition in order that a foliation of second type becomes a generalized curve foliation following [6, Theorem 1.2(b)]:

Corollary 3.6. *Let $\mathcal{F}_W \in \mathrm{Fol}(f)$ be a foliation defined by W as in* (5)*. Suppose that $\mathcal{F}_W \in \mathrm{Fol}(f)$ is of the second type. Then $\mathcal{F}_W \in \mathrm{Fol}(f)$ is a generalized curve foliation if and only if*

$$\left(1 + \sum_{m(r+1)+n(s+1)=mn} a_{r,s}\xi^{p(r+1)+q(s+1)-qn}\right) \notin \mathbb{Q}^- \cup \{0\}.$$

Proof. It is a consequence of [6, Theorems 1.2, 3.4]. □

Remark 3.7. From Theorem 3.4, we can deduce [6, Proposition 5.7]: let n, m be two positive integers which are not coprime. Consider $f(x,y) = y^n - x^m$ and $\mathcal{F}_W \in \mathrm{Fol}(f)$, where $W = \mathrm{d}f + \Delta'(x,y)(nx\mathrm{d}y - my\mathrm{d}x)$ with $\Delta'(x,y) \in \mathbb{C}\{x,y\}$. If we suppose that $i_0(\Delta', f) > mn - m - n$ then from the proof of Lemma 3.3 we get that $\mathrm{ord}_v(G) > 0$. Hence $-d(1 + \sum_{r,s} a_{r,s}\xi^{p(r+1)+q(s+1)-qn}) = -d$. In particular, the foliation $\mathcal{F}_W$ is generalized curve.

In Example 3.5, we have a family of nondicritical generalized curve foliations $\mathcal{F}_W$ with $i_0(\Delta', f) = mn - m - n$ when $(1+a\xi) \notin \mathbb{Q}^- \cup \{0\}$.

4. Weierstrass 1-Forms

In this section, we introduce, using the Weierstrass division of power series, a distinguished equation for a given 1-form, with respect to any polynomial having as principal coefficient a unit of $\mathbb{C}\{x\}$, called *Weierstrass 1-forms*. These expressions turn out useful for calculations and in particular in this work for the study (and characterizations) of generalized curve foliations. The Weierstrass 1-forms are well-defined for any 1-form with respect to any polynomial in $\mathbb{C}\{x\}[y]$ with a unit as leading coefficient. Nevertheless, in this paper we are using the Weierstrass 1-forms associated with 1-forms defining nondicritical foliations.

Let $f = \sum_{i=0}^{n} a_i(x)y^{n-i} \in \mathbb{C}\{x\}[y]$ be a polynomial (not necessarily irreducible) of degree $\deg_y(f) = n$, $a_0(0) \neq 0$ and $W \in \Omega^1_{\mathbb{C}^2,0}$ be a 1-form. Here we do not suppose that $C : \{f(x,y) = 0\}$ is the union of separatrices of the foliation $\mathcal{F}_W$.

We will assume that $\deg_y(0) = -\infty$.

In the following lemma, we obtain an equation of W in function of f and $\mathrm{d}f$.

Lemma 4.1. *If $W \in \Omega^1_{\mathbb{C}^2,0}$ and $f(x,y) = \sum_{i=0}^{n} a_i(x)y^{n-i} \in \mathbb{C}\{x\}[y]$ with $\deg_y(f) = n > 1$ and $a_0(0) \neq 0$, then there exist unique $h, p \in \mathbb{C}\{x,y\}$ and $A, B \in \mathbb{C}\{x\}[y]$ with $\deg_y(B) < n-1$ and $\deg_y(A) < n$ such that*

$$W = h\mathrm{d}f + pf\mathrm{d}x + A\mathrm{d}x + B\mathrm{d}y. \tag{10}$$

Proof. Put $W = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y \in \Omega^1_{\mathbb{C}^2,0}$. After the Weierstrass division of S by f_y, there exist $h \in \mathbb{C}\{x,y\}$ and $B \in \mathbb{C}\{x\}[y]$ such that $S = hf_y + B$, where $\deg_y B < \deg_y f_y = \deg_y f - 1 = n-1$ and $W = R\mathrm{d}x + hf_y\mathrm{d}y + B\mathrm{d}y$. Since $\mathrm{d}f = f_x\mathrm{d}x + f_y\mathrm{d}y$ then $W = (R - hf_x)\mathrm{d}x + h\mathrm{d}f + B\mathrm{d}y$. Now, by the Weierstrass division of $R - hf_x$ by f there exist $p \in \mathbb{C}\{x,y\}$ and $A \in \mathbb{C}\{x\}[y]$ such that $R - hf_x = pf + A$ with $\deg_y A < \deg_y f = n$ and

$$W = h\mathrm{d}f + pf\mathrm{d}x + A\mathrm{d}x + B\mathrm{d}y.$$

Suppose that $h_1\mathrm{d}f + p_1f\mathrm{d}x + A_1\mathrm{d}x + B_1\mathrm{d}y = h_2\mathrm{d}f + p_2f\mathrm{d}x + A_2\mathrm{d}x + B_2\mathrm{d}y$ with $h_i, p_i \in \mathbb{C}\{x,y\}$ and $A_i, B_i \in \mathbb{C}\{x\}[y]$ with $\deg_y(B_i) < n-1$ and $\deg_y(A_i) < n$ for $i = 1, 2$. Then $(h_1 - h_2)f_y = B_2 - B_1$ and $(h_1 - h_2)f_x + (p_1 - p_2)f = A_2 - A_1$.

As $\deg_y(B_i) < n - 1 = \deg_y(f_y)$, we get $h_1 = h_2$ and $B_1 = B_2$. Similarly, $\deg_y(A_i) < n = \deg_y(f)$ implies that $p_1 = p_2$ and $A_1 = A_2$. □

Definition 4.2. Let $W \in \Omega^1_{\mathbb{C}^2,0}$ and $f = \sum_{i=0}^{n} a_i(x)y^{n-i} \in \mathbb{C}\{x\}[y]$ with $\deg_y(f) = n > 1$ and $a_0(0) \neq 0$. The *Weierstrass form* of W with respect to f is $h\mathrm{d}f + pf\mathrm{d}x + A\mathrm{d}x + B\mathrm{d}y$, where $h, p \in \mathbb{C}\{x,y\}$ and $A, B \in \mathbb{C}\{x\}[y]$ are unique and satisfying $\deg_y(B) < n-1$ and $\deg_y(A) < n$.

We can rewrite (10) as

$$W = h\mathrm{d}f + pf\mathrm{d}x + \omega, \tag{11}$$

where $\omega = \sum A_{i,j}x^{i-1}y^j\mathrm{d}x + \sum B_{i,j}x^iy^{j-1}\mathrm{d}y = A\mathrm{d}x + B\mathrm{d}y$ for some $A_{i,j}, B_{i,j} \in \mathbb{C}$. Since $\deg_y B < n-1$ then $B_{0,n} = 0$.

Observe that, in (11), h, p and ω depend on f.

Moreover if $W = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y \in \Omega^1_{\mathbb{C}^2,0}$, where $R(x,y), S(x,y) \in \mathbb{C}\{x\}[y]$ and $f \in \mathbb{C}\{x\}[y]$ with $\deg_y(f) > \max\{\deg_y(R), \deg_y(S)+1\}$ then, in (11), $h = p = 0$ and the Weierstrass form of W is ω.

Example 4.3. Consider the prenormal form $W = \mathrm{d}f + (\Delta(x,y) + f(x,y)g(x,y))(nx\mathrm{d}y - my\mathrm{d}x)$ given by Loray (see (5)), where $f(x,y) = y^n - x^m$ and $\Delta(x,y) = \sum_{0 \le i \le m-2,\, 0 \le j \le n-2} a_{i,j}x^iy^j$.

Applying the Weierstrass division there are $\varphi \in \mathbb{C}\{x,y\}$ and $B \in \mathbb{C}\{x\}[y]$ such that $(\Delta(x,y) + f(x,y)g(x,y))nx = \varphi f_y + B$, $\deg_y B < n-1$. Moreover $\varphi(0,0) = 0$. Hence $W = \mathrm{d}f - myg f\mathrm{d}x + \varphi(\mathrm{d}f - f_x\mathrm{d}x) + B\mathrm{d}y - my\Delta(x,y)\mathrm{d}x$.

Similarly there are $\psi \in \mathbb{C}\{x,y\}$ and $A \in \mathbb{C}\{x\}[y]$ such that $-\varphi f_x - my\Delta(x,y) = \psi f + A$ with $\deg_y A < n$. Hence the Weierstrass form of W is

$$W = (1+\varphi)\mathrm{d}f + (\psi - myg)f\mathrm{d}x + A\mathrm{d}x + B\mathrm{d}y,$$

with $h(x,y) := 1+\varphi$ and $p := \psi - myg$. In this example, $h(0,0) = 1$, so h is a unit of $\mathbb{C}\{x,y\}$.

Remark 4.4. Note that the Kähler differential module of the local ring $\mathcal{O}_f := \mathbb{C}\{x,y\}/(f)$ is $\Omega_f := \Omega^1_{\mathbb{C}^2,0}/J$, where $J := \mathbb{C}\{x,y\}\mathrm{d}f + f\Omega^1_{\mathbb{C}^2,0}$. So, given $W \in \Omega^1_{\mathbb{C}^2,0}$, the 1-form ω in (11) represents the class of W in Ω_f. In particular, if f is irreducible and $\varphi(t) = (x(t), y(t))$ is a parametrization of $f(x,y) = 0$ then $\varphi^*(W) = \varphi^*(\omega)$ and $\mathrm{ord}_t\varphi^*(W) = \mathrm{ord}_t\varphi^*(\omega)$. The set $\Lambda = \{\mathrm{ord}_t\varphi^*(W) + 1 : W \in \Omega^1_{\mathbb{C}^2,0}\}$ is an important analytic invariant of $f(x,y) = 0$ (see [8, 19]).

Proposition 4.5. *Let $W \in [\mathrm{Fol}(f)]$ as (11), where h is a unit of $\mathbb{C}\{x,y\}$ and $f(x,y) = \sum_{i=0}^n a_i(x)y^{n-i} \in \mathbb{C}\{x\}[y]$ with $n = \deg_y(f) = \mathrm{ord}f$ and $a_0(0) \neq 0$. Then $\mathcal{F}_W \in \mathrm{Fol}(f)$, that is C is the union of separatrices of $\mathcal{F}_W$.*

Proof. Given that $\deg_y(f) = \mathrm{ord}f$, the multiplicity of W equals the multiplicity of $h\mathrm{d}f + \omega$. Since h is a unit we get $\mathrm{mult}\,(W) = \min\{\mathrm{ord}(f_x + A), \mathrm{ord}(f_y + B)\} \leq \mathrm{ord}(f_y + B)$. But $B_{0,n} = 0$ and $\mathrm{ord}f = \deg_y(f)$, so $\mathrm{ord}(f_y + B) \leq n-1$. Hence

$$\mathrm{mult}\,(W) \leq n-1. \tag{12}$$

Suppose that $\mathcal{F}_W$ has other separatrix $g(x,y) = 0$. By Theorem 2.2, we have $\mathrm{mult}\,(W) \geq \mathrm{ord}(fg) - 1 > n-1$, which is a contradiction after the inequality (12). □

In Section 5, we will see how the Weierstrass 1-forms allow us to give new a characterization of generalized curve foliations with one separatrix and sufficient conditions for the general case. We can also characterize the second type foliations using the Weierstrass 1-forms. To do this, we first remember the characterization given by Mattei and Salem.

Theorem 4.6 ([11, Théorème 3.1.9]). *Let W be a 1-form. Suppose that $\mathcal{F}_W$ is a non dicritical foliation and $C : \{f(x,y) = 0\}$ is its union of separatrices. Then $\mathcal{F}_W$ is a second type foliation if and only if* $\mathrm{mult}(W) = \mathrm{mult}(\mathrm{d}f)$.

Let $f(x,y) = \sum_{i=0}^n a_i(x)y^{n-i} \in \mathbb{C}\{x\}[y]$ with $n = \deg_y(f) = \mathrm{ord}f$ and $a_0(0) \neq 0$. Consider a 1-form $W \in \Omega^1_{\mathbb{C}^2,0}$ in its Weierstrass form as (11), where $\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$. Let us relate some algebraic aspects of the 1-forms W, ω and the *parameters* h and p in the Weierstrass form.

Note that $\frac{W\wedge \mathrm{d}f}{\mathrm{d}x\wedge \mathrm{d}y} \in (f)$ if and only if $\frac{\omega\wedge \mathrm{d}f}{\mathrm{d}x\wedge \mathrm{d}y} \in (f)$. More specifically, g is the cofactor of W, that is $W \wedge \mathrm{d}f = gf\mathrm{d}x \wedge \mathrm{d}y$, if and only if the cofactor of ω is $g - pf_y$.

In addition, remark that:

- If $h(0,0) = 0$ then $\mathrm{mult}\,(W) = \mathrm{mult}\,(\mathrm{d}f)$ if and only if $\mathrm{mult}\,(\omega) = \mathrm{mult}\,(\mathrm{d}f)$.
- If $h(0,0) \neq 0$ then $\mathrm{mult}\,(W) = \mathrm{mult}\,(\mathrm{d}f)$ if and only if $\mathrm{mult}\,(\omega) \geq \mathrm{mult}\,(\mathrm{d}f)$.

As, by Theorem 4.6, the equality $\mathrm{mult}\,(W) = \mathrm{mult}\,(\mathrm{d}f)$ characterizes the 1-forms that define second type foliations, we can read this property using ω.

In (11) we remarked that for any Weierstrass form $W = h\mathrm{d}f + pf\mathrm{d}x + \omega$, where $\omega = \sum A_{i,j}x^{i-1}y^j\mathrm{d}x + \sum B_{i,j}x^iy^{j-1}\mathrm{d}y = A\mathrm{d}x + B\mathrm{d}y$ for some $A_{i,j}, B_{i,j} \in \mathbb{C}$, we get $B_{0,n} = 0$.

If $f \in K(n,m)$ and $W \in [\mathrm{Fol}(f)]$ we can obtain additional information about the coefficient A.

We can write, without lost of generality

$$f(x,y) = y^n - x^m + \sum_{in+jm>nm} a_{i,j}x^iy^j. \tag{13}$$

Lemma 4.7. *Let $f \in K(n,m)$. If $W \in [\mathrm{Fol}(f)]$, with W as in* (11), *then $A_{m,0} = 0$.*

Proof. Consider $f(x,y)$ as in (13). Since $C\colon\{f(x,y) = 0\}$ is the separatrix of $\mathcal{F}_W$ and $\omega = W - h\mathrm{d}f - pf\mathrm{d}x$ then $f(x,y) = 0$ is also a separatrix of w. Consider a parametrization of $f(x,y) = 0$ given by $(x(t), y(t)) = (t^n, t^m + \cdots)$, where $\cdots$ means terms of greater degree. Therefore $\sum A_{i,j}x(t)^{i-1}y(t)^j\mathrm{d}x(t) + \sum B_{i,j}x(t)^iy(t)^{j-1}\mathrm{d}y(t) = 0$. Suppose that $A_{m,0} \neq 0$. Since $A_{m,0}x(t)^{m-1}\mathrm{d}x(t) = nA_{m,0}t^{nm-1}$ then there is $(i_0, j_0) \in \operatorname{supp}\mathcal{A} \cup \operatorname{supp}\mathcal{B}$ such that $ni_0 + mj_0 = nm$, where $\mathcal{A} = \sum A_{i,j}x(t)^{i-1}y(t)^j\mathrm{d}x(t) - nA_{m,0}t^{nm-1}$ and $\mathcal{B} = \sum B_{i,j}x(t)^iy(t)^{j-1}\mathrm{d}y(t)$. But n and m are coprime, so $(i_0, j_0) \in \{(m,0),(0,n)\}$, which is a contradiction since these points are not in $\operatorname{supp}\mathcal{A} \cup \operatorname{supp}\mathcal{B}$. □

We finish this section by studying the behavior of the Weierstrass 1-forms after a blowing-up.

Let $f = \sum_{i=0}^n a_i(x)y^{n-i} \in \mathbb{C}\{x\}[y]$ be a polynomial (not necessarily irreducible) with $\deg_y(f) = \mathrm{ord}f = n$, $a_0(0) \neq 0$ and $W \in \Omega^1_{\mathbb{C}^2,0}$. Considering the Weierstrass form of W respect to f, we have $W = h\mathrm{d}f + pf\mathrm{d}x + \omega$, with $\omega = A\mathrm{d}x + B\mathrm{d}y$.

Denote $q^{(1)}$ (respectively, $V^{(1)}$) the strict transform of any $q \in \mathbb{C}\{x,y\}$ (respectively, on any 1-form V) by the canonical quadratic transformation $(x,y) \to (x,xy)$ defining a blowing-up.

Lemma 4.8. *Under the above conditions, we have*

$$\begin{aligned} W^{(1)} &= x^{n-v-\nu(h)}h^{(1)}(x,y)\mathrm{d}(f^{(1)}) \\ &\quad + [nx^{n-v-1+\nu(h)}h^{(1)}(x,y) + x^{n-v+\nu(p)}p^{(1)}(x,y)]f^{(1)}\mathrm{d}x + x^{v_0-v}\omega^{(1)}, \end{aligned}$$

where $v = \mathrm{mult}(W)$, $v_0 = \mathrm{mult}(\omega)$ *and* $\nu(\cdot)$ *denotes the order of an element of* $\mathbb{C}\{x, y\}$.

Proof. If $\mathrm{mult}(W) < n - 1$ we have $p = h = 0$ in the Weierstrass form and the result is evident.

We consider now $\mathrm{mult}(W) = v \geq n - 1$ and $\mathrm{mult}(\omega) = v_0$.

Since $\deg_y(f) = \mathrm{ord} f = n$ and $a_0(0) \neq 0$ we get $\mathrm{ord}(a_i(x)) \geq i$ for all $i = 0, \ldots, n$ and consequently we get

$$(f^{(1)})_y = (f_y)^{(1)} \quad \text{and} \quad x(f^{(1)})_x = x^{s-n+1}(f_x)^{(1)} + y(f_y)^{(1)} - nf^{(1)}, \tag{14}$$

where $s = \mathrm{ord}(f_x) \geq n - 1$. In particular, we have

$$(\mathrm{d}f)^{(1)} = x\mathrm{d}(f^{(1)}) + nf^{(1)}\mathrm{d}x. \tag{15}$$

Similarly, a calculation shows that

$$\omega^{(1)} = \frac{1}{x^{\nu_0}}((A(x, xy) + yB(x, xy))\mathrm{d}x + xB(x, xy)\mathrm{d}y). \tag{16}$$

Now from Equations (14)–(16) we have

$$\begin{aligned} W^{(1)} = {} & x^{n-v-\nu(h)}h^{(1)}(x, y)\mathrm{d}(f^{(1)}) + [nx^{n-v-1+\nu(h)}h^{(1)}(x, y) \\ & + x^{n-v+\nu(p)}p^{(1)}(x, y)]f^{(1)}\mathrm{d}x + x^{v_0-v}\omega^{(1)}. \end{aligned}$$

□

With the notations of Lemma 4.8 we notice that $\deg_y(xB(x, xy)) = \deg_y(B) < n - 1 = \deg_y((f^{(1)})_y) = \deg_y(f^{(1)}) - 1$ and

$$\begin{aligned} \deg_y(A(x, xy) + yB(x, xy)) & \leq \min\{\deg_y(A(x, xy), \deg_y(yB(x, xy))\} \\ & < n = \deg_y(f^{(1)}). \end{aligned}$$

Remark 4.9. If $W \in \mathrm{Fol}(f)$ and $\nu = n - 1$, that is, W define a second type foliation then

$$\begin{aligned} W^{(1)} = {} & x^{\nu(h)+1}h^{(1)}(x, y)\mathrm{d}(f^{(1)}) \\ & + (nx^{\nu(h)}h^{(1)}(x, y) + x^{\nu(p)+1}p^{(1)}(x, y))f^{(1)}\mathrm{d}x + x^{v_0-v}\omega^{(1)}. \end{aligned}$$

5. Characterization of Generalized Curve Foliations

In this section, we present our main results. For that we need the notion of *GSV-index*.

Let $f(x, y) \in \mathbb{C}\{x, y\}$ and $W \in [\mathrm{Fol}(f)]$. By [9, p. 198] in the irreducible case and [16, (1.1) Lemma] in the reduced case, there are $g, k \in \mathbb{C}\{x, y\}$ and a 1-form η such that $gW = k\mathrm{d}f + f\eta$, with f and k coprime.

Definition 5.1. With the above notations, the GSV-index of W with respect to $C:\{f(x,y)=0\}$ is

$$GSV(W,C) := \frac{1}{2\pi i}\int_{\partial C}\frac{g}{k}\mathrm{d}\left(\frac{k}{g}\right).$$

Suppose that $f(x,y)$ is irreducible and consider a parametrization $(x(t),y(t))$ of C. Write $W = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y$ and $\eta = p(x,y)\mathrm{d}x + q(x,y)\mathrm{d}y$. We have

$$W = \left(\frac{k}{g}f_x + \frac{f}{g}p\right)\mathrm{d}x + \left(\frac{k}{g}f_y + \frac{f}{g}q\right)\mathrm{d}y.$$

On the other hand,

$$\frac{k}{g}(x(t),y(t)) = \frac{(\frac{k}{g}f_y + \frac{f}{g}q)}{f_y}(x(t),y(t)),$$

so the number of zeroes of k restricted to C minus the number of zeroes of g restricted to C equals $\mathrm{ord}_t\frac{S}{f_y}(x(t),y(t))$. Hence by Rouché–Hurwitz theorem, we have

$$GSV(W,C) = \mathrm{ord}_t\frac{S}{f_y}(x(t),y(t)). \tag{17}$$

A similar calculation as before shows that

$$GSV(W,C) = \mathrm{ord}_t\frac{R}{f_x}(x(t),y(t)).$$

Now, consider $C:\{f(x,y)=0\}$ for $f = f_1\cdot f_2$ reducible. After [2, p. 532], we have

$$GSV(W,C) = GSV(W,C_1) + GSV(W,C_2) - 2i_0(f_1,f_2), \tag{18}$$

where $C_i:\{f_i(x,y)=0\}$.

Cavalier and Lehmann gave a characterization of generalized curve foliations using the GSV-index:

Theorem 5.2 ([5, Théorème 3.3]). *Let $C:\{f(x,y)=0\}$ be a reduced curve and $\mathcal{F}_W \in \mathrm{Fol}(f)$ a nondicritical foliation. Then $\mathcal{F}_W$ is generalized curve if and only if $GSV(W,C)=0$.*

Let $f = \sum_{i=0}^{n} a_i(x)y^{n-i} \in \mathbb{C}\{x\}[y]$, reduced with $a_0(0)\neq 0$ and $f = f_1\cdots f_r$ is its factorization into irreducible factors. Consider $C:\{f(x,y)=0\}$ and $C_j:\{f_j(x,y)=0\}$, for $1\leq j\leq r$.

The following proposition stablishes a characterization of nondicritical generalized curve foliations in terms of the polar S of the foliation $\mathcal{F}_W$ and the *polar* f_y of the separatrix $C:\{f(x,y)=0\}$.

Proposition 5.3. *Fix $f = \sum_{i=0}^{n} a_i(x)y^{n-i} \in \mathbb{C}\{x\}[y]$, reduced with $a_0(0)\neq 0$. Let $\mathcal{F}_W \in \mathrm{Fol}(f)$ with $W = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y$. Then $\mathcal{F}_W$ is a generalized curve foliation if and only if $i_0(S,f) = \mu(f) + i_0(f,x) - 1 = i_0(f_y,f)$.*

Proof. Let $(x_i(t), y_i(t))$ be a parametrization of $C_i : \{f_i(x,y) = 0\}$ for $1 \leq i \leq r$. By (17) we have $GSV(W, C_i) = \mathrm{ord}_t(\frac{S(x_i(t),y_i(t))}{(f_i)_y(x_i(t),y_i(t))}) = i_0(S, f_i) - i_0((f_i)_y, f_i)$. Hence, after (18) we get

$$GSV(W, C) = \sum_{i=1}^{r} i_0(S, f_i) - \sum_{i=1}^{r} i_0((f_i)_y, f_i) - 2 \sum_{1 \leq i < j \leq r} i_0(f_i, f_j).$$

Since $i_0(S, f) = \sum_{i=1}^{r} i_0(S, f_i)$, using Teissier formula (see (2)) we have $i_0((f_i)_y, f_i) = \mu(f_i) + i_0(f_i, x) - 1$. Consequently

$$GSV(W, C) = i_0(S, f) - \sum_{i=1}^{r} \mu(f_i) - i_0(f, x) + r - 2 \sum_{1 \leq i < j \leq r} i_0(f_i, f_j).$$

Now by the Milnor formula for reduced curves (see [18, Theorem 6.5.1]), we get

$$GSV(W, C) = i_0(S, f) - \mu(f) - i_0(f, x) + 1.$$

By hypothesis f is reduced and x does not divide f, so $\mu(f) - i_0(f, x) + 1$ is finite and after Theorem 5.2, $\mathcal{F}_W$ is a generalized curve foliation if and only if

$$i_0(S, f) = \mu(f) + i_0(f, x) - 1 = i_0(f, f_y). \qquad \square$$

In the following theorem, we give information of a generalized curve foliation by means of the Weierstrass form with respect to the union of its separatrices.

Theorem 5.4. *Fix $f = \sum_{i=0}^{n} a_i(x) y^{n-i} \in \mathbb{C}\{x\}[y]$, reduced with $a_0(0) \neq 0$. Let $\mathcal{F}_W \in \mathrm{Fol}(f)$ and $h\mathrm{d}f + pf\mathrm{d}x + A\mathrm{d}x + B\mathrm{d}y$ the Weierstrass form of W with respect to f.*

(a) *If $\mathcal{F}_W$ is a generalized curve foliation then h is a unit and $i_0(B, f) \geq i_0(f, f_y)$.*
(b) *If h is a unit and $i_0(B, f) > i_0(f, f_y)$ then $\mathcal{F}_W$ is a generalized curve foliation.*

Proof. First we suppose that $\mathcal{F}_W$ is a generalized curve foliation which union of separatrices is $C : \{f(x,y) = 0\}$. Hence, by Theorem 2.4, we have that $\mathcal{NP}(W) = \mathcal{NP}(f)$. Moreover, since $h\mathrm{d}f + pf\mathrm{d}x + A\mathrm{d}x + B\mathrm{d}y$ is the Weierstrass form of $W = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y$ with respect to f we get $B_{0,n} = 0$, in particular we conclude that h is a unit. On the other hand, by Proposition 5.3, $i_0(f_y, f) = i_0(S, f) = i_0(hf_y + B, f)$. We finish the proof after properties of the intersection number.

If h is a unit and $i_0(B, f) > i_0(f, f_y)$ then $i_0(hf_y, f) = i_0(f_y, f) < i_0(B, f)$. So, $i_0(S, f) = i_0(hf_y + B, f) = i_0(f_y, f)$ and, by Proposition 5.3, $\mathcal{F}_W$ is a generalized curve foliation. $\square$

The following example illustrates that if h is a unit and $i_0(B, f) = i_0(f_y, f)$ then we cannot conclude that $\mathcal{F}_W$ is a generalized curve foliation.

Example 5.5. Consider $W = \mathrm{d}(y^3 - x^6) + axy(3x\mathrm{d}y - 6y\mathrm{d}x)$, where $a \in \mathbb{C}^*$ given in Example 3.5.

In this case, $W = R(x,y)\mathrm{d}x + S(x,y)\mathrm{d}y$ where $R(x,y) = -6x(x^4 + ay^2), S(x,y) = 3y(y+ax^2)$ and $i_0(S,f) = i_0(f_y,f) = 12$ if and only if $a^3 \neq -1$. Hence, after Proposition 5.3, $\mathcal{F}_W$ is a generalized curve foliation if and only if $a^3 \neq -1$.

On the other hand, the Weierstrass form of W with respect to $f = y^3 - x^6$ is given by $h\mathrm{d}f + pf\mathrm{d}x + A\mathrm{d}x + B\mathrm{d}y$ with $h = 1$, $p = 0$, $A = -6axy^2$ and $B = 3ax^2y$. So, for $a^3 = -1$ we get h a unit, $i_0(B,f) = 12 = i_0(f_y,f)$ but $\mathcal{F}_W$ is not a generalized curve foliation.

In fact, as we state in Theorem 3.4, in addition to the intersection number hypothesis we need a condition on the coefficient $a \in \mathbb{C}^*$.

The situation presented in the above example does not occur for the case of foliation with just one separatrix as we will show in the following subsection.

5.1. *The irreducible case*

Let $f \in \mathbb{C}\{x\}[y]$ an irreducible monic polynomial of degree $n = \mathrm{ord} f$ with semigroup $\Gamma(f) = \langle v_0, v_1, v_2, \ldots, v_g\rangle$ $(v_0 = n)$. Remember that we denote $e_i = \gcd(v_0, \ldots, v_i)$ for $i \in \{0, \ldots, g\}$ and $n_i = \frac{e_{i-1}}{e_i}$ for $i \in \{1, \ldots, g\}$. In addition, as $\deg_y f = \mathrm{ord} f = n$ we get $i_0(f,x) = v_0$. By convention we put $n_0 = 1$. We say that $f_k \in \mathbb{C}\{x\}[y]$ is a *k-semiroot* of the polynomial f if f_k is monic, $\deg_y(f_k) = \frac{v_0}{e_k} = n_0 n_1 \cdots n_k$ and $i_0(f_k, f) = v_{k+1}$ for $k = 0, \ldots, g$ where $v_{g+1} = \infty$. Note that $f_g = f$. The notion of semiroot is a generalization of the *characteristic approximate roots* introduced and studied by Abhyankar and Moh in [1].

Applying [12, Corollary 5.4] to the polynomial $B \in \mathbb{C}\{x\}[y]$ in (10), it can be uniquely written as a finite sum of the form

$$B = \sum_{\text{finite}} a_\alpha(x) f_0^{\alpha_1} \cdots f_{g-1}^{\alpha_g}, \tag{19}$$

where $a_\alpha(x) \in \mathbb{C}\{x\}$, f_k are k-semiroots of f and $0 \leq \alpha_j < n_j$ for $j = 1, \ldots, g$ and $\alpha_{g+1} \in \mathbb{N}$. Since $\deg_y B < \deg_y f - 1$ the polynomial $f_g = f$ does not appear as a factor in the terms of the right-hand side of (19).

As a k-semiroot f_k is irreducible and admits semigroup $\langle \frac{v_0}{e_k} = \frac{n}{e_k}, \frac{v_1}{e_k} = \frac{m}{e_k}, \ldots, \frac{v_k}{e_k}\rangle$ it follows that its Newton polygon has a single compact face with vertices $(0, \frac{m}{e_k})$ and $(\frac{n}{e_k}, 0)$ and consequently $v_{n,m}(f_k) = \frac{nm}{e_k}$.

In this way, using the relations (1) we get

$$\begin{aligned} i_0(f, f_k) = v_{k+1} &= \frac{nm}{e_k} + \sum_{i=2}^{k+1} \frac{n_i \cdots n_{k+1}}{n_{k+1}} (\beta_i - \beta_{i-1}) \\ &= v_{n,m}(f_k) + \sum_{i=2}^{k+1} \frac{e_{i-1}}{e_k} (\beta_i - \beta_{i-1}). \end{aligned} \tag{20}$$

In particular, $i_0(f, f_k) \geq v_{n,m}(f_k)$ with equality if and only if $k = 0$.

Remark 5.6. For any two distinct terms $T_\alpha := a_\alpha(x) f_0^{\alpha_1} \cdots f_{g-1}^{\alpha_g}$ and $T_{\alpha'} := a_{\alpha'}(x) f_0^{\alpha'_1} \cdots f_{g-1}^{\alpha'_g}$ of (19) we get $i_0(T_\alpha, f) \neq i_0(T_{\alpha'}, f)$.

In addition, remark that we cannot have $\alpha_i = n_i - 1$ for all $i = 1, \ldots, g$. Indeed, if this is the case, we get

$$\begin{aligned}\deg_y(B) &= \sum_{i=1}^{g} (n_i - 1) \deg_y(f_{i-1}) = \sum_{i=1}^{g} (n_i - 1) n_0 n_1 \cdots n_{i-1} \\ &= \sum_{i=1}^{g} n_0 n_1 \cdots n_{i-1} n_i - \sum_{i=1}^{g} n_0 n_1 \cdots n_{i-1} \\ &= n_0 n_1 \cdots n_g - n_0 = v_0 - 1 = n - 1,\end{aligned}$$

which is a contradiction.

As a consequence of (20) and Remark 5.6 we have that $i_0(f, B) \geq v_{n,m}(f)$ with equality if and only if $f \in K(n, m)$.

Lemma 5.7. *With the above notations, if f is irreducible and $B \in \mathbb{C}\{x\}[y]$ with $\deg_y B < n - 1$ then $i_0(B, f) \neq i_0(f_y, f)$.*

Proof. Suppose that $i_0(f_y, f) = i_0(B, f)$. After Remark 5.6, there exists a unique g-tuple $(\alpha_1, \ldots, \alpha_g)$ with $0 \leq \alpha_i < n_i$ such that

$$i_0(B, f) = i_0(a_\alpha(x) f_0^{\alpha_1} \cdot \ldots \cdot f_{g-1}^{\alpha_g}, f) = \sum_{i=1}^{g} \alpha_i v_i + \lambda_0 v_0,$$

where $\lambda_0 = \operatorname{ord}_x a_\alpha(x)$. Now, by (2) we get $i_0(f_y, f) = \mu(f) + i_0(f, x) - 1 = \mu(f) + v_0 - 1 = \sum_{i=1}^{g} (n_i - 1) v_i$ (see, for example [7, Proposition 7.5(ii), p. 102] for the last equality). In this way, we have

$$\sum_{i=1}^{g} (n_i - 1) v_i = i_0(f_y, f) = i_0(B, f) = \sum_{i=1}^{g} \alpha_i v_i + \lambda_0 v_0,$$

that is $\sum_{i=1}^{g} (n_i - 1 - \alpha_i) v_i - \lambda_0 v_0 = 0$. But, this implies that $\lambda_0 = 0$ and $\alpha_i = n_i - 1$ for all $i = 1, \ldots, g$, which is a contradiction. Hence, $i_0(f_y, f) \neq i_0(B, f)$. □

Remark 5.8. Observe that Lemma 5.7 is not true for f reduced (non-irreducible): consider $f(x, y) = y^2 - x^2$ and $B = x^\alpha$. We have $\deg_y(B) = 0 < 1 = \deg_y(f) - 1$, $i_0(f_y, f) = 2$ and $i_0(B, f) = 2\alpha$. So, for $\alpha = 1$ we get $i_0(f_y, f) = i_0(B, f)$ and $i_0(f_y, f) \neq i_0(B, f)$ for $\alpha \neq 1$.

Corollary 5.9. *With the above notations, for f irreducible we have, $i_0(B, f) > i_0(f_y, f)$ if and only if $i_0(hf_y + B, f) = \mu(f) + i_0(f, x) - 1$, for any unit $h \in \mathbb{C}\{x, y\}$.*

Proof. Suppose that $i_0(B,f) > i_0(f_y,f) = i_0(hf_y,f)$ since h is a unit. Then

$$i_0(hf_y + B, f) = \min\{i_0(hf_y,f), i_0(B,f)\} = i_0(f_y,f),$$

and by Teissier's lemma (see (2)) we have $i_0(hf_y + B, f) = \mu(f) + i_0(f,x) - 1$. Now we suppose that $i_0(hf_y + B, f) = \mu(f) + i_0(f,x) - 1$ for h unit, that is, by Teissier's lemma, $i_0(hf_y + B, f) = i_0(f, f_y)$ and by Lemma 5.7 we conclude $i_0(B,f) > i_0(f_y,f)$. □

Remark 5.10. If W is written as (10) and $W \in [\mathrm{Fol}(f)]$ then $A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y \in [\mathrm{Fol}(f)]$. We must have $i_0(A,f) + v_0 = i_0(B,f) + v_1$. Hence by (2) and (3) we get $i_0(A,f) - i_0(f_x,f) = i_0(B,f) - i_0(f_y,f)$. So, $i_0(B,f) > i_0(f_y,f)$ if and only if, $i_0(A,f) > i_0(f_x,f)$. Moreover, if f is irreducible then, by Lemma 5.7, we have $i_0(A,f) \neq i_0(f_x,f)$. So, $i_0(hf_x + A, f) = \min\{i_0(f_x,f), i_0(A,f)\}$ for any unit $h \in \mathbb{C}\{x,y\}$. Consequently $i_0(hf_y + B, f) = \mu(f) + i_0(f,x) - 1$ if and only if $i_0(hf_x + A, f) = \mu(f) + i_0(f,y) - 1$ for any unit $h \in \mathbb{C}\{x,y\}$.

The following theorem provides us a characterization of generalized curve foliations with a single separatrix in terms of its Weierstrass form to respect this separatrix.

Theorem 5.11. *Let $f(x,y) \in \mathbb{C}\{x\}[y]$ be irreducible, $\mathcal{F}_W \in \mathrm{Fol}(f)$, where $h\mathrm{d}f + pf\mathrm{d}x + A\mathrm{d}x + B\mathrm{d}y$ is the Weierstrass form of W with respect to f. Then $\mathcal{F}_W$ is a generalized curve foliation if and only if $h \in \mathbb{C}\{x,y\}$ is a unit and $i_0(B,f) > i_0(f_y,f)$.*

Proof. It is a consequence of Theorem 5.4 and Corollary 5.9. □

The following corollary gives us a characterization of generalized curve foliations with a single separatrix of genus 1 in terms of the weighted order (see Definition 3.1).

Corollary 5.12. *Let $f \in K(n,m)$, $\mathcal{F}_W \in \mathrm{Fol}(f)$, where $h\mathrm{d}f + pf\mathrm{d}x + A\mathrm{d}x + B\mathrm{d}y$ is the Weierstrass form of W with respect to f. Then $\mathcal{F}_W$ is a generalized curve foliation if and only if $h \in \mathbb{C}\{x,y\}$ is a unit and $v_{n,m}(\omega) > nm$, where $\omega = A\mathrm{d}x + B\mathrm{d}y$.*

Proof. By Remark 5.10 the equality $i_0(hf_y + B, f) = \mu(f) + i_0(f,x) - 1$ is equivalent to $i_0(hf_x + A, f) = \mu(f) + i_0(f,y) - 1$ for any unit $h \in \mathbb{C}\{x,y\}$. It follows, by Corollary 5.9, that this is equivalent to claim $i_0(f_y,f) < i_0(B,f)$ and $i_0(f_x,f) < i_0(A,f)$. As $f \in K(n,m)$, we have $\mu(f) = nm - n - m + 1$, $i_0(f_y,f) = nm - m$ and $i_0(f_x,f) = nm - n$ (see (2) and (3)). So, $nm < i_0(B,f) + m$ and $nm < i_0(A,f) + n$. Since $i_0(H,f) = v_{n,m}(H)$, for any $H \in \mathbb{C}\{x\}[y]$ with $\deg_y H < n = \deg_y f$ then $i_0(hf_y + B, f) = \mu(f) + i_0(f,x) - 1$ is equivalent to $nm < \min\{v_{n,m}(B) + m, v_{n,m}(A) + n\} = v_{n,m}(\omega)$.We finish the proof using Theorem 5.11 and Corollary 5.9. □

Example 5.13. Let $W_c = \mathrm{d}(y^3 - x^4) + cx^2y(3x\mathrm{d}y - 4y\mathrm{d}x)$, $c \in \mathbb{C}^*$. In this case $n = 3$, $m = 4$ and the Weierstrass form of W_c is $h\mathrm{d}f + pf\mathrm{d}x + \omega$ with $\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = cx^2y(3x\mathrm{d}y - 4y\mathrm{d}x)$, $h = 1$ and $p = 0$.

As h is a unit, we conclude that $\mathcal{F}_{W_c}$ is a generalized curve foliation using Theorem 5.11 since $i_0(B, f) = 13 > 9 = \mu(f) + i_0(f, x) - 1$ and by Corollary 5.12 since $\upsilon_{3,4}(\omega) = 17 > 12 = nm$.

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