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A note on the plane curve singularities in positive characteristic

by

Evelia R. García Barroso[(1)], Arkadiusz Płoski[(2)]

Abstract

Given an algebroid plane curve $f = 0$ over an algebraically closed field of characteristic $p \geq 0$ we consider the Milnor number $\mu(f)$, the delta invariant $\delta(f)$ and the number $r(f)$ of its irreducible components. Put $\bar{\mu}(f) = 2\delta(f) - r(f) + 1$. If $p = 0$ then $\bar{\mu}(f) = \mu(f)$ (the Milnor formula). If $p > 0$ $\mu(f)$ is not an invariant and $\bar{\mu}(f)$ plays the role of $\mu(f)$. Let $\mathcal{N}_f$ be the Newton polygon of f. We define the numbers $\mu(\mathcal{N}_f)$ and $r(\mathcal{N}_f)$ which can be computed by explicit formulas. The aim of this note is to give a simple proof of the inequality $\bar{\mu}(f) - \mu(\mathcal{N}_f) \geq r(\mathcal{N}_f) - r(f) \geq 0$ due to Boubakri, Greuel and Markwig. We also prove that $\bar{\mu}(f) = \mu(\mathcal{N}_f)$ when f is non-degenerate.

Key Words: Milnor number, Newton polygon, non-degeneracy.
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1 Introduction

The main objective of this note is to give a new and simple proof of [3, Proposition 7].The paper is organised as follows. Section 2 is a survey of prerequisites from the theory of algebroid curves (see [13]). We define the invariant $\bar{\mu}$ which plays the crucial role in this note. In Section 3 we use the Newton polygons that are vital to the proof of the main result. In Section 4 we make use of the Newton polygon $\mathcal{N}_f$, associated with a formal power series $f \in K[[x, y]]$ (K is an algebraically closed field of arbitrary characteristic), to compute the invariant $\bar{\mu}(f)$ and the number $r(f)$ of irreducible components of the curve $f(x, y) = 0$. We define the numbers $\mu(\mathcal{N}_f)$ and $r(\mathcal{N}_f)$ which are combinatorial counterparts of $\bar{\mu}(f)$ and $r(f)$. Suppose that f is a reduced power series. We give a new proof of [3, Proposition 7] which states

$$\bar{\mu}(f) - \mu(\mathcal{N}_f) \geq r(\mathcal{N}_f) - r(f) \geq 0. \tag{1}$$

On the other hand, under the assumption of non-degeneracy introduced by Beelen and Pellikaan [2, Definition 3.14] we prove that

$$\bar{\mu}(f) = \mu(\mathcal{N}_f) \quad \text{and} \quad r(f) = r(\mathcal{N}_f). \tag{2}$$

The inequality (1) generalizes [14, Theorem 1.2] where (1) is proved when the characteristic of the field K is zero and f is convenient. Section 5 is devoted to the proofs of (1) and (2).

2 Prerequisites

Let K be an algebraically closed field of arbitrary characteristic. Let $f \in K[[x,y]]$ be a non-zero power series without constant term. The power series f is reduced if it has not multiple factors.

In what follows we consider the equisingularity invariants of a reduced plane curve $\{f(x,y)=0\}$ (see [13]): $r(f)$ is the number of irreducible factors of f, $c(f)=\dim_K \overline{\mathcal{O}_f}/\mathcal{C}$ is the *degree of the conductor*, where $\mathcal{O}_f = K[[x,y]]/(f)$, $\overline{\mathcal{O}_f}$ is the integral closure of $\mathcal{O}_f$ in the total quotient ring of $\mathcal{O}_f$ and $\mathcal{C}$ is the *conductor* of $\mathcal{O}_f$, that is the largest ideal in $\mathcal{O}_f$ which remains an ideal in $\overline{\mathcal{O}_f}$. Finally the *delta invariant* of f is $\delta(f)=\dim_K \overline{\mathcal{O}_f}/\mathcal{O}_f$. Since $\mathcal{O}_f$ is Gorenstein we get $c(f)=2\delta(f)$.

If $f \in K[[x,y]]$ is irreducible then the *semigroup of values* of $f(x,y)$, denoted by $\Gamma(f)$, is defined as the set of intersection multiplicities $i_0(f,g)=\dim_K K[[x,y]]/(f,g)$ where g runs over all power series in $K[[x,y]]$ such that $g \not\equiv 0 \pmod f$. This semigroup is numerical, that is $\mathbb{N}\backslash\Gamma(f)$ is a finite set. Denote by c the *conductor* of $\Gamma(f)$, that is, the smallest element of $\Gamma(f)$ such that $c+N \in \Gamma(f)$ for any nonnegative integer N. The semigroup $\Gamma(f)$ admits a minimal system of generators $v_0 < v_1 < \cdots < v_g$ such that $\gcd(v_0,\ldots,v_g)=1$. We write $\Gamma(f)=\langle v_0,\ldots,v_g\rangle$. Put $e_i := \gcd(v_0,\ldots,v_i)$ for $0 \le i \le h$ and $n_i = \frac{e_{i-1}}{e_i}$ for $1 \le i \le g$.

If f is irreducible then $c(f)$ equals to the conductor c of the semigroup $\Gamma(f)$. Consequently $c(f)=\sum_{k=1}^{g}(n_k-1)v_k - v_0 + 1$.

Let $\mu(f)$ be the *Milnor number* of f defined as the codimension of the ideal generated by $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, that is $\mu(f)=i_0\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. The *invariant Milnor number* of f is defined to be $\bar{\mu}(f)=2\delta(f)-r(f)+1=c(f)-r(f)+1$ (see[6]). If $p=0$ then $\bar{\mu}(f)=\mu(f)$ (the Milnor formula). If $p>0$ $\mu(f)$ is not an invariant and $\bar{\mu}(f)$ plays the role of $\mu(f)$. Melle and Wall [11], based on a result of Deligne [4], proved that $\mu(f) \ge \bar{\mu}(f)$.

Any plane reduced curve $\{f(x,y)=0\}$ is called a *tame singularity* if $\mu(f)=\bar{\mu}(f)$. If the characteristic of K is zero any singularity of plane reduced curve is tame.

Proposition 2.1.

1. *For any unit $u \in K[[x,y]]$ we get $\bar{\mu}(uf)=\bar{\mu}(f)$.*
2. *For every reduced power series $f \in K[[x,y]]$ we have $\bar{\mu}(f) \ge 0$ and $\bar{\mu}(f)=0$ if and only if* $\mathrm{ord} f = 1$.
3. *Let $f = g_1 \cdots g_s$ be a reduced power series where $g_i \in K[[x,y]]$ are pairwise coprime. Then*
$$\bar{\mu}(f)+s-1=\sum_{i=1}^{s}\bar{\mu}(g_i)+2\sum_{1\le i<j\le s} i_0(g_i,g_j).$$

Proof. See [5, Proposition 1.2, Remark 2.2]. □

If the characteristic of K is positive then, in general, we have $\mu(uf) \neq \mu(f)$ (see [3, page 63]).

3 Newton polygons and plane curve singularities

A segment $S \subset \mathbb{R}^2$ is a *Newton edge* if its vertices (α,β), (α',β') lie in $\mathbb{N}^2$ and $\alpha < \alpha'$, $\beta' < \beta$. Put $|S|_1 = \alpha' - \alpha$, $|S|_2 = \beta - \beta'$, $r(S) = \gcd(|S|_1, |S|_2)$ If S, T are two Newton edges we define $[S,T] := \min\{|S|_1|T|_2, |S|_2|T|_1\}$. If $\frac{|S|_1}{|S|_2} < \frac{|T|_1}{|T|_2}$ then $[S,T] = |S|_1|T|_2$ (see Figure 1).

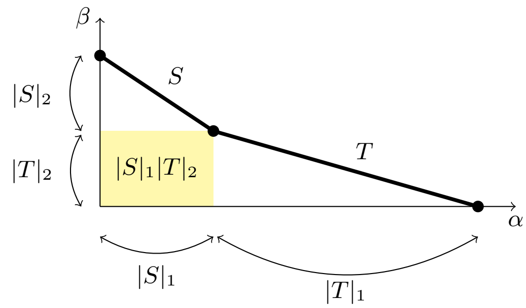

Figure 1: $[S,T] = |S|_1|T|_2$

Let K be an algebraically closed field of characteristic $p \geq 0$. Consider $f \in K[[x,y]]$ a nonzero power series without constant term. Write $f = \sum_{\alpha,\beta} c_{\alpha\beta}x^\alpha y^\beta$. The *support* of f is $\operatorname{supp} f = \{(\alpha,\beta) \in \mathbb{N}^2 \,:\, c_{\alpha\beta} \neq 0\}$. The *Newton diagram* $\Delta(f)$ of f is the convex hull of $\operatorname{supp} f + (\mathbb{R}_{\geq 0})^2$. The *Newton polygon* $\mathcal{N}_f$ of f is the set of compact faces of the boundary of $\Delta(f)$. We put $|\mathcal{N}_f|_1 = \sum_{S\in\mathcal{N}_f} |S|_1$, $|\mathcal{N}_f|_2 = \sum_{S\in\mathcal{N}_f} |S|_2$, $[\mathcal{N}_f, \mathcal{N}_f] = \sum_{S,T\in\mathcal{N}_f}[S,T]$ and $r(\mathcal{N}_f) = \sum_{S\in\mathcal{N}_f} r(S) + k + l$, where k, l are maximal such that $x^k y^l$ divides f.

A power series $f \in K[[x,y]]$ is *convenient* if $f(x,0)f(0,y) \neq 0$; otherwise we will say that f is *non-convenient*. When f is convenient the curve $f(x,y) = 0$ does not contain the axes. Hence there is an edge $F \in \mathcal{N}_f$ with the vertex $(m,0)$, where $m = \operatorname{ord} f(x,0)$ and there is $E \in \mathcal{N}_f$ with the vertex $(0,n)$ where $n = \operatorname{ord} f(0,y)$. The edges F and E are not necessarily different.

Let $f(x,y) = \sum_{\alpha,\beta} c_{\alpha\beta}x^\alpha y^\beta \in K[[x,y]]$. Recall that the *order* of f is $\operatorname{ord} f = \min\{\alpha+\beta \,:\, c_{\alpha\beta} \neq 0\}$ and the *initial part* of f is $\operatorname{in} f = \sum_{\alpha+\beta=\operatorname{ord} f} c_{\alpha\beta}x^\alpha y^\beta$.

For any segment $S \in \mathcal{N}_f$ we put $\operatorname{in}(f,S) = \sum_{(\alpha,\beta)\in S} c_{\alpha\beta}x^\alpha y^\beta$. Let $x^{\alpha(S)}y^{\beta(S)}$ be the monomial of highest degree dividing $\operatorname{in}(f,S)$. Then $\operatorname{in}(f,S) = x^{\alpha(S)}y^{\beta(S)}\overline{\operatorname{in}}(f,S)$ where $\overline{\operatorname{in}}(f,S)$ is a convenient power series. We say that f is *non-degenerate* if $\overline{\operatorname{in}}(f,S)$ is reduced for every $S \in \mathcal{N}_f$, that is it does not have multiple factors.

Remark 3.1. *A power series f is non-degenerate if and only if for any segment $S \in \mathcal{N}_f$*

the solutions of the system

$$\begin{cases} \frac{\partial}{\partial x}\mathrm{in}(f,S)=0 \\ \frac{\partial}{\partial y}\mathrm{in}(f,S)=0 \\ \mathrm{in}(f,S)=0 \end{cases}$$

are contained in $\{xy=0\}$ *(see [8, Proposition 3.5]). On the other hand* f *is non-degenerate in the strong sense (Kouchnirenko [9]) if the solutions of the system*

$$\begin{cases} \frac{\partial}{\partial x}\mathrm{in}(f,S)=0 \\ \frac{\partial}{\partial y}\mathrm{in}(f,S)=0 \end{cases}$$

are contained in $\{xy=0\}$ *for any segment* $S\in\mathcal{N}_f$. *In zero characteristic both definitions are equivalent (see [2, Remark 3.15]). Nevertheless if the characteristic of* K *is* $p>0$ *then the power series* $f(x,y)=x^p+y^{p+1}$ *is non-degenerate but it is not non-degenerate in the strong sense.*

Assume that f is a convenient power series. Recall that $m=\mathrm{ord}f(x,0)$ and $n=\mathrm{ord}f(0,y)$. We put

$$\mu(\mathcal{N}_f)=[\mathcal{N}_f,\mathcal{N}_f]-|\mathcal{N}_f|_1-|\mathcal{N}_f|_2+1 \tag{3}$$

and

$$\delta(\mathcal{N}_f)=\frac{1}{2}\left(\mu(\mathcal{N}_f)+r(\mathcal{N}_f)-1\right).$$

Note that

- $\mu(\mathcal{N}_f)=2$(area of the polygon bounded by $\mathcal{N}_f$ and the axes) $-n-m+1$, which is called the *Newton number* of f.
- $r(\mathcal{N}_f)=$ (number of integer points on $\mathcal{N}_f$) -1, and
- $\delta(\mathcal{N}_f)=$ number of integer points lying below $\mathcal{N}_f$ but not on the axes. This is a consequence of Pick's formula.

If f is a reduced power series (not necessarily convenient) then we define:

$$\mu(\mathcal{N}_f)=\sup_{m\in\mathbb{N}}\{\mu(\mathcal{N}_{f_m})\;:\;f_m=f+x^m+y^m\}. \tag{4}$$

Like in the case of convenient power series we put

$$\delta(\mathcal{N}_f)=\frac{1}{2}\left(\mu(\mathcal{N}_f)+r(\mathcal{N}_f)-1\right) \tag{5}$$

for any reduced power series.
Observe that if f is convenient then the two definitions of $\mu(\mathcal{N}_f)$, (3) and (4), coincide.
Let $f\in K[[x,y]]$ be a reduced power series and let $x^{d_1}y^{d_2}$ be the monomial of highest degree dividing f. We have $f=x^{d_1}y^{d_2}g$ where $g\in K[[x,y]]$ is a convenient power series or a unit. Since f is reduced $d_1,d_2\le 1$ and $(d_1,d_2)=(0,0)$ if and only if f is convenient. We have $[\mathcal{N}_f,\mathcal{N}_f]=2$(the area between $\mathcal{N}_f$ and the lines $x-d_1=0$, $y-d_2=0$).

The following nice formula is due to Lenarcik:

Lemma 3.2. *([12, Proposition 61]) Let f be a reduced power series of order bigger than one. Let A_1 be the area limited by $\mathcal{N}_f$ and the lines $x-1=0$ and $y-1=0$. If $(m_1,1),(1,n_1)\in\mathcal{N}_f$ then $\mu(\mathcal{N}_f)=2A_1+m_1+n_1-1$.*

Lemma 3.3. *Let A be the area between the Newton polygon of $f=x^{d_1}y^{d_2}g\in K[[x,y]]$ and the lines $x-d_1=0$ and $y-d_2=0$. Let $m=\mathrm{ord} f(x,0)$, $n=\mathrm{ord} f(0,y)$ (by convention* $\mathrm{ord}0=+\infty$*). Then*

$$A=\begin{cases} A_1+\frac{m+m_1-1}{2}+\frac{n+n_1-1}{2}, \;\; |\mathcal{N}_f|_1=m, \;\; |\mathcal{N}_f|_2=n & \text{if } (d_1,d_2)=(0,0)\\ A_1+\frac{m_1+m-2}{2}, \;\; |\mathcal{N}_f|_1=m-1, \;\; |\mathcal{N}_f|_2=n_1 & \text{if } (d_1,d_2)=(1,0)\\ A_1+\frac{n_1+n-2}{2}, \;\; |\mathcal{N}_f|_1=m_1, \;\; |\mathcal{N}_f|_2=n-1 & \text{if } (d_1,d_2)=(0,1)\\ A_1, \;\; |\mathcal{N}_f|_1=m_1-1, \;\; |\mathcal{N}_f|_2=n_1-1 & \text{if } (d_1,d_2)=(1,1). \end{cases} \quad (6)$$

Proof. It is a consequence of Lemma 3.2. ∎

Lemma 3.4. *([15, p.146]) Let $f=x^{d_1}y^{d_2}g\in K[[x,y]]$ be a reduced power series with $g(0,0)=0$. Then*

$$\mu(\mathcal{N}_f)=\begin{cases} [\mathcal{N}_f,\mathcal{N}_f]-|\mathcal{N}_f|_1-|\mathcal{N}_f|_2+1 & \text{if } (d_1,d_2)=(0,0)\\ [\mathcal{N}_f,\mathcal{N}_f]-|\mathcal{N}_f|_1+|\mathcal{N}_f|_2 & \text{if } (d_1,d_2)=(1,0)\\ [\mathcal{N}_f,\mathcal{N}_f]+|\mathcal{N}_f|_1-|\mathcal{N}_f|_2 & \text{if } (d_1,d_2)=(0,1)\\ [\mathcal{N}_f,\mathcal{N}_f]+|\mathcal{N}_f|_1+|\mathcal{N}_f|_2+1 & \text{if } (d_1,d_2)=(1,1). \end{cases} \quad (7)$$

Proof. We have $[\mathcal{N}_f,\mathcal{N}_f]=2A$ and by Lemma 3.2, $\mu(\mathcal{N}_f)=2A_1+m_1+n_1-1$. Use Lemma 3.3. ∎

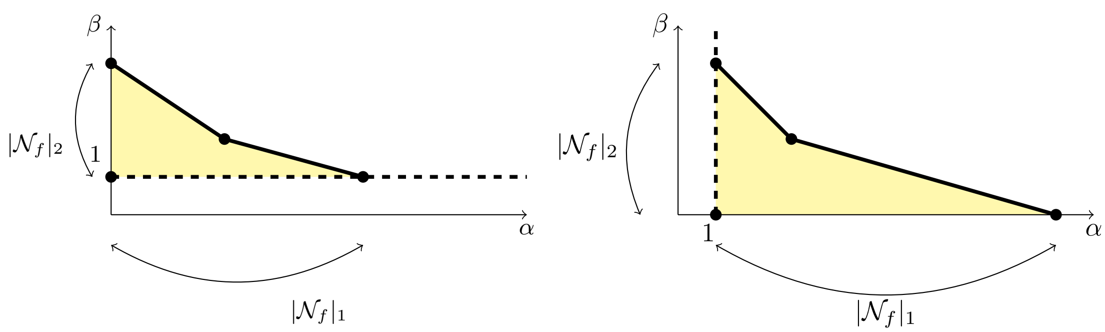

Figure 2: $f(x,y)=xg(x,y)$ and $f(x,y)=yg(x,y)$

A power series f will be called *elementary* if f is convenient and $\mathcal{N}_f$ contains only one edge S. The pair $(m,n)=(|S|_1,|S|_2)=(\mathrm{ord} f(x,0),\mathrm{ord} f(0,y))$ is by definition the *bidegree* of f and we will denote it by $\mathrm{bideg}(f)$. In what follows we write $\mathrm{In} f=\mathrm{in}(f,S)$. After [13,

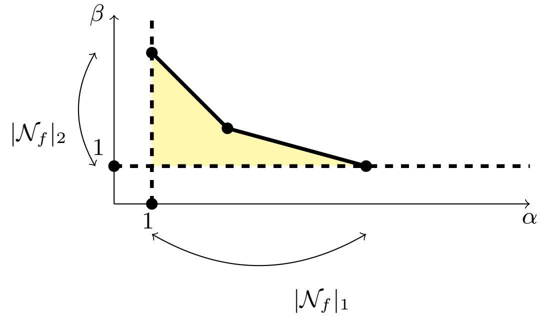

Figure 3: $f(x,y) = xyg(x,y)$

Chapter 2], every convenient irreducible power series is elementary. If f, g are elementary of bidegree (m,n) resp. (m',n') such that $\frac{m}{n} = \frac{m'}{n'}$, then fg is elementary of bidegree $(m+m', n+n')$. Moreover, $\mathrm{In} fg = \mathrm{In} f \cdot \mathrm{In}\, g$.

Lemma 3.5. *If $f \in K[[x,y]]$ is elementary of bidegree (m,n) and $d = \gcd(m,n)$ then $\mathrm{In} f(x,y) = F(x^{m/d}, y^{n/d})$ where $F = F(u,v)$ is a homogeneous form of degree d. Moreover, if f is irreducible then $\mathrm{In} f(x,y) = \left(ax^{m/d} + by^{n/d}\right)^d$.*

Proof. The polynomial $\mathrm{In} f$ is a linear combination of monomials $x^\alpha y^\beta$, where $\alpha n + \beta m = nm$. It is easy to check that $\alpha = \frac{m}{d}\alpha_1$, $\beta = \frac{m}{d}\beta_1$ for some $\alpha_1, \beta_1 \in \mathbb{N}$. Moreover, $\alpha_1 + \beta_1 = d$. Therefore $x^\alpha y^\beta = x^{\frac{m}{d}\alpha_1} y^{\frac{n}{d}\beta_1}$ and $\mathrm{In} f = F(x^{\frac{m}{d}} y^{\frac{n}{d}})$, where $F = F(u,v)$ is a homogeneous polynomial of degree d. Let $F(u,v) = \prod_{i=1}^{s}(a_i u + b_i v)^{e_i}$ where $a_i b_j \neq a_j b_i$ for $i \neq j$. Then $\mathrm{In} f = F(x^{\frac{m}{d}} y^{\frac{n}{d}}) = \prod_{i=1}^{s}(a_i x^{\frac{m}{d}} + b_i y^{\frac{n}{d}})^{e_i}$. By Hensel's lemma (see [1, Lemma A.1], [10]) we get $f(x,y) = g_1(x,y)\cdots g_s(x,y) \in K[[x,y]]$ where $\mathrm{In} g_i = (a_i x^{\frac{m}{d}} + b_i y^{\frac{n}{d}})^{e_i}$ for $i \in \{1,\ldots,s\}$. If f is irreducible then $s = 1$, $d = e_1$, $f = g_1$ and $\mathrm{In} f = \mathrm{In}\, g_1 = (a_1 x^{\frac{m}{d}} + b_1 y^{\frac{n}{d}})^d$. ∎

Corollary 3.6. *If f is non-degenerate, convenient and irreducible power series of bidegree (m,n) then $\gcd(n,m) = 1$.*

Lemma 3.7 (Newton factorization)**.** *Let $f \in K[[x,y]]$ be convenient. Then $f = \prod_{S \in \mathcal{N}_f} f_S$ in $K[[x,y]]$ where f_S are elementary. Moreover, the bidegree of f_S is $(|S|_1, |S|_2)$ and $\mathrm{In} f_S = c \cdot \overline{\mathrm{in}}(f,S)$, for some $c \in K \backslash \{0\}$.*

Proof. Firstly we prove that any convenient power series is a product of elementary power series. If f is elementary of bidegree (m,n) then we put $I(f) = \frac{m}{n}$. Let $f = f_1 \cdots f_r$ be the factorization into irreducible factors of a convenient power series. Let $\{I(f_i) \;:\; 1 \leq i \leq r\} = \{\omega_j \;:\; 1 \leq j \leq s\}$ where $\omega_1 < \omega_2 < \cdots < \omega_s$. For any $j \in \{1,\ldots,s\}$ we put $A_j := \{k \in \{1,\ldots,s\} \;:\; I(f_k) = \omega_j\}$ and $g_j := \prod_{i \in A_j} f_i$. Then g_j is an elementary power series and $f = g_1 \cdots g_s$ with $I(g_i) < I(g_j)$ for any $i \neq j$. Let $\mathrm{bideg}(g_k) = (m_k, n_k)$. Since

$\frac{m_1}{n_1} < \cdots < \frac{m_s}{n_s}$ the points $v_k = \left(\sum_{i=1}^{k} m_i, \sum_{i=k+1}^{s} n_i\right)$ with $k \in \{1, \ldots, s\}$ (by convention the empty sum equals zero) are vertices of $\mathcal{N}_f$. Let $S^{(k)}$ be the segment of $\mathcal{N}_f$ with vertices v_{k-1} and v_k for $k \in \{1, \ldots, s\}$, so $(|S^{(k)}|_1, |S^{(k)}|_2) = (m_k, n_k)$. If $S \in \mathcal{N}_f$ then $S = S^{(k)}$ for some $k \in \{1, \ldots, s\}$ and we put $f_S = g_k$. Therefore $f = \prod_{S \in \mathcal{N}_f} f_S$ where f_S are elementary, $\text{bideg}(f_S) = (|S|_1, |S|_2)$ and $\text{In} f_S = c \cdot \overline{\text{in}}(f, S)$ for some $c \in K \backslash \{0\}$. □

Corollary 3.8. *If $f \in K[[x, y]]$ is non-degenerate then f_S are non-degenerate for any $S \in \mathcal{N}_f$.*

For any two power series $f, g \in K[[x, y]]$ we put $i_0(f, g) := \dim_K K[[x, y]]/(f, g)$ and call it the *intersection multiplicity* of f and g.

Lemma 3.9. *If $\mathcal{N}_f = \{S\}$ and $\mathcal{N}_g = \{T\}$ are elementary then $i_0(f, g) \geq [S, T]$ with equality if and only if S and T are not parallel or the system of equations* $\text{In} f = 0$, $\text{In} g = 0$ *has the unique solution $(x, y) = (0, 0)$.*

Proof. Put $\text{bideg}(f) := (m, n)$ and $\text{bideg}(g) := (m_1, n_1)$. We have to check that $i_0(f, g) \geq \min\{mn_1, m_1 n\}$ with equality if and only if $\frac{m}{n} = \frac{m_1}{n_1}$ or the system of equations $\text{In} f = 0$, $\text{In} g = 0$ has the only solution $(x, y) = (0, 0)$. Put $f(x, y) = \sum_{ij} a_{ij} x^i y^j$. Let $\overrightarrow{w} = (n, m) \in \mathbb{N}_+^2$. Then $\text{ord}_{\overrightarrow{w}}(f) := \inf\{ni + jm \; : \; a_{ij} \neq 0\} = nm$ and $\text{in}_{\overrightarrow{w}} f := \sum_{in+jm=nm} a_{ij} x^i y^j = \text{In} f$. Let us distinguish two cases.

Case 1: $\frac{m}{n} \neq \frac{m_1}{n_1}$. We may assume $\frac{m}{n} < \frac{m_1}{n_1}$. Then $\text{ord}_{\overrightarrow{w}}(g) = mn_1 = \min\{mn_1, m_1 n\}$ and $\text{in}_{\overrightarrow{w}} g = c y^{n_1}$ for $c \neq 0$. Therefore the system of equations $\text{in}_{\overrightarrow{w}} f = 0$ and $\text{in}_{\overrightarrow{w}} g = 0$ has the unique solution $(x, y) = (0, 0)$ and we get

$$i_0(f, g) = \frac{\text{ord}_{\overrightarrow{w}} f \text{ord}_{\overrightarrow{w}} g}{mn} = \text{ord}_{\overrightarrow{w}} g = mn_1 = \min\{mn_1, m_1 n\},$$

by [6, Lemma A.1].

Case 2: $\frac{m}{n} = \frac{m_1}{n_1}$. We check $\text{ord}_{\overrightarrow{w}}(g) = mn_1$ and $\text{in}_{\overrightarrow{w}} g = \text{In} g$. Again by [6, Lemma A.1] we get $i_0(f, g) \geq \text{ord}_{\overrightarrow{w}} g = mn_1 = nm_1$ with equality if the system $\text{In} f = 0$, $\text{In} g = 0$ has the unique solution $(x, y) = (0, 0)$. □

4 Main result

The following theorem is the main result of this note:

Theorem 4.1. *Let $f \in K[[x, y]]$ be a reduced power series. Then*

1. $\bar{\mu}(f) - \mu(\mathcal{N}_f) \geq r(\mathcal{N}_f) - r(f) \geq 0$.
2. *If f is non-degenerate then $\bar{\mu}(f) = \mu(\mathcal{N}_f)$ and $r(\mathcal{N}_f) = r(f)$.*

The first statement of Theorem 4.1 was proved in [3, Proposition 7]. We provide a new and simple proof of it. The proof of Theorem 4.1 is given in Section 5.

As an immediate consequence of Theorem 4.1 we have

Corollary 4.2. *([3, Lemma 4]) Let $f \in K[[x,y]]$ be a reduced power series. We have $r(f) \leq r(\mathcal{N}_f)$ and if f is non-degenerate then $r(f) = r(\mathcal{N}_f)$.*

Corollary 4.3. *([2, Proposition 3.17], [3, Proposition 5]) Let $f \in K[[x,y]]$. We have $\delta(\mathcal{N}_f) \leq \delta(f)$ and if f is non-degenerate then $\delta(\mathcal{N}_f) = \delta(f)$.*

Proof. From the definition of the invariant Milnor number of f and the equality (5) we have $\bar{\mu}(f) - \mu(\mathcal{N}_f) = 2(\delta(f) - \delta(\mathcal{N}_f)) + r(\mathcal{N}_f) - r(f)$. We use Theorem 4.1. □

Corollary 4.4. *([3, Theorem 9]) Let $f \in K[[x,y]]$ be a reduced power series. If f is strongly non-degenerate then f is tame, i.e., $\mu(f) = \bar{\mu}(f)$.*

Proof. By Kouchnirenko's planar theorem [3, Proposition 4] we have $\mu(f) = \mu(\mathcal{N}_f)$. On the other hand by Theorem 4.1 we get $\bar{\mu}(f) = \mu(\mathcal{N}_f)$. Therefore $\mu(f) = \bar{\mu}(f)$. □

5 Proof of the main result

We begin with the proof of Theorem 4.1 for convenient power series. Firstly we consider the case of elementary power series. Let $f \in K[[x,y]]$ be an elementary power series of bidegree (m,n). Let $d := \gcd(m,n)$. Then the theorem reduces to the following statement:

$$\bar{\mu}(f) - (n-1)(m-1) \geq d - r(f) \geq 0. \tag{8}$$

If f is non-degenerate then $\bar{\mu}(f) = (n-1)(m-1)$ and $r(f) = d$.

We distinguish two cases. Suppose first that f is irreducible, that is $r(f) = 1$.

Lemma 5.1. *Let $f \in K[[x,y]]$ be irreducible with semigroup of values $\Gamma(f) = \langle v_0, v_1, \ldots, v_h \rangle$. If c is the conductor of $\Gamma(f)$ then $c \geq (v_0 - 1)(v_1 - 1) + \gcd(v_0, v_1) - 1$. The equality $c = (v_0 - 1)(v_1 - 1)$ holds if and only if $\gcd(v_0, v_1) = 1$.*

Proof. Let us define Puiseux characteristic sequence $b_0, b_1, \ldots, b_h$ by putting $b_0 = v_0$, $b_k = v_k - \sum_{i=1}^{k-1}(n_i - 1)v_i$ for $k \in \{1, \ldots, h\}$. Note that $\gcd(b_0, \ldots, b_k) = e_k$ for $k \in \{0, \ldots, h\}$ and $b_0 < b_1 < \cdots < b_h$. Moreover $c = \sum_{k=1}^{h}(e_{k-1} - e_k)(b_k - 1)$ (see for example [13, Chapter 3, p. 58]. If $e_1 = 1$ then $c = (e_0 - e_1)(b_1 - 1) = (b_0 - 1)(b_1 - 1)$. Therefore we may assume that $h > 1$. We have

$$\begin{aligned} c &= (e_0 - e_1)(b_1 - 1) + \sum_{k=2}^{h}(e_{k-1} - e_k)(b_k - 1) \\ &\geq (e_0 - e_1)(b_1 - 1) + \sum_{k=2}^{h}(e_{k-1} - e_k)(b_2 - 1) \\ &= (e_0 - e_1)(b_1 - 1) + (e_1 - 1)(b_2 - 1) \\ &= (e_0 - e_1)(b_1 - 1) + (e_1 - 1)(b_2 - b_1 + b_1 - 1) \\ &= (e_0 - 1)(b_1 - 1) + (e_1 - 1)(b_2 - b_1) \\ &\geq (b_0 - 1)(b_1 - 1) + e_1 - 1, \quad \text{since } b_2 - b_1 \geq 1. \end{aligned}$$

□

Suppose that $r(f) = 1$. Since $\bar{\mu}(f) = c(f) = c$ we have, by Lemma 5.1, $\bar{\mu}(f) \geq (v_0 - 1)(v_1 - 1) + \gcd(v_0, v_1) - 1$. The power series f being unitangent we have $m = \mathrm{ord} f(0, y) = \mathrm{ord} f$ or $n = \mathrm{ord} f(x, 0) = \mathrm{ord} f$. Assume that $m = \mathrm{ord} f$. Then $m \leq n \leq v_1$ (see [7]). If the axis $y = 0$ has maximal contact with the curve $f(x, y) = 0$ then $n = v_1$ and by Lemma 5.1 we get

$$\bar{\mu}(f) \geq (v_0 - 1)(v_1 - 1) + \gcd(v_0, v_1) - 1 = (m - 1)(n - 1) + d - 1 \geq 0.$$

If $n < v_1$ then $n \equiv 0 \pmod{m}$, $d = \gcd(m, n) = m$ and we get

$$\begin{aligned} \bar{\mu}(f) &\geq (v_0 - 1)(v_1 - 1) = (m - 1)(v_1 - n + n - 1) \\ &= (m - 1)(n - 1) + (v_1 - 1)(m - 1) \\ &\geq (m - 1)(n - 1) + m - 1 = (m - 1)(n - 1) + d - 1. \end{aligned}$$

If $m = n$ then $\bar{\mu}(f) \geq n(n - 1)$ (see [13, p. 88]).
Suppose that f is non-degenerate. Then, by Corollary 3.6, $d = \gcd(n, m) = 1$. Consequently, by Lemma 5.1, $\bar{\mu}(f) = (m - 1)(n - 1) + d - 1$.

Suppose now that f is elementary but $r(f) > 1$. Recall that any irreducible convenient power series is elementary.

Lemma 5.2. *Let f be an elementary power series with* $\mathrm{bideg}(f) = (m, n)$ *and $f = f_1 \cdots f_r$ the factorization of f into irreducible factors with* $\mathrm{bideg}(f_i) = (m_i, n_i)$*. If $d = \gcd(m, n)$ and $d_i = \gcd(m_i, n_i)$ then*

1. $\frac{m_i}{d_i} = \frac{m}{d}$ *and* $\frac{n_i}{d_i} = \frac{n}{d}$ *for any* $i \in \{1, \ldots, r\}$.
2. $\sum_{i=1}^{r} d_i = d$.

Moreover, $r \leq d$ with equality if f is non-degenerate.

Proof. By Lemma 3.5 we have $\mathrm{In} f(x,y) = F(x^{m/d}, y^{n/d})$ for some homogeneous polynomial F of degree d. Since f_i are elementary $\mathrm{In} f(x,y) = \mathrm{In} f_1(x,y) \cdots \mathrm{In} f_r(x,y)$. By Lemma 3.5 $\mathrm{In} f_i(x,y) = (a_i x^{\frac{m_i}{d_i}} + b_i y^{\frac{n_i}{d_i}})^{d_i}$ for some $a_i, b_i \in K$. Then $a_i x^{\frac{m_i}{d_i}} + b_i y^{\frac{n_i}{d_i}}$ is an irreducible factor of $F(x^{m/d}, y^{n/d})$, which implies $\frac{m_i}{d_i} = \frac{m}{d}$ and $\frac{n_i}{d_i} = \frac{n}{d}$ for any $i \in \{1, \dots, r\}$. Since $f(x,0) = \prod_{i=1}^r f_i(x,0)$ we have $m = \mathrm{ord} f(x,0) = \sum_{i=1}^r \mathrm{ord} f_i(x,0) = \sum_{i=1}^r m_i = \sum_{i=1}^r d_i \frac{m}{d}$ whence $\sum_{i=1}^r d_i = d$. Obviously $r \leq d$. If f is non-degenerate then f_i are non-degenerate and $d_i = 1$ for $i \in \{1, \dots, r\}$ by Corollary 3.6. Therefore $r = d$. □

By the third statement of Proposition 2.1 we get

$$\bar{\mu}(f) + r - 1 = \sum_{i=1}^{r} \bar{\mu}(f_i) + 2 \sum_{1 \leq i < j \leq r} i_0(f_i, f_j).$$

By the irreducible elementary case we have $\bar{\mu}(f_i) \geq \left(\frac{m}{d} d_i - 1\right)\left(\frac{n}{d} d_i - 1\right) + (d_i - 1)$. Moreover, by Lemma 3.9, $i_0(f_i, f_j) \geq \dfrac{mn}{d^2} d_i d_j$. Therefore we get

$$\begin{aligned} \bar{\mu}(f) + r - 1 &\geq \sum_{i=1}^{r} \left[\left(\frac{m}{d} d_i - 1\right)\left(\frac{n}{d} d_i - 1\right) + (d_i - 1)\right] + 2 \sum_{1 \leq i < j \leq r} \frac{mn}{d^2} d_i d_j \\ &= \frac{mn}{d^2} \left(\sum_{i=1}^{r} d_i^2 + 2 \sum_{1 \leq i < j \leq r} d_i d_j \right) + \left(\frac{-n-m}{d} + 1 \right) \sum_{i=1}^{r} d_i \\ &= mn - n - m + d. \end{aligned}$$

Whence $\bar{\mu}(f) + r - 1 \geq (n-1)(m-1) + d - 1$ which implies the inequality (8). If f is non-degenerate then $d_i = 1$ for $i \in \{1, \dots, r\}$, $\bar{\mu}(f_i) = \left(\frac{m}{d} - 1\right)\left(\frac{n}{d} - 1\right)$ and $i_0(f_i, f_j) = \frac{mn}{d^2}$ and the inequalities become equalities. Moreover, $r(f) = r = d$ by Lemma 5.2.

Let us prove now the general case, that is $\sigma := \sharp \mathcal{N}_f > 1$. Let $f = \prod_{S \in \mathcal{N}_f} f_S$ be the Newton factorization of f. By the third statement of Proposition 2.1 we get

$$\bar{\mu}(f) + \sigma - 1 = \sum_{S \in \mathcal{N}_f} \bar{\mu}(f_S) + \sum_{S \neq T} i_0(f_S, f_T) = \sum_{S \in \mathcal{N}_f} \bar{\mu}(f_S) + \sum_{S \neq T} [S, T],$$

where S and T are not parallel. Since f_S is elementary of bidegree $(|S|_1, |S|_2)$ we get

$$\bar{\mu}(f_S) \geq (|S|_1 - 1)(|S|_2 - 1) + \gcd(|S|_1, |S|_2) - r(f_S).$$

A simple calculation shows that

$$\bar{\mu}(f) + \sigma - 1 \geq [\mathcal{N}_f, \mathcal{N}_f] - |\mathcal{N}_f|_1 - |\mathcal{N}_f|_2 + \sigma + r(\mathcal{N}_f) - r(f).$$

Therefore $\bar{\mu}(f) \geq \mu(\mathcal{N}_f) + r(\mathcal{N}_f) - r(f)$. If f is non-degenerate then f_S is non-degenerate. Thus $\bar{\mu}(f_S) = \mu(\mathcal{N}_{f_S})$ and $r(f_S) = r(\mathcal{N}_{f_S}) = \gcd(|S|_1, |S|_2)$. Using the Newton factorization we get $\bar{\mu}(f) = \mu(\mathcal{N}_f)$. Obviously $r(f) = \sum_S r(f_S) = \sum_S \gcd(|S|_1, |S|_2) = r(\mathcal{N}_f)$.

It remains to prove Theorem 4.1 for non-convenient power series.

Let $f(x,y) = x^{d_1}y^{d_2}g(x,y)$ where $g = g(x,y)$ is a convenient reduced power series or a unit. We assume that $g(0,0) = 0$ (if $g(0,0) \neq 0$ then $\overline{\mu}(f) = \mu(\mathcal{N}_f)$ and $r(f) = \mu(\mathcal{N}_f)$).

Because the length of a segment is the same on parallel axes we have

$$|\mathcal{N}_f|_i = |\mathcal{N}_g|_i \text{ for } i = 1,2,\ [\mathcal{N}_f, \mathcal{N}_f] = [\mathcal{N}_g, \mathcal{N}_g] \text{ and } r(\mathcal{N}_f) - r(f) = r(\mathcal{N}_g) - r(g). \tag{9}$$

Since we have already proved Theorem 4.1 for convenient power series we get

$$\overline{\mu}(g) - \mu(\mathcal{N}_g) \geq r(\mathcal{N}_g) - r(g) \geq 0, \tag{10}$$

and the equalities $\overline{\mu}(g) = \mu(\mathcal{N}_g)$ and $r(\mathcal{N}_g) = r(g)$ holding for non-degenerate g.

By Proposition 2.1 we get

$$\begin{aligned} \bar{\mu}(f) + 2 &= \bar{\mu}(x) + \bar{\mu}(y) + \bar{\mu}(g) + 2i_0(g,x) + 2i_0(g,y) + 2i_0(x,y) \\ &= \bar{\mu}(g) + 2\mathrm{ord}g(0,y) + 2\mathrm{ord}g(x,0) + 2, \end{aligned}$$

and

$$\begin{aligned} \bar{\mu}(f) &= \bar{\mu}(g) + 2|\mathcal{N}_g|_1 + 2|\mathcal{N}_g|_2 \\ &\geq \mu(\mathcal{N}_g) + r(\mathcal{N}_g) - r(g) + 2|\mathcal{N}_g|_1 + 2|\mathcal{N}_g|_2 \\ &= [\mathcal{N}_g, \mathcal{N}_g] + |\mathcal{N}_g|_1 + |\mathcal{N}_g|_2 + 1 + r(\mathcal{N}_g) - r(g) \\ &= [\mathcal{N}_f, \mathcal{N}_f] + |\mathcal{N}_f|_1 + |\mathcal{N}_f|_2 + 1 + r(\mathcal{N}_f) - r(f) \\ &\geq \mu(\mathcal{N}_f) + r(\mathcal{N}_f) - r(f). \end{aligned}$$

If f is non-degenerate then g is non-degenerate and we get $\bar{\mu}(f) = \mu(\mathcal{N}_f)$ and $r(\mathcal{N}_f) = r(f)$.

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(1) Departamento de Matemáticas, Estadística e I. O., IMAULL,
Universidad de La Laguna, Apartado de Correos 456, 38200 La Laguna, Tenerife, España
E-mail: `ergarcia@ull.es`

(2) Department of Mathematics and Physics, Kielce University of Technology,
Al. 1000 L PP7, 25-314 Kielce, Poland
E-mail: `matap@tu.kielce.pl`