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A FORMULA FOR THE DEGREE OF SINGULARITY OF PLANE ALGEBROID CURVES

EVELIA R. GARCÍA BARROSO

Departamento de Matemáticas, Estadística e I.O.
IMAULL, Universidad de La Laguna
38200 La Laguna, Tenerife, España
ORCID: 0000-0001-7575-2619 E-mail: ergarcia@ull.es

ARKADIUSZ PŁOSKI

Department of Mathematics and Physics
Kielce University of Technology
25-314 Kielce, Poland
ORCID: 0000-0001-7596-0928 E-mail: matap@tu.kielce.pl

Abstract. Let $\mathcal{O}_f$ be the local ring of an algebroid reduced curve $\{f = 0\}$ over an algebraically closed field K, $\overline{\mathcal{O}}_f$ its integral closure in the total quotient ring of $\mathcal{O}_f$ and $\mathcal{C}_f$ the conductor of $\mathcal{O}_f$ in $\overline{\mathcal{O}}_f$. The codimension $c(f) = \dim_K \overline{\mathcal{O}}_f/\mathcal{C}_f$ is called the *degree of singularity* of the curve $\{f = 0\}$. Suppose that the Newton polygon $\mathcal{N}_f$ of the curve $\{f = 0\}$ intersects the axes at the points $(m, 0), (0, n)$ and put $c(\mathcal{N}_f) = 2$, (area of the polygon bounded by $\mathcal{N}_f$ and the axes) $+$ (number of integer points on $\mathcal{N}_f$) $- m - n - 1$. We prove that there exists a factorization $f = f_1 \cdots f_s$ of f in $K[[x, y]]$ such that $c(f) = c(\mathcal{N}_f) + \sum_{i=1}^{s} c(\tilde{f}_i)$, where $\{\tilde{f}_i = 0\}$ is obtained as a composition of quadratic transforms of the curve $\{f_i = 0\}$. The proof is effective: the Newton polygon $\mathcal{N}_f$ and the initial parts of f corresponding to the compact edges of $\mathcal{N}_f$ determine the Newton polygons of f_i and the number of quadratic transforms necessary to compute $\tilde{f}_i$. As application of our result we give a formula for the Milnor number of f.

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1. Introduction. Let $K[[x,y]]$ be the ring of formal power series with coefficients in an algebraically closed field K of arbitrary characteristic. For any non-zero power series $f=\sum_{\alpha\beta} c_{\alpha\beta}x^\alpha y^\beta \in K[[x,y]]$ we put $\operatorname{ord} f := \inf\{\alpha+\beta \;:\; c_{\alpha\beta}\neq 0\}$ and $\operatorname{in} f := \sum_{\alpha+\beta=\operatorname{ord} f} c_{\alpha\beta}x^\alpha y^\beta$. By convention $\operatorname{ord} 0 = +\infty$ and $\operatorname{in} 0 = 0$. Observe that $\operatorname{ord} f = 0$ if and only if $f(0,0)=c_{00}\neq 0$.

Let $f\in K[[x,y]]$ be a nonzero power series without constant term. An *algebroid curve* $\{f=0\}$ is defined to be the ideal generated by f in $K[[x,y]]$. The *intersection multiplicity* $i_0(f,g)$ of the curves $\{f=0\}$ and $\{g=0\}$ is equal to the codimension of the ideal generated by the power series $f,g\in K[[x,y]]$.

If the power series f is reduced, that is, without multiple factors, (resp. irreducible) the curve $\{f=0\}$ is called reduced (resp. irreducible or *branch*).

We denote by $r(f)$ the number of irreducible factors (counted with multiplicities) of the formal power series $f\in K[[x,y]]$. The curve $\{f=0\}$ is *singular* if $\operatorname{ord} f>1$. The aim of this note is to study the degree of singularity $c(f)$ (called also the degree of conductor) of a reduced curve $\{f=0\}$. If $\operatorname{char} K=0$ then $c(f)$ satisfies Milnor's formula $i_0\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=c(f)-r(f)+1$ which is not valid in positive characteristic. We define $\mu(f):=c(f)-r(f)+1$ in arbitrary characteristic (see [10], [5]).

The paper is organized as follows. In Section 2 we collect together the basic properties of $c(f)$. Section 3 is devoted to the main results (Theorem A and Theorem B). In Section 4 we construct a modification $\{\tilde{f}=0\}$ of the branch $\{f=0\}$ by means of Hamburger–Noether expansion. The proofs (of Theorem A and Theorem B) are given in Sections 5, 6, 7 in the case of convenient (in the sense of Kouchnirenko) power series. In Section 8 we consider the case of non-convenient power series. The paper ends with a formula for the Milnor number which implies the inequality for $\mu(f)$ due to Boubakri, Greuel and Markwig (see [2]).

We refer the reader to our paper [7] in which we give an overview of the properties of Newton polygons. In absence of Puiseux' theorem in positive characteristic we use a factorization of the formal power series in terms of the Newton polygon.

2. Degree of singularity. For any reduced curve $\{f=0\}$ we put $\mathcal{O}_f=K[[x,y]]/(f)$. Let $\overline{\mathcal{O}}_f$ be the integral closure of $\mathcal{O}_f$ in the total quotient ring of $\mathcal{O}_f$. Let $\mathcal{C}_f:=\overline{\mathcal{O}}_f:\mathcal{O}_f$ be the conductor of $\overline{\mathcal{O}}_f$ in $\mathcal{O}_f$. The codimension $c(f)=\dim_K \overline{\mathcal{O}}_f/\mathcal{C}_f$ is the *degree of singularity* of the reduced curve $\{f=0\}$. The following two properties are basic (see [11, Chapter 4, Section 1]):

(2.1) $c(f)=0$ if and only if $\operatorname{ord} f=1$, that is, if the curve $\{f=0\}$ is non-singular.

(2.2) If $f=g_1\cdots g_s$ is a reduced power series where the factors g_i, for $i\in\{1,\dots,s\}$ are pairwise coprime, then $c(f)=\sum_{i=1}^{s} c(g_i)+\sum_{i\neq j} i_0(g_i,g_j)$.

A curve $\{f=0\}$ is *unitangent* if $\operatorname{in} f=(ax+by)^{\operatorname{ord} f}$ for some $a,b\in K$. The line $ax+by=0$ is called the *tangent line* to the curve $\{f=0\}$. We recall two well-known properties (see [11, Chapters 1,2]):

(2.3) Every branch is unitangent.

(2.4) For any non-zero power series $f \in K[[x,y]]$ without constant term there is a factorization $f = f_1 \cdots f_t$ such that the curves $\{f_i = 0\}$, for $i \in \{1,\dots,t\}$, are unitangent and the curves $\{f_i = 0\}$ and $\{f_j = 0\}$, for $i \neq j$, have different tangent lines. This factorization of f is called the *tangential factorization* of f.

Let $\{f = 0\}$ be a unitangent curve with $n = \operatorname{ord} f$. We distinguish two possible cases:

(i) $f = c(y - ax)^n +$ higher order terms, where $a, c \in K$, $c \neq 0$, and
(ii) $f = cx^n +$ higher order terms, with $c \in K \backslash \{0\}$.

Let us define a power series $f_1 = f_1(x_1, y_1) \in K[[x_1, y_1]]$ by putting $f_1(x_1,y_1) = x_1^{-n} f(x_1, ax_1 + x_1 y_1)$ in the case (i) and $f_1(x_1,y_1) = y_1^{-n} f(x_1 y_1, y_1)$ in the case (ii).

The power series $Q(f) := f_1$ is called the *(strict) quadratic transform* of f. Using the quadratic transforms we can compute the degree of singularity (see [6, Proposition 4.7]):

(2.5) If f is irreducible then $c(f) = (\operatorname{ord} f)(\operatorname{ord} f - 1) + c(Q(f))$.
(2.6) If $f = f_1 \cdots f_t$ is the tangential factorization of f then $c(f) = (\operatorname{ord} f)(\operatorname{ord} f - 1) + \sum_{i=1}^{t} c(Q(f_i))$.

Let f be a unitangent power series. We say that $Q^{(i)}(f)$ is well-defined, for $i \in \mathbb{N}$, if $i = 0$ (by definition $Q^{(0)}(f) = f$) or $Q^{(i-1)}(f)$ is well-defined and unitangent. Then we set $Q^{(i)}(f) := Q(Q^{(i-1)}(f))$.

If f is irreducible then $Q^{(i)}(f)$ are well-defined for all $i \in \mathbb{N}$. Let $f = f_1 \cdots f_r$ be a factorization of f such that $\{f_i = 0\}$ are irreducible having the same tangent. Then f is unitangent and $Q(f) = Q(f_1) \cdots Q(f_r)$. Applying this property inductively, we get

(2.7) If $Q^{(l)}(f_1), \dots, Q^{(l)}(f_r)$ have the same tangent for $0 \le l \le k-1$ ($k \ge 1$) then $Q^{(k)}(f)$ is well-defined and $Q^{(k)}(f) = Q^{(k)}(f_1) \cdots Q^{(k)}(f_r)$.

3. Main results. Let $\overrightarrow{w} = (n, m)$ be a pair of strictly positive integers. In the sequel we call $\overrightarrow{w}$ a *weight*. Let $f = \sum c_{\alpha\beta} x^\alpha y^\beta \in K[[x,y]]$ be a non-zero power series. Then

- the $\overrightarrow{w}$ *-order* of f is $\operatorname{ord}_{\overrightarrow{w}} f = \inf\{\alpha n + \beta m \,:\, c_{\alpha\beta} \neq 0\}$,
- the $\overrightarrow{w}$*-initial form* of f is $\operatorname{in}_{\overrightarrow{w}} f = \sum_{\alpha n + \beta m = w} c_{\alpha\beta} x^\alpha y^\beta$, where $w = \operatorname{ord}_{\overrightarrow{w}} f$.

A non-zero power series $f \in K[[x,y]]$ is called *quasi-unitangent* (with respect to the weight $\overrightarrow{w}$) if $\operatorname{in}_{\overrightarrow{w}} f = (ax^{m/d} + by^{n/d})^d$ for some $n, m \in \mathbb{N}$, $d = \gcd(m,n)$ and $a, b \in \mathbb{K}\backslash\{0\}$. The binomial curve $\{ax^{m/d} + by^{n/d} = 0\}$ is called the *quasi-tangent* to the curve $\{f = 0\}$ with respect to $\overrightarrow{w}$.

A formal power series $f \in K[[x,y]]$ is *convenient* if $f(0,0) = 0$ and $f(0,y)f(x,0) \neq 0$. Every quasi-unitangent power series is convenient.

Lemma 3.1 ([7, Lemmas 4 and 5], [8, Propositions 2.5 and 2.6]).

(i) *Every convenient irreducible formal power series f is quasi-unitangent with respect to the weight $\overrightarrow{w} = (\operatorname{ord} f(x,0), \operatorname{ord} f(0,y))$.*
(ii) *For every non-zero power series f, without constant term, there is a factorization $f = f_0 f_1 \cdots f_s$ such that f_0 is a monomial, the power series f_i, $1 \le i \le s$, are quasi-unitangent, and for $i \neq j$ the curves $\{f_i = 0\}$ and $\{f_j = 0\}$ have different quasi-tangents.*

In what follows we call $f = f_0 f_1 \cdots f_s$ the *quasi-tangential factorization* of f.

Let $m, n \in \mathbb{N}$ with $m \geq n > 0$. Recall that the *continued fraction expansion* of the ratio $\frac{m}{n}$ is denoted by $\frac{m}{n} = [h_0, h_1, \dots, h_q]$, where the sequence $h_0, h_1, \dots, h_q \in \mathbb{N}$ is uniquely determined by the Euclidean algorithm:

$$\begin{aligned} m &= h_0 n + n_1 \\ n &= h_1 n_1 + n_2 \\ &\vdots \\ n_{q-2} &= h_{q-1} n_{q-1} + n_q \\ n_{q-1} &= h_q n_q. \end{aligned} \tag{1}$$

Moreover $n_q = \gcd(m, n)$. If n divides m then the Euclidean algorithm is reduced to $m = h_0 n$ and the continued fraction expansion of $\frac{m}{n}$ is $[h_0]$. Let $k(m, n) := h_0 + h_1 + \cdots + h_q$.

The following theorem is the first main result of this note:

Theorem A. *Let $\{f = 0\}$ be a reduced quasi-unitangent curve with respect to the weight $\overrightarrow{w} = (n, m)$. Suppose that $m \geq n$ and put $d = \gcd(m, n)$, $k := k(m, n)$. Then $Q^{(k)}(f)$ is well-defined,* $\operatorname{ord} Q^{(k-1)}(f) = d$ *and $c(f) = mn - m - n + d + c(Q^{(k)}(f))$. Moreover $r(Q^{(k)}(f)) = r(f)$.*

In what follows we put $\tilde{f} = Q^{(k)}(f)$ and call $\tilde{f}$ the *modification* of f.

The proof of Theorem A is given in Sections 5 and 6 of this paper.

Example 3.2. Let $f = x^m + y^n +$ the terms of weight $> mn$. If $d = \gcd(m, n) = 1$ then f is irreducible and $c(f) = mn - m - n + 1$ since $\tilde{f} = Q^{(k)}(f) = Q(Q^{(k-1)}(f))$ and $\operatorname{ord} Q^{(k-1)}(f) = 1$.

Write $f = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha y^\beta$. The *support* of f is $\operatorname{supp} f = \{(\alpha, \beta) \in \mathbb{N}^2 \ : \ c_{\alpha\beta} \neq 0\}$. The *Newton diagram* $\Delta(f)$ of f is the convex hull of $\operatorname{supp} f + (\mathbb{R}_{\geq 0})^2$. The *Newton polygon* $\mathcal{N}_f$ of f is the set of compact faces of the boundary of $\Delta(f)$. The power series $f \in K[[x, y]]$ is convenient if $\mathcal{N}_f$ intersects the axes at the points $(m, 0)$ and $(0, n)$.

For every convenient power series $f \in K[[x, y]]$ we define

$$\begin{aligned} c(\mathcal{N}_f) = {} & 2\,(\text{area of the polygon bounded by } \mathcal{N}_f) \text{ and the axes} \\ & + \ (\text{number of integer points on } \mathcal{N}_f) - m - n - 1. \end{aligned}$$

Note that if $\mathcal{N}_f$ is a segment joining the points $(m, 0)$ and $(0, n)$ then

$$c(\mathcal{N}_f) = 2.\left(\frac{1}{2} mn\right) + (d + 1) - m - n - 1 = mn - n - m + d.$$

Theorem B. *Let $f \in K[[x, y]]$ be a reduced convenient power series and let $f = f_1 \cdots f_s$ be a quasi-tangential factorization of f. Then $c(f) = c(\mathcal{N}_f) + \sum_{i=1}^{s} c(\tilde{f}_i)$.*

Example 3.3. Let $f = x^m + y^n +$ the terms of weight $> mn$ with $d := \gcd(m, n) \not\equiv 0$ (mod char K). Then f is reduced and $c(f) = mn - m - n + d$. Indeed, since $d \not\equiv 0$ (mod char K), we have $x^m + y^n = \prod_{i=1}^{d} (x^{m/d} - c_i y^{n/d})$, where $c_i \neq c_j$ for $i \neq j$. By Hensel's Lemma (see [1, Appendix A]) we get $f = \prod_{i=1}^{d} f_i$ with $f_i = x^{m/d} - c_i y^{n/d} + \cdots$. By Example 3.2 the power series $\tilde{f}_i$ are nonsingular and we get $c(f) = c(\mathcal{N}_f) + 0 = mn - m - n + d$.

If $K = \mathbb{C}$ then two versions of Theorem B are known: in [9] the author gives formulae for the invariants $\delta(f)$ (*the double point number*) and $\mu(f)$ of plane curve singularities in terms of the local toric modifications; in [3] the Newton transformations (which are not birational) are used to the same purpose.

4. Modifications of a branch. To study the modification of a branch $\{f = 0\}$ we use its *Hamburger–Noether expansion*. Let $m = \operatorname{ord} f(x, 0)$, $n = \operatorname{ord} f(0, y)$ and we assume that f is convenient and $d = \gcd(m, n)$. Assume that $n < m$ and n does not divide m. Let $(x(t), y(t))$ be a good parametrization of the branch $\{f = 0\}$. Then $\operatorname{ord} x(t) = n$ and $\operatorname{ord} y(t) = m$. By [4, p. 83, 95] there exists a sequence of power series $z_{-1}(t), z_0(t), \ldots, z_q(t)$ such that

$$\begin{cases} z_{-1}(t) &=& y(t), \\ z_0(t) &=& x(t), \\ \vdots \\ z_{i-2}(t) &=& (z_{i-1}(t))^{h_{i-1}} \cdot z_i(t) \quad \text{for } i = 1, \ldots, q, \\ z_{q-1}(t) &=& (\text{a unit}) \cdot (z_q(t))^{h_q}. \end{cases} \tag{2}$$

Let $n_i = \operatorname{ord} z_i(t)$ for $i \in \{-1, 0, 1, \ldots, q\}$. Then, by (1) we get:

$$\begin{cases} n_{-1} &=& m \\ n_0 &=& n \\ n_{i-2} &=& h_{i-1}n_{i-1} + n_i \quad \text{for } i = 1, \ldots, q, \\ n_{q-1} &=& h_q n_q. \end{cases} \tag{3}$$

We have $n_q = d$.

LEMMA 4.1. *With the above notations we have*

(1) $\sum_{i=0}^{q} h_i n_i = m + n - d$.
(2) $\sum_{i=0}^{q} h_i (n_i)^2 = mn$.

Proof. Observe that

$$\sum_{i=0}^{q} h_i n_i = \sum_{i=1}^{q+1} h_{i-1} n_{i-1} = \sum_{i=1}^{q} h_{i-1} n_{i-1} + h_q n_q = \sum_{i=1}^{q} (n_{i-2} - n_i) + n_{q-1}$$
$$= n_{-1} + n_0 - n_q = m + n - d.$$

On the other hand

$$\sum_{i=0}^{q} h_i (n_i)^2 = \sum_{i=1}^{q+1} h_{i-1} (n_{i-1})^2 = \sum_{i=1}^{q} (h_{i-1} n_{i-1}) n_{i-1} + h_q (n_q)^2$$
$$= \sum_{i=1}^{q} (n_{i-2} - n_i) n_{i-1} + n_{q-1} n_q$$
$$= n_{-1} n_0 = mn. \ \blacksquare$$

The following lemma is implicit in [4, Remark 3.3.5, p. 84].

LEMMA 4.2. *Consider the notations of* (3). *Let* $H_i = h_0 + \cdots + h_i$ *for* $i \in \{0, 1, \ldots, q\}$. *We have:*

(1) $\operatorname{ord} Q^{(0)}(f) = \cdots = \operatorname{ord} Q^{(h_0-1)}(f) = n_0$,
(2) $\operatorname{ord} Q^{(H_{i-1})}(f) = \cdots = \operatorname{ord} Q^{(H_{i-1}+h_i-1)}(f) = n_i$ *for* $i \in \{1,\dots,q\}$.

Proof. By (2) we get $Q^{(0)}(f)(z_0(t), z_{-1}(t)) = f(x(t), y(t)) = 0$. Therefore

$$Q^{(i)}(f)\left(z_0(t), \frac{z_{-1}(t)}{(z_0(t))^i}\right) = 0$$

for $i \in \{0,1,\dots,h_0\}$. Since $\frac{z_{-1}(t)}{(z_0(t))^i} = (z_0(t))^{h_0-i} z_1(t)$ we have

$$Q^{(i)}(f)\left(z_0(t), (z_0(t))^{h_0-i} z_1(t)\right) = 0$$

and $\operatorname{ord} Q^{(i)}(f) = \operatorname{ord} z_0(t) = n_0$ for $i \in \{0,1,\dots,h_0-1\}$ and $Q^{(h_0)}(f)(z_0(t), z_{-1}(t)) = 0$. Hence, the first part of lemma follows. Likewise we check the second part:

- $Q^{(H_{i-1})}(f)(z_{i-1}(t), z_i(t)) = 0$ if $i \in \{1,\dots,q\}$ is even,
- $Q^{(H_{i-1})}(f)(z_i(t), z_{i-1}(t)) = 0$ if $i \in \{1,\dots,q\}$ is odd.

Moreover, $\operatorname{ord} Q^{(H_{i-1}+j)}(f) = Q^{(H_{i-1})}(f)$ for $j \in \{1,\dots,h_i-1\}$ and the lemma follows. ∎

5. Proof of Theorem A (the case of one branch). Let $\{f=0\}$ be a branch. We keep the notations and assumptions of Section 4.

LEMMA 5.1. *Suppose that* $m = kn$, $k \ge 1$. *Then*

$$c(f) = kn(n-1) + c(Q^{(k)}(f)) \text{ and } \operatorname{ord} Q^{(k-1)}(f) = n.$$

Proof. We proceed by induction on the number k. If $k=1$ then, by (2.5), $c(f) = (\operatorname{ord} f)(\operatorname{ord} f - 1) + c(Q(f))$. Let $k > 1$ and suppose that the lemma is true for $k-1$. By [7, Lemma 4, p. 10] we have $f(x,y) = c(y^k + ax)^n + \sum_{k\alpha+\beta>kn} a_{\alpha\beta} x^\alpha y^\beta$. Let (x_1, y_1) be new variables. Putting $(x,y) = (x_1y_1, y_1)$ we get

$$\begin{aligned} f(x_1y_1, y_1) &= c(y_1^k + ax_1y_1)^n + \sum_{k\alpha+\beta>kn} a_{\alpha\beta}(x_1y_1)^\alpha y_1^\beta \\ &= y_1^n\Big[c(y_1^{k-1} + ax_1)^n + \sum_{k\alpha+\beta>kn} a_{\alpha\beta} x_1^\alpha y_1^{\alpha+\beta-n}\Big] \\ &= y_1^n f_1(x_1, y_1), \qquad f_1 = Q(f). \end{aligned}$$

By the inductive assumption $c(Q(f)) = (k-1)n(n-1) + c(Q^{(k)}(f))$. By the case $k=1$ we get $c(f) = n(n-1) + c(Q(f)) = kn(n-1) + c(Q^{(k)}(f))$. It is easy to see that $\operatorname{ord} Q^{(k-1)}(f) = n$. ∎

LEMMA 5.2. *Suppose that* $n < m$ *and* n *does not divide* m. *Then*

(1) $c(Q^{(0)}(f)) = h_0 n_0(n_0 - 1) + c(Q^{(h_0)}(f))$,
(2) $c(Q^{(H_{i-1})}(f)) = h_i n_i(n_i - 1) + c(Q^{(H_i)}(f))$ *for* $i \in \{1,\dots,q\}$.

Proof. The first part of the lemma follows using h_0 times the equality (2.5). Similarly, using h_i times (2.5) to the power series $Q^{(H_{i-1})}(f)$ we get the second part of the lemma by the second part of Lemma 4.2. ∎

Now we can pass to the proof of Theorem A when the curve $\{f=0\}$ is a branch.

We may assume $n \leq m$. We distinguish two cases. First, we suppose that n divides m, that is, $m = kn$ for some $k \geq 1$. We have $(m-1)(n-1)+d-1 = (kn-1)(n-1)+(n-1) = kn(n-1)$ and the theorem follows from Lemma 5.1. Now suppose that n does not divide m (in particular $n < m$). Using Lemma 5.2 we get

$$c(f) = \sum_{i=0}^{q} h_i n_i(n_i - 1) + c(Q^{(k)}(f)),$$

where $k = H_q = h_0 + \cdots + h_q$. Therefore by Lemma 4.1

$$c(f) = \sum_{i=0}^{q} h_i (n_i)^2 - \sum_{i=0}^{q} h_i n_i + c(Q^{(k)}(f)) = mn - (m+n-d) + c(Q^{(k)}(f)).$$

Moreover, $\operatorname{ord} Q^{(k-1)}(f) = \operatorname{ord} Q^{(H_{q-1}+h_q-1)}(f) = n_q = d$. ■

EXAMPLES 5.3. (1) Let $f(x,y) = (y^2+x^3)^2 + x^7y$. Here, we have $m = \operatorname{ord} f(x,0) = 6$, $n = \operatorname{ord} f(0,y) = 4$ and $d = 2$. Moreover, the continued fraction expansion of $\frac{6}{4} = [h_0, h_1] = [1,2]$ and $k = h_0 + h_1 = 3$. By Theorem A:

$$c(f) = (6-1)(4-1) + (2-1) + c(Q^{(3)}(f)) = 5.3 + 1 + 0 = 16$$

since $\{Q^{(3)}(f) = 0\} = \{-x_3 + y_3^3 + x_3 y_3 = 0\}$ is non-singular.

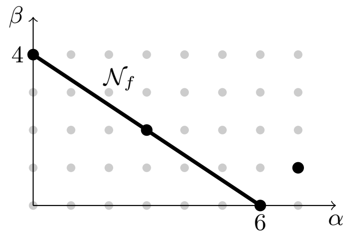

Fig. 1. Newton polygon of $f = (y^2+x^3)^2 + x^7y$

(2) Let $f(x,y) = (y^2+x)^n + y^{2n+1}$. We have $m = 2n$, $k = 2$ and $c(f) = 2n(n-1) + c(Q^{(2)}(f))$. Hence $\{Q(f) = 0\} = \{(x_1+y_1)^n + y_1^{n+1} = 0\}$ and $\{Q^{(2)}(f) = Q(Q(f)) = 0\}$ is non-singular. Therefore $c(Q^{(2)}(f)) = 0$ and $c(f) = 2n(n-1)$.

6. Proof of Theorem A (the general case). In this section we will prove the general case of Theorem A.

LEMMA 6.1. *Let $f, g \in K[[x,y]]$ be irreducible with* $\operatorname{ord} f(x,0) = m$, $\operatorname{ord} f(0,y) = n$, $\operatorname{ord} g(x,0) = m'$ *and* $\operatorname{ord} g(0,y) = n'$. *We have*

(1) *$i_0(f,g) \geq \min\{mn', m'n\}$ with equality if and only if the branches $\{f = 0\}$ and $\{g = 0\}$ have different quasi-tangents.*
(2) *If $i_0(f,g) > \min\{mn', m'n\}$ then $mn' = m'n$ and $i_0(f,g) = \min\{mn', m'n\} + i_0(Q^{(k)}(f), Q^{(k)}(g))$, where $k = k(m,n)$. Moreover, the branches $Q^{(i)}(f)$ and $Q^{(i)}(g)$ have the same tangent for $i < k$.*

Proof. The first part of the lemma is proved in [7, Lemma 6, p. 12]. Let us prove the second part. Suppose that $i_0(f,g) > \min\{mn', m'n\}$. Since $\{f=0\}$ and $\{g=0\}$ have the same quasi-tangent then $mn' = m'n$. Let $d = \gcd(m,n)$ and $d' = \gcd(m',n')$. The Euclidean algoritms for $\left(\frac{m}{d}, \frac{n}{d}\right)$ and $\left(\frac{m'}{d'}, \frac{n'}{d'}\right)$ are identical since $\frac{m}{d} = \frac{m'}{d'}$ and $\frac{n}{d} = \frac{n'}{d'}$:

$$\begin{cases} \frac{n_{-1}}{d} = \frac{m}{d} \\ \frac{n_0}{d} = \frac{n}{d} \\ \vdots \\ \frac{n_{i-2}}{d} = h_{i-1}\frac{n_{i-1}}{d} + \frac{n_i}{d} \\ \vdots \\ \frac{n_{q-1}}{d} = h_q \end{cases} \qquad \begin{cases} \frac{n'_{-1}}{d'} = \frac{m'}{d'} \\ \frac{n'_0}{d'} = \frac{n'}{d'} \\ \vdots \\ \frac{n'_{i-2}}{d'} = h_{i-1}\frac{n'_{i-1}}{d'} + \frac{n'_i}{d'} \\ \vdots \\ \frac{n'_{q'-1}}{d'} = h_{q'}. \end{cases}$$

By the unicity of the Euclidean algorithms we get $q = q'$ and $\frac{n_i}{d} = \frac{n'_i}{d'}$ for $i \in \{-1, 0, 1, \dots, q\}$ $\left(\frac{n_q}{d} = \frac{n'_q}{d'} = 1\right)$ and $k(m,n) = k(m',n')$. Using the second part of Lemma 4.1 we get

$$\begin{aligned} \sum_{i=0}^{q} h_i n_i n'_i &= \sum_{i=0}^{q} (h_i n_i d')\frac{n_i}{d} = \frac{d'}{d}\sum_{i=0}^{q} h_i (n_i)^2 \\ &= \frac{d'}{d}mn = d'm\frac{n'}{d'} = mn'. \end{aligned}$$

Therefore the assumption $i_0(f,g) > mn'$ implies $i_0(f,g) > \sum_{i=0}^{q} h_i n_i n'_i$. By Max Noether theorem (see [11, Lemma 5.1, pp. 90-91]) we get that $Q^{(i)}(f)$ and $Q^{(i)}(g)$ have a common tangent and $i_0(f,g) = \sum_{i=0}^{q} h_i n_i n'_i + i_0(Q^{(k)}(f), Q^{(k)}(g))$. ■

Let us pass to the proof of Theorem A in the general case.

Suppose that $f = c(y^{n/d} - ax^{m/d})^d + \cdots \in K[[x,y]]$, where c is a non-zero constant and $a \in K$. Put $d = \gcd(m,n)$, $k = k(m,n)$. Let $f = f_1 \cdots f_r$ be the factorization of f into irreducible factors. Let $\overrightarrow{w}_i = (n_i, m_i)$. Then $\mathrm{in}_{\overrightarrow{w}_i} f_i = c_i(y^{n_i/d_i} - a_i x^{m_i/d_i})^{d_i}$ for $i \in \{1, \dots, r\}$, $a_i, c_i \in K$, $c_i \neq 0$, which implies $\frac{n_i}{d_i} = \frac{n}{d}$, $\frac{m_i}{d_i} = \frac{m}{d}$ for $i \in \{1, \dots, r\}$ and $\sum_{i=1}^{r} d_i = d$. Moreover, $k\left(\frac{m_i}{d_i}, \frac{n_i}{d_i}\right) = k\left(\frac{m}{d}, \frac{n}{d}\right) = k(m,n) = k$.

According to the irreducible case of Theorem A we have

$$c(f_i) = m_i n_i - m_i - n_i + d_i + c(Q^{(k)}(f_i)) \text{ for } i \in \{1, \dots, r\}. \tag{4}$$

On the other hand, by the second part of Lemma 6.1 we get

$$i_0(f_i, f_j) = m_i n_j + i_0(Q^{(k)}(f_i), Q^{(k)}(f_j)), \tag{5}$$

for any $i, j \in \{1, \dots, r\}$, $i \neq j$.

Using (2.2), (4) and (5) we get

$$\begin{aligned}
c(f) &= \sum_{i=1}^{r} c(f_i) + 2 \sum_{1\le i<j\le r} i_0(f_i, f_j) \\
&= \sum_{i=1}^{r} (m_i n_i - m_i - n_i + d_i + c(Q^{(k)}(f_i))) \\
&\quad + 2 \sum_{1\le i<j\le r} (m_i n_j + i_0(Q^{(k)}(f_i), Q^{(k)}(f_j))) \\
&= mn - m - n + d + \sum_{i=1}^{r} c(Q^{(k)}(f_i)) + 2 \sum_{1\le i<j\le r} i_0(Q^{(k)}(f_i), Q^{(k)}(f_j)) \\
&= mn - m - n + d + c(Q^{(k)}(f_1)\cdots Q^{(k)}(f_r)) \\
&= mn - m - n + d + c(Q^{(k)}(f)),
\end{aligned}$$

where the last equality follows from (2.7). To end the proof observe that $\operatorname{ord} Q^{(k-1)}(f) = \operatorname{ord}(Q^{(k-1)}(f_1)\cdots Q^{(k-1)}(f_r)) = d_1 + \cdots + d_r = d$. Moreover

$$r(Q^{(k)}(f)) = r(Q^{(k)}(f_1)\cdots Q^{(k)}(f_r)) = r = r(f). \ \blacksquare$$

EXAMPLE 6.2. Let $f(x,y) = (y^2 - x^3)^2 - x^7 \in K[[x,y]]$. Here $n = 4$, $m = 6$, $d = 2$, $k = k(6,4) = 1+2 = 3$ and $Q^{(3)}(f) = y_3^2 - x_3^2(1+y_3)$. Hence $c(f) = (m-1)(n-1) + d - 1 + c(Q^{(k)}(f)) = 16 + c(Q^{(3)}(f)) = 16 + 2 = 18$. Observe that we can compute $c(Q^{(3)}(f))$ using (2.2) or Theorem B.

7. Proof of Theorem B.

Let $f(x,y) \in K[[x,y]]$ be a convenient power series, with $m = \operatorname{ord} f(x,0)$, $n = \operatorname{ord} f(0,y)$ and $d = \gcd(m,n)$. Let us recall some notations introduced in [7, Section 3]. A segment $S \subseteq \mathbb{R}^2$ is a *Newton edge* if its vertices (α,β), (α',β') lie in $\mathbb{N}^2$ and $\alpha < \alpha'$, $\beta' < \beta$. Let $|S|_1 = \alpha' - \alpha$, $|S|_2 = \beta - \beta'$, $r(S) = \gcd(|S|_1, |S|_2)$. If S, T are two Newton edges we define $[S,T] := \min\{|S|_1|T|_2, |S|_2|T|_1\}$. If $\frac{|S|_1}{|S|_2} < \frac{|T|_1}{|T|_2}$ then $[S,T] = |S|_1|T|_2$.

We put $|\mathcal{N}_f|_1 = \sum_{S\in\mathcal{N}_f} |S|_1$, $|\mathcal{N}_f|_2 = \sum_{S\in\mathcal{N}_f} |S|_2$, $[\mathcal{N}_f, \mathcal{N}_f] = \sum_{S,T\in\mathcal{N}_f} [S,T]$ and $r(\mathcal{N}_f) = \sum_{S\in\mathcal{N}_f} r(S) + k + l$, where k, l are maximal such that $x^k y^l$ divides f.
If the Newton polygon $\mathcal{N}_f$ intersects the axes at the points $(m,0)$ and $(0,n)$ then $|\mathcal{N}_f|_1 = m$, $|\mathcal{N}_f|_2 = n$ and $r(\mathcal{N}_f) = \sum_{S\in\mathcal{N}_f} r(S)$.

We have

$$\begin{aligned}
c(\mathcal{N}_f) &= 2(\text{area of the polygon bounded by } \mathcal{N}_f \text{ and the axes}) \\
&\qquad + (\text{number of integer points on } \mathcal{N}_f) - n - m - 1 \\
&= [\mathcal{N}_f, \mathcal{N}_f] + (r(\mathcal{N}_f) + 1) - m - n - 1 \\
&= [\mathcal{N}_f, \mathcal{N}_f] + r(\mathcal{N}_f) - m - n.
\end{aligned}$$

If $\mathcal{N}_f$ contains only the edge S then $c(f) = [S,S] - |S|_1 - |S|_2 + r(S) = nm - n - m + d$.

LEMMA 7.1. *Suppose that $f \in K[[x,y]]$ is reduced and its Newton polygon $\mathcal{N}_f$ contains exactly one edge S. Then $c(f) = [S,S] - |S|_1 - |S|_2 + r(S) + \sum_{i=1}^{s} c(\tilde{f}_i)$, where $f = f_1 \cdots f_s$ is a quasi-tangential factorization of f.*

Proof. Let $\overrightarrow{w} = (n, m)$ and $d = \gcd(n, m)$. By Hensel's lemma (see [1, Appendix A]) we get $f = f_1 \cdots f_s$ with $\mathrm{in}_{\overrightarrow{w}} f_i = (a_i x^{m/d} + b_i y^{n/d})^{e_i}$ for some $e_i \in \mathbb{N}$ with $d = \sum_{i=1}^{s} e_i$, $i \in \{1, \dots, s\}$ and $a_i b_j - a_j b_i \neq 0$. Theorem A implies that

$$c(f_i) = \binom{\frac{m}{d} e_i}{}\binom{\frac{n}{d} e_i}{} - \left(\frac{m}{d} e_i\right) - \left(\frac{n}{d} e_i\right) + e_i + c(\tilde{f}_i),$$

for $i \in \{1, \dots, s\}$. By the first part of Lemma 6.1 we get $i_0(f_i, f_j) = \left(\frac{m}{d} e_i\right)\left(\frac{n}{d} e_j\right)$ for $i \neq j$. Combining the above equalities with (2.2) we get

$$\begin{aligned} c(f) &= \sum_{i=1}^{s} c(f_i) + 2 \sum_{1<i<j\leq s} i_0(f_i, f_j) \\ &= \sum_{i=1}^{s} \frac{mn}{d^2} e_i^2 - \sum_{i=1}^{s} \frac{m}{d} e_i - \sum_{i=1}^{s} \frac{n}{d} e_i + \sum_{i=1}^{s} e_i + \sum_{i=1}^{s} c(\tilde{f}_i) + 2 \sum_{1<i<j\leq s} \frac{mn}{d^2} e_i e_j \\ &= mn - m - n + d + \sum_{i=1}^{s} c(\tilde{f}_i). \ \blacksquare \end{aligned}$$

Now we can check the general case of Theorem B. Let $f = \prod_{S \in \mathcal{N}_f} f_S$ be the Newton factorization of the power series f [7, Section 3, Lemma 5]. By Lemma 7.1

$$c(f_S) = [S, S] - |S|_1 - |S|_2 + r(S) + \sum_{i \in I(S)} c(\tilde{f}_i),$$

where $f_S = \prod_{i \in I(S)} f_i$ is a quasi-tangential factorization of f. If $S, T \in \mathcal{N}_f$ are not parallel then $i_0(f_S, f_T) = [S, T]$ (see [7, Lemma 6]). Therefore, by (2.2),

$$\begin{aligned} c(f) &= \sum_{S \in \mathcal{N}_f} c(f_S) + \sum_{S \neq T} i_0(f_S, f_T) \\ &= \sum_{S \in \mathcal{N}_f} \Big([S, S] - |S|_1 - |S|_2 + r(S) + \sum_{i \in I(S)} c(\tilde{f}_i) \Big) + \sum_{S \neq T} [S, T] \\ &= [\mathcal{N}_f, \mathcal{N}_f] - |\mathcal{N}_f|_1 - |\mathcal{N}_f|_2 + r(\mathcal{N}_f) + \sum_{S \in \mathcal{N}_f} \sum_{i \in I(S)} c(\tilde{f}_i). \end{aligned}$$

8. Theorem B in the non-convenient case. Let $f(x, y) \in K[[x, y]]$ be a reduced power series. Assume that $\mathcal{N}_f \neq \emptyset$. Then, the quasi-tangential factorization of f has the form $f = x^{d_1} y^{d_2} f_1 \cdots f_s$, where $d_1, d_2 \in \{0, 1\}$ and $g := f_1 \cdots f_s$ is convenient.

Since the Newton polygons $\mathcal{N}_f$ and $\mathcal{N}_g$ are parallel then $|\mathcal{N}_f|_i = |\mathcal{N}_g|_i$, for $i \in \{1, 2\}$ and $[\mathcal{N}_f, \mathcal{N}_f] = [\mathcal{N}_g, \mathcal{N}_g]$. Recall that $r(\mathcal{N}_f) = \sum_{S \in \mathcal{N}_f} r(S) + d_1 + d_2 = r(\mathcal{N}_g) + d_1 + d_2$.

We define

$$c(\mathcal{N}_f) = \begin{cases} [\mathcal{N}_f, \mathcal{N}_f] - |\mathcal{N}_f|_1 - |\mathcal{N}_f|_2 + r(\mathcal{N}_f) & \text{if } (d_1, d_2) = (0, 0) \\ [\mathcal{N}_f, \mathcal{N}_f] - |\mathcal{N}_f|_1 + |\mathcal{N}_f|_2 + r(\mathcal{N}_f) - 1 & \text{if } (d_1, d_2) = (1, 0) \\ [\mathcal{N}_f, \mathcal{N}_f] + |\mathcal{N}_f|_1 - |\mathcal{N}_f|_2 + r(\mathcal{N}_f) - 1 & \text{if } (d_1, d_2) = (0, 1) \\ [\mathcal{N}_f, \mathcal{N}_f] + |\mathcal{N}_f|_1 + |\mathcal{N}_f|_2 + r(\mathcal{N}_f) & \text{if } (d_1, d_2) = (1, 1). \end{cases} \tag{6}$$

THEOREM 8.1. *Let f be a reduced power series and let $f = x^{d_1} y^{d_2} f_1 \cdots f_s$ be a quasi-tangential factorization of f. Then $c(f) = c(\mathcal{N}_f) + \sum_{i=1}^{s} c(\tilde{f}_i)$.*

Proof. Since $g := f_1 \cdots f_s$ is a convenient power series, by Theorem B we get $c(g) = c(\mathcal{N}_g) + \sum_{i=1}^{s} c(\tilde{f}_i)$. Suppose that $f = xyg$ that is $(d_1, d_2) = (1, 1)$ (the proof for the cases $(d_1, d_2) \in \{(1, 0), (0, 1)\}$ is analogous). Then, by (2.2)

$$\begin{aligned}
c(f) = c(xyg) &= c(x) + c(y) + c(g) + 2(i_0(x, g) + i_0(y, g) + i_0(x, y)) \\
&= c(g) + 2(|\mathcal{N}_g|_1 + |\mathcal{N}_g|_2 + 1) = c(\mathcal{N}_g) + \sum_{i=1}^{s} c(\tilde{f}_i) + 2(|\mathcal{N}_g|_1 + |\mathcal{N}_g|_2 + 1) \\
&= [\mathcal{N}_g, \mathcal{N}_g] - |\mathcal{N}_g|_1 - |\mathcal{N}_g|_2 + r(\mathcal{N}_g) + 2(|\mathcal{N}_g|_1 + |\mathcal{N}_g|_2 + 1) + \sum_{i=1}^{s} c(\tilde{f}_i) \\
&= [\mathcal{N}_g, \mathcal{N}_g] + |\mathcal{N}_g|_1 + |\mathcal{N}_g|_2 + r(\mathcal{N}_g) + 2 + \sum_{i=1}^{s} c(\tilde{f}_i) \\
&= [\mathcal{N}_f, \mathcal{N}_f] + |\mathcal{N}_f|_1 + |\mathcal{N}_f|_2 + r(\mathcal{N}_f) + \sum_{i=1}^{s} c(\tilde{f}_i) \\
&= c(\mathcal{N}_f) + \sum_{i=1}^{s} c(\tilde{f}_i). \ \blacksquare
\end{aligned}$$

COROLLARY 8.2 (cf. [8, Theorem 3.12] where a similar result is proved). *Suppose that $f = x^{d_1} y^{d_2} f_1 \cdots f_s \in K[[x, y]]$ is reduced. Then $c(f) \geq c(\mathcal{N}_f)$ with equality if and only if all modifications $\tilde{f}_i$, for $i \in \{1, \ldots, s\}$ are non-singular.*

9. A formula for the Milnor number. We keep the notations and assumptions of Section 8. If $f = x^{d_1} y^{d_2} f_1 \cdots f_s$ is a quasi-tangential factorization of the reduced power series $f \in K[[x, y]]$ then we put $s(f) := d_1 + d_2 + s$. Observe that $r(f) = d_1 + d_2 + \sum_{i=1}^{s} r(f_i) \geq d_1 + d_2 + \sum_{i=1}^{s} 1 = s(f)$ with equality if and only if the factors f_i are irreducible.

We define the *Milnor number* $\mu(f)$ to be (see [10], [5], [7])

$$\mu(f) := c(f) - r(f) + 1. \tag{7}$$

Let

$$\mu(\mathcal{N}_f) = c(\mathcal{N}_f) - r(\mathcal{N}_f) + 1. \tag{8}$$

The following theorem, for $K = \mathbb{C}$, is essentially due to Gwoździewicz [9, Corollary 5].

THEOREM 9.1. *Let $f = x^{d_1} y^{d_2} f_1 \cdots f_s$ be a quasi-tangential factorization of the reduced formal power series f. Then*

$$\mu(f) = \mu(\mathcal{N}_f) + r(\mathcal{N}_f) - s(f) + \sum_{i=1}^{s} \mu(\tilde{f}_i).$$

Proof. By Theorem 8.1 $c(f) = c(\mathcal{N}_f) + \sum_{i=1}^{s} c(\tilde{f}_i)$. By (7) (for f_i) and (8) we get

$$\mu(f) + r(f) - 1 = \mu(\mathcal{N}_f) + r(\mathcal{N}_f) - 1 + \sum_{i=1}^{s} (\mu(\tilde{f}_i) + r(\tilde{f}_i) - 1),$$

which implies

$$\mu(f) + r(f) = \mu(\mathcal{N}_f) + r(\mathcal{N}_f) + \sum_{i=1}^{s} \mu(\tilde{f}_i) + \sum_{i=1}^{s} r(\tilde{f}_i) - s.$$

Since $\sum_{i=1}^{s} r(\tilde{f}_i) = \sum_{i=1}^{s} r(f_i) = r(f) - d_1 - d_2$ the theorem follows. ∎

The following corollary is a bit stronger than [2, Proposition 9].

Corollary 9.2 ([7]). *Under the assumptions of Theorem 9.1 we have*

$$\mu(f) - \mu(\mathcal{N}_f) \geq r(\mathcal{N}_f) - r(f),$$

with equality if and only if the modifications $\tilde{f}_i$, $1 \leq i \leq s$, are non-singular.

Proof. By Theorem 9.1 we get $\mu(f) - \mu(\mathcal{N}_f) \geq r(\mathcal{N}_f) - s(f) \geq r(\mathcal{N}_f) - r(f)$ since $s(f) \leq r(f)$. Suppose that $\tilde{f}_i$ are non-singular for $1 \leq i \leq s$. Then $\mu(\tilde{f}_i) = 0$ and $r(\tilde{f}_i) = 1$ for $1 \leq i \leq s$. Consequently, $r(f) = d_1 + d_2 + \sum_{i=1}^{s} r(f_i) = d_1 + d_2 + \sum_{i=1}^{s} r(\tilde{f}_i) = d_1 + d_2 + s = s(f)$ and by Theorem 9.1 $\mu(f) - \mu(\mathcal{N}_f) = r(\mathcal{N}_f) - r(f)$. On the other hand if $\mu(f) - \mu(\mathcal{N}_f) = r(\mathcal{N}_f) - r(f)$ then again by Theorem 9.1 $r(f) - s(f) + \sum_{i=1}^{s} \mu(\tilde{f}_i) = 0$ which implies $\mu(\tilde{f}_i) = 0$ for $1 \leq i \leq s$. ∎

Remark 9.3. For any reduced formal power series $f \in K[[x, y]]$ we have $r(\mathcal{N}_f) - r(f) \geq 0$ (see for example [7]). Consequently, by Corollary 9.2 we get $\mu(f) \geq \mu(\mathcal{N}_f)$ with equality if and only if the series f_i are irreducible with non-singular modifications $\tilde{f}_i$.

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