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On Milnor and Tjurina Numbers of Foliations

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Abstract

We study the relationship between the Milnor and Tjurina numbers of a singular foliation $\mathcal{F}$, in the complex plane, with respect to a balanced divisor of separatrices $\mathcal{B}$ for $\mathcal{F}$. For that, we associate with $\mathcal{F}$ a new number called the χ-number and we prove that it is a C^1 invariant for holomorphic foliations. We compute the polar excess number of $\mathcal{F}$ with respect to a balanced divisor of separatrices $\mathcal{B}$ for $\mathcal{F}$, via the Milnor number of the foliation, the multiplicity of some hamiltonian foliations along the separatrices in the support of $\mathcal{B}$ and the χ-number of $\mathcal{F}$. On the other hand, we generalize, in the plane case and the formal context, the well-known result of Gómez-Mont given in the holomorphic context, which establishes the equality between the GSV-index of the foliation and the difference between the Tjurina number of the foliation and the Tjurina number of a set of separatrices of $\mathcal{F}$. Finally, we state numerical relationships between some classic indices, as Baum–Bott, Camacho–Sad, and variational indices of a singular foliation and its Milnor and Tjurina numbers; and we obtain a bound for the sum of Milnor numbers of the local separatrices of a holomorphic foliation on the complex projective plane.

Keywords Tjurina number · Milnor number · χ-number · Dicritical foliation

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1 Introduction

The Milnor and Tjurina numbers are classical invariants in the theory of complex analytic hypersurfaces with isolated singularity. The Milnor number of a holomorphic function germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ with an isolated singularity is exactly the rank of the middle homology group of the Milnor fiber of f, equal to the number of spheres in its bouquet decomposition. The notion of the Milnor number for hypersurfaces was introduced in Milnor (1969, Sect. 7). The Tjurina number is the dimension of the base space of a semi-universal deformation of the hypersurface. Semi-universal deformations were studied in Tjurina (1969) and as far as our knowledge reaches, the name Tjurina number appears for the first time in Greuel (1980). The Milnor number is a topological invariant and the Tjurina number is an analytic invariant of the singularity. There is an abundant and varied bibliography on the Milnor number of singular hypersurfaces for the classical concept, as well as for the more recent relative concept, known as Bruce–Roberts Milnor number (see for example the recent papers Bivià-Ausina et al. 2024; Barbosa et al. 2024). Moreover, the Milnor number is related to polar multiplicities; see, for instance, Carvalho et al. (2022) and the references therein.

The Tjurina number has not been studied that well, perhaps because it is an analytical invariant, but in recent years new studies have been published (see for example Alberich-Carramiñana et al. 2021; Genzmer and Hernandes 2020; Wang 2020; Almirón 2022). In the context of singular foliations, the notion of Milnor number appears for the first time with that name in Camacho et al. (1984), although this notion is found in previous works such as that of van den Essen (1979), where a new proof of Seidenberg's theorem is given. As in the case of hypersurfaces, the Milnor number of a one-dimensional holomorphic foliation is a topological invariant and there is a varied bibliography on it. However, the concept of the Tjurina number of a foliation has been less studied and according to our knowledge, always related to the Gómez-Mont–Seade–Verjovsky index, after Gómez-Mont (1998). We emphasize that Gómez-Mont did not use the terminology of the Tjurina number of a foliation, such a name appears for the first time in Cano et al. (2019, p. 159). In Licanic (2004, Corollary 2.7) there is a bound for the Tjurina number of an $\mathcal{F}$-invariant curve C as a function of the Tjurina number of $\mathcal{F}$ with respect to C. Some verifications on the relationship between the Milnor and Tjurina numbers of non-dicritical foliations and their total union of separatrices are collected in Fernández Sánchez et al. (2022).

In this work we deepen into the study of the Tjurina number of a singular foliation along a reduced curve of separatrices and its relationship with the Milnor number and other invariants and indices associated with the foliation such as the polar excess number, Baum–Bott, Camacho–Sad, and variational indices. Taking into account that a singular foliation could admit infinitely many separatrices—dicritical foliation—the Tjurina number of a foliation will be associated with a balanced divisor of separatrices. The notion of balanced divisor of separatrices for a foliation was introduced by Genzmer (2007) and we recall it in Definition 2.1. This is a geometric object formed by a finite set of separatrices, choosing all isolated separatrices and some separatrices from the ones associated to dicritical components, with weights, possibly negative (those that correspond to poles). In the non-dicritical case, this notion coincides with the total union of separatrices of the foliation. A balanced divisor of separatrices provides a control of the algebraic multiplicity of the foliation and of its separatrices (see Proposition 2.4). In Foliation Theory it is not easy to determine whether an invariant associated with a foliation is topological (remember that, for example, it is not yet known if the algebraic multiplicity is). However, it has been shown that some indices are analytical invariants, such as the Baum–Bott index (see Cerveau and Lins Neto 2013, Remark 3.2), Camacho–Sad and Variational indices (a good exercise for beginners in the world of foliations). We hope that this article contributes for understanding the dicritical foliations and the invariants associated with them.

The paper is organized as follows. In Sect. 2, we recall some preliminary notions that are necessary in the paper. The first motivation for this work was understanding the relationship between the Milnor and Tjurina numbers of a singular foliation $\mathcal{F}$ and a balanced divisor of separatrices for $\mathcal{F}$. It is for this purpose that in Sect. 3 we associate a new number $\chi_p(\mathcal{F})$ to any (dicritical or non dicritical) singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$. This number is defined in function of the algebraic multiplicities and excess tangency indices of the strict transforms of $\mathcal{F}$ at infinitely near points of p. Hence $\chi_p(\mathcal{F})$ is a C^1 invariant for holomorphic foliations. We study the properties of the χ-number in Proposition 3.1. In particular, we prove that it is a nonnegative integer number and any foliation of algebraic multiplicity bigger than 1 is of the second type, a concept that we introduce below, if and only if the χ-number equals zero.

Section 4 is devoted to the indices associated with a foliation making use of a generic polar curve of it, as the polar intersection and the polar excess. Proposition 4.2 provides a formula to compute the polar excess number of a foliation with respect to the zero divisor of a reduced balanced divisor of separatrices, and as a consequence, we obtain a characterization of generalized curve foliations, which generalizes (Cano et al. 2019, Proposition 2) to the dicritical context. In Proposition 4.7, we establish the relationship between the Milnor number of a foliation, the multiplicity of the foliation along the separatrices (of a balanced divisor) and the χ-number of the foliation, generalizing (Cano et al. 2019, Corollary 2) to dicritical foliations. As a consequence we give a new proof, using foliations, of the well-known relationship between the Milnor number of a reduced plane curve and the Milnor numbers of its irreducible components (see Proposition 4.8). Theorem A is the main result in this section and one of the main results in this paper. In Theorem A we compute the polar excess number $\Delta_p(\mathcal{F}, \mathcal{B})$ of a singular foliation $\mathcal{F}$, with respect to a balanced divisor of separatrices $\mathcal{B}$, via the Milnor number of the foliation, $\mu_p(\mathcal{F})$, the multiplicity of some hamiltonian foliations along the separatrices in the support of $\mathcal{B}$, $\mu_p(dF_B, B)$, and the χ-number of $\mathcal{F}$. More precisely

Theorem A *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then*

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}) - \sum_B a_B \mu_p(dF_B, B) + \deg(\mathcal{B}) - 1 - \chi_p(\mathcal{F}),$$

where for each separatrix B, F_B is a balanced divisor of separatrices for $\mathcal{F}$ adapted to B. Moreover, if $\mathcal{F}$ is a foliation of the second type, then

$$\Delta_p(\mathcal{F},\mathcal{B}) = \mu_p(\mathcal{F}) - \sum_B a_B \mu_p(dF_B, B) + \deg(\mathcal{B}) - 1.$$

As a consequence, in Corollary 4.10, we give a new characterization of generalized curve foliations in the non-dicritical case.

In Sect. 5, we study the Gómez-Mont–Seade–Verjovsky index (GSV-index). In particular, in Corollary 5.3, we compute the GSV-index of a foliation $\mathcal{F}$ with respect to the zero divisor of a reduced balanced divisor of separatrices for $\mathcal{F}$. In Proposition 5.4, we generalize, to the dicritical case (Cano et al. 2019, Proposition 4) which establishes the equality between the GSV-index of a foliation $\mathcal{F}$ (containing perhaps purely formal branches) with respect to an $\mathcal{F}$-invariant curve $C : f(x, y) = 0$ and the intersection numbers of C with generic polar curves of $\mathcal{F}$ and of df.

We finish this section establishing, in Proposition 5.7, a relationship between the GSV-index and the multiplicity of a foliation along a fixed separatrix.

In Sect. 6, we introduce the notion of Tjurina number of a singular foliation $\mathcal{F}$ along a reduced curve of separatrices C, denoted by $\tau_p(\mathcal{F}, C)$. Gómez-Mont proved, for a singular foliation with a set of convergent separatrices C, that the difference between the Tjurina number of the foliation and the Tjurina number of C, $\tau_p(C)$, equals to the GSV-index (see Gómez-Mont 1998, Theorem 1). In Proposition 6.2, we show that this result also holds, in the formal context, for the Tjurina number of a singular foliation along a reduced curve of separatrices. As a consequence we get the next corollary for non-dicritical foliations:

Corollary B *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and C is the total union of separatrices of $\mathcal{F}$. Then*

$$\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C) + \chi_p(\mathcal{F}).$$

Moreover, if $\mathcal{F}$ is of second type then $\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C)$.

The main result in this section, and another of the main results in the paper, is Theorem C, where given a balanced divisor of separatrices $\mathcal{B} = \sum_B a_B B$ of the singular foliation $\mathcal{F}$, we compute the difference of the Milnor of $\mathcal{F}$ and the sum $T_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B \tau_p(\mathcal{F}, B)$ of Tjurina numbers of $\mathcal{F}$ along the components of $\mathcal{B}$:

Theorem C *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then*

$$\begin{aligned}\mu_p(\mathcal{F}) - T_p(\mathcal{F},\mathcal{B}) = &\sum_B a_B[\mu_p(dF_B, B) - \tau_p(B)] - \deg(\mathcal{B}) + 1 + \chi_p(\mathcal{F})\\ &- \sum_B a_B[i_p(B, (F_B)_0 \backslash B) - i_p(B, (F_B)_\infty)],\end{aligned}$$

where F_B is a balanced divisor of separatrices for $\mathcal{F}$ adapted to B.

As a consequence, we precise this relationship for second type foliations in Corollary 6.6 and for non-dicritical foliations in:

Corollary D *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C = \cup_{j=1}^{\ell} C_j$ is the total union of separatrices of $\mathcal{F}$. Then*

$$\mu_p(\mathcal{F}) - T_p(\mathcal{F}, C) = \mu_p(C) - \sum_{j=1}^{\ell} \tau_p(C_j) + \chi_p(\mathcal{F}) - \sum_{j=1}^{\ell} i_p(C_j, C \backslash C_j),$$

where $\tau_p(C_j)$ is the Tjurina number of C_j.

We complete this section with several examples. In particular, in Example 6.5, we construct a family of dicritical foliations which are not of the second type. We finish Sect. 6 stating numerical relationships between some classic indices, such as Baum–Bott, Camacho–Sad, and variational indices, of a singular foliation and the Milnor and Tjurina numbers. Finally, in Sect. 7, we obtain a bound for the sum of Milnor numbers of the local separatrices of a holomorphic foliation on the complex projective plane.

2 Basic Tools

In order to fix the terminology and the notation, we recall some basic concepts of local Foliation Theory. Unless we specify otherwise, throughout this text $\mathcal{F}$ denotes a germ of a singular (holomorphic or formal) foliation at $(\mathbb{C}^2, p)$. In local coordinates (x, y) centered at p, the foliation is given by a (holomorphic or formal) 1-form

$$\omega = P(x, y)dx + Q(x, y)dy, \tag{2.1}$$

or by its dual vector field

$$v = -Q(x, y)\frac{\partial}{\partial x} + P(x, y)\frac{\partial}{\partial y}, \tag{2.2}$$

where $P(x, y), Q(x, y) \in \mathbb{C}[[x, y]]$ are relatively prime, where $\mathbb{C}[[x, y]]$ is the ring of complex formal power series in two variables. The *algebraic multiplicity* $\nu_p(\mathcal{F})$ is the minimum of the orders $\nu_p(P)$, $\nu_p(Q)$ at p of the coefficients of a local generator of $\mathcal{F}$.

Let $f(x, y) \in \mathbb{C}[[x, y]]$. We say that $C : f(x, y) = 0$ is *invariant* by $\mathcal{F}$ or $\mathcal{F}$-*invariant* if

$$\omega \wedge df = (f.h) dx \wedge dy,$$

for some $h \in \mathbb{C}[[x, y]]$. If C is an irreducible $\mathcal{F}$-invariant curve then we say that C is a *separatrix* of $\mathcal{F}$. The separatrix C is analytic if f is convergent. We denote by $\mathrm{Sep}_p(\mathcal{F})$ the set of all separatrices of $\mathcal{F}$. When $\mathrm{Sep}_p(\mathcal{F})$ is a finite set we say that the foliation $\mathcal{F}$ is *non-dicritical* and we call *total union of separatrices* of $\mathcal{F}$ to the union of all elements of $\mathrm{Sep}_p(\mathcal{F})$. Otherwise we say that $\mathcal{F}$ is a *dicritical* foliation.

A point $p \in \mathbb{C}^2$ is a *reduced* or *simple* singularity for $\mathcal{F}$ if the linear part $\mathrm{D}v(p)$ of the vector field v in (2.2) is non-zero and has eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ fitting in one of the two following cases:

(1) $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}^+$ (in this case we say that p is a *non-degenerate* or *complex hyperbolic* singularity).

(2) $\lambda_1 \neq 0$ and $\lambda_2 = 0$ (in this case we say that p is a *saddle-node* singularity).

The reduction process of the singularities of a codimension one singular foliation over an ambient space of dimension two was achieved by Seidenberg (1968). Van den Essen gave a new proof in van den Essen (1979).

A singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ is a *generalized curve foliation* if it has no saddle-nodes in its reduction process of singularities, that is, the case (1). This concept was defined by Camacho–Lins Neto–Sad (Camacho et al. 1984, p. 144). In this case, there is a system of coordinates (x, y) in which $\mathcal{F}$ is induced by the equation

$$\omega = x(\lambda_1 + a(x, y))dy - y(\lambda_2 + b(x, y))dx, \tag{2.3}$$

where $a(x, y), b(x, y) \in \mathbb{C}[[x, y]]$ are non-units, so that $\mathrm{Sep}_p(\mathcal{F})$ is formed by two transversal analytic branches given by $\{x = 0\}$ and $\{y = 0\}$. In the case (2), up to a formal change of coordinates, the saddle-node singularity is given by a 1-form of the type

$$\omega = x^{k+1}dy - y(1 + \lambda x^k)dx, \tag{2.4}$$

where $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}_{>0}$ are invariants after formal changes of coordinates (see Martinet and Ramis 1982, Proposition 4.3). The curve $\{x = 0\}$ is an analytic separatrix, called *strong*, whereas $\{y = 0\}$ corresponds to a possibly formal separatrix, called *weak* or *central*.

Given a foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ we follow (Fernández-Pérez and Mol 2019, p. 1115) to introduce the set $\mathcal{I}_p(\mathcal{F})$ of *infinitely near points* of $\mathcal{F}$ at p. This is defined in a recursive way along the reduction process of the singularities of $\mathcal{F}$. We do as follows. Given a sequence of blow-ups $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$—a possibly intermediate step in the reduction process, where $\mathcal{D} = \pi^{-1}(p)$ and $\tilde{X}$ is the ambient space containing $\mathcal{D}$—and a point $q \in \mathcal{D}$ we set:

- if $\tilde{\mathcal{F}}$ is $\mathcal{D}$-reduced at q, i.e. $q \in \mathcal{D}$ is a reduced singularity for $\tilde{\mathcal{F}}$, then $\mathcal{I}_q(\tilde{\mathcal{F}}) = \{q\}$;
- if $\tilde{\mathcal{F}}$ is $\mathcal{D}$-singular but not $\mathcal{D}$-reduced at q, we perform a blow-up $\sigma : (\hat{X}, \hat{\mathcal{D}}) \to (\tilde{X}, \mathcal{D})$ at q, where $\hat{\mathcal{D}} = \sigma^{-1}(\mathcal{D}) = (\sigma^*\mathcal{D}) \cup D$ and $D = \sigma^{-1}(q)$ (here $\sigma^*\mathcal{D}$ denotes the strict transform of $\mathcal{D}$). If $q_1, \ldots, q_\ell$ are all $\hat{\mathcal{D}}$-singular points of $\hat{\mathcal{F}} = \sigma^*\tilde{\mathcal{F}}$ on D, then
$$\mathcal{I}_q(\tilde{\mathcal{F}}) = \{q\} \cup \mathcal{I}_{q_1}(\hat{\mathcal{F}}) \cup \ldots \cup \mathcal{I}_{q_\ell}(\hat{\mathcal{F}}).$$

In order to simplify notation, we settle that a numerical invariant for a foliation $\mathcal{F}$ at $q \in \mathcal{I}_p(\mathcal{F})$ actually means the same invariant computed for the strict transform of $\mathcal{F}$ at q.

For a fixed reduction process of singularities $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ for $\mathcal{F}$, a component $D \subset \mathcal{D}$ can be:

- *non-dicritical*, if D is $\tilde{\mathcal{F}}$-invariant. In this case, D contains a finite number of simple singularities. Each non-corner singularity of D carries a separatrix transversal to D, whose projection by π is a curve in $\mathrm{Sep}_p(\mathcal{F})$. Remember that a corner singularity of D is an intersection point of D with other irreducible component of $\mathcal{D}$.
- *dicritical*, if D is not $\tilde{\mathcal{F}}$-invariant. The reduction process of singularities gives that D may intersect only non-dicritical components of $\mathcal{D}$ and $\tilde{\mathcal{F}}$ is everywhere transverse to D. The π-image of a local leaf of $\tilde{\mathcal{F}}$ at each non-corner point of D belongs to $\mathrm{Sep}_p(\mathcal{F})$.

Let σ be the blow-up of the reduction process of singularities π that generated the component $D \subset \mathcal{D}$. We say that σ is *non-dicritical* (respectively *dicritical*) if D is non-dicritical (respectively dicritical).

Denote by $\mathrm{Sep}_p(D) \subset \mathrm{Sep}_p(\mathcal{F})$ the set of separatrices whose strict transform by π intersects the component $D \subset \mathcal{D}$. If $B \in \mathrm{Sep}_p(D)$ with D non-dicritical, B is said to be *isolated*. Otherwise, it is said to be a *dicritical separatrix*. This determines the decomposition $\mathrm{Sep}_p(\mathcal{F}) = \mathrm{Iso}_p(\mathcal{F}) \cup \mathrm{Dic}_p(\mathcal{F})$, where notations are self-evident. The set $\mathrm{Iso}_p(\mathcal{F})$ is finite and contains all purely formal separatrices. It subdivides further in two classes: *weak* separatrices—those arising from the weak separatrices of saddle-nodes—and *strong* separatrices—corresponding to strong separatrices of saddle-nodes and separatrices of non-degenerate singularities. On the other hand, if $\mathrm{Dic}_p(\mathcal{F})$ is non-empty then it is an infinite set of analytic separatrices. Observe that a foliation $\mathcal{F}$ is *dicritical* when $\mathrm{Sep}_p(\mathcal{F})$ is infinite, which is equivalent to saying that $\mathrm{Dic}_p(\mathcal{F})$ is non-empty. Otherwise, $\mathcal{F}$ is *non-dicritical*.

Throughout the text, we adopt the language of *divisors* of formal curves. More specifically, a *divisor of separatrices* for a foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ is a formal sum

$$\mathcal{B} = \sum_{B \in \mathrm{Sep}_p(\mathcal{F})} a_B \cdot B, \tag{2.5}$$

where the coefficients $a_B \in \mathbb{Z}$ are zero except for finitely many $B \in \mathrm{Sep}_p(\mathcal{F})$. The set of separatrices $\{B : a_B \neq 0\}$ appearing in (2.5) is called the *support* of the divisor $\mathcal{B}$ and it is denoted by $\mathrm{supp}(\mathcal{B})$. The *degree* of the divisor $\mathcal{B}$ is by definition $\deg \mathcal{B} = \sum_{B \in \mathrm{supp}(\mathcal{B})} a_B$. We denote by $\mathrm{Div}_p(\mathcal{F})$ the set of all these divisors of separatrices, which turns into a group with the canonical additive structure. We follow the usual terminology and notation:

- $\mathcal{B} \geq 0$ denotes an *effective* divisor, one whose coefficients are all non-negative;
- there is a unique decomposition $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$, where $\mathcal{B}_0, \mathcal{B}_\infty \geq 0$ are respectively the *zero* and *pole* divisors of $\mathcal{B}$;
- the *algebraic multiplicity* of $\mathcal{B}$ is $\nu_p(\mathcal{B}) = \sum_{B \in \mathrm{supp}(\mathcal{B})} \nu_p(B)$.

Given a foliation $\mathcal{F}$ and a formal meromorphic equation $F(x, y) = \prod_{i=1}^{s} f_i(x, y)^{a_i}$, whose irreducible components define separatrices $B_i : f_i(x, y) = 0$ of $\mathcal{F}$, we associate the divisor $(F) = \sum_i a_i \cdot B_i$. A curve of separatrices C, associated with a reduced equation $F(x, y)$, is identified to the divisor (F) and we write $C = (F)$. Such an effective divisor is named *reduced* if all its coefficients are either 0 or 1. In general, $\mathcal{B} \in \mathrm{Div}_p(\mathcal{F})$ is reduced if both $\mathcal{B}_0$ and $\mathcal{B}_\infty$ are reduced divisors. A divisor $\mathcal{B}$ is said to be *adapted* to a curve of separatrices C if $\mathcal{B}_0 - C \geq 0$.

Following Genzmer (2007, p. 5) and Genzmer and Mol (2018, Definition 3.1), we remember the following notion:

Definition 2.1 A *balanced divisor of separatrices* for $\mathcal{F}$ is a divisor of the form

$$\mathcal{B} = \sum_{B \in \mathrm{Iso}_p(\mathcal{F})} B + \sum_{B \in \mathrm{Dic}_p(\mathcal{F})} a_B \cdot B,$$

where the coefficients $a_B \in \mathbb{Z}$ are non-zero except for finitely many $B \in \mathrm{Dic}_p(\mathcal{F})$, and, for each dicritical component $D \subset \mathcal{D}$, the following equality is respected:

$$\sum_{B \in \mathrm{Sep}_p(D)} a_B = 2 - \mathrm{Val}(D).$$

The integer $\mathrm{Val}(D)$ stands for the *valence* of a component $D \subset \mathcal{D}$ in the reduction process of singularities, that is, it is the number of components of $\mathcal{D}$ intersecting D other from D itself.

Observe that the notion of balanced divisor of separatrices generalizes to dicritical foliations the notion of total union of separatrices for non-dicritical foliations.

A balanced divisor $\mathcal{B} = \sum_B a_B B$ of separatrices of $\mathcal{F}$ is called *primitive* if $a_B \in \{-1, 1\}$ for any $B \in \mathrm{supp}(\mathcal{B})$. A *balanced equation of separatrices* is a formal meromorphic function $F(x, y)$ whose associated divisor $C = C_0 - C_\infty$ is a balanced divisor. A balanced equation is *reduced*, *primitive* or *adapted* to a curve C if the same is true for the underlying divisor.

Remember that the *intersection number* of two formal curves C and D at $(\mathbb{C}^2, p)$ is by definition

$$i_p(C, D) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(g, h),$$

where $C : g(x, y) = 0$, $D : h(x, y) = 0$, and (g, h) denotes the ideal generated by g and h in $\mathbb{C}[[x, y]]$. The intersection number for formal curves at $(\mathbb{C}^2, p)$ is canonically extended in a bilinear way to divisors of curves.

Let $\mathcal{F}$ be a foliation at $(\mathbb{C}^2, p)$ given by a 1-form as in (2.1), with reduction process $\pi : (\tilde{X}, \mathcal{D}) \to (\mathbb{C}^2, p)$ and let $\tilde{\mathcal{F}} = \pi^* \mathcal{F}$ be the strict transform of $\mathcal{F}$. Denote by $\mathsf{Sing}(\cdot)$ the set of singularities of a foliation. A saddle-node singularity $q \in \mathsf{Sing}(\tilde{\mathcal{F}})$ is said to be a *tangent saddle-node* if its weak separatrix is contained in the exceptional divisor $\mathcal{D}$, that is, the weak separatrix is an irreducible component of $\mathcal{D}$.

We have the following definition given by Mattei–Salem (Mattei and Salem 2004, Définition 3.1.4) to non-dicritical case and used by Genzmer (2007) for arbitrary foliations:

Definition 2.2 A foliation is *in the second class* or is *of second type* if there are no tangent saddle-nodes in its reduction process of singularities.

Let B be a separatrix of $\mathcal{F}$ at p. Suppose that $\{y = 0\}$ is the tangent cone of B, then we may choose one of its primitive Puiseux parametrizations $\gamma(t) = (t^n, \phi(t))$ at p such that $n = \nu_p(B)$, where $\nu_p(B)$ denotes the algebraic multiplicity of B. The *tangency index of* $\mathcal{F}$ *along* B *at* p (or *weak index* in Fernández-Pérez and Mol 2019, p. 1114) is

$$\operatorname{Ind}_p(\mathcal{F}, B) := \operatorname{ord}_t Q(\gamma(t)).$$

The tangency index $\operatorname{Ind}_p(\mathcal{F}, B)$ does not depend on the chosen parametrization of B because by properties of the multiplicity number we get the equality $\operatorname{ord}_t Q(\gamma(t)) = i_p(Q, B)$. The foliation $\mathcal{F}$ given by the 1-form defined in (2.4) verifies $\operatorname{Ind}_p(\mathcal{F}, B) = k + 1 > 1$, where $B : \{y = 0\}$.

The tangency index was defined in Camacho et al. (1984, p. 159), where the authors denomine it *multiplicity of* $\mathcal{F}$ *along* B *at* p and denoted by $\mu_{\mathcal{F}}(B, p)$. In the same paper the authors define a similar notion, the index with respect to a vector field Z (see p. 152) and denote it by $Ind_p(Z/B)$ which coincides with the multiplicity of $\mathcal{F}$ along B at p that we introduce in (4.7) and we denote by $\mu_p(\mathcal{F}, B)$. The reader should pay attention to it to avoid confusion. We adopt the notation given by Genzmer (2007), instead of the original given in Camacho et al. (1984) since $\mu_p(\mathcal{F}, B)$ resembles a Milnor number, which will be studied in Sect. 4.

Given a component $D \subset \mathcal{D}$, we denote by $\nu(D)$ its multiplicity, which coincides with the algebraic multiplicity of a curve E at $(\mathbb{C}^2, p)$ whose strict transform $\pi^* E$ meets D transversally outside a corner of $\mathcal{D}$. The following invariant is a measure of the existence of tangent saddle-nodes in the reduction process of singularities of a foliation:

Definition 2.3 The *tangency excess* of the foliation $\mathcal{F}$ is defined as $\xi_p(\mathcal{F}) = 0$, when p is a reduced singularity, and, in the non-reduced case, as the number

$$\xi_p(\mathcal{F}) = \sum_{q \in SN(\mathcal{F})} \nu(D_q)(\operatorname{Ind}_q(\tilde{\mathcal{F}}, \tilde{B}) - 1),$$

where $SN(\mathcal{F})$ stands for the set of tangent saddle-nodes on $\mathcal{D}$, $\tilde{B}$ is the weak separatrix passing by $q \in SN(\mathcal{F})$, and D_q is the component of $\mathcal{D}$ containing $\tilde{B}$. By (2.4), we observe that $\operatorname{Ind}_q(\tilde{\mathcal{F}}, \tilde{B}) = k + 1 > 1$.

Remark that $\xi_p(\mathcal{F}) \geq 0$ and, by definition, $\xi_p(\mathcal{F}) = 0$ if and only if $SN(\mathcal{F}) = \emptyset$, that is, if and only if $\mathcal{F}$ is of second type. In several papers (see for example Fernández-Pérez and Mol 2019; Cabrera and Mol 2022) the tangency excess of $\mathcal{F}$ is denoted by $\tau_p(\mathcal{F})$. In this paper, we denote it by $\xi_p(\mathcal{F})$ since we keep the letter τ for the Tjurina number of a curve or a foliation.

The following proposition proved by Genzmer (see Genzmer 2007, Proposition 2.4) will be very useful in this paper:

Proposition 2.4 *Let* $\mathcal{F}$ *be a singular foliation at* $(\mathbb{C}^2, p)$ *and* $\mathcal{B}$ *a balanced divisor of separatrices for* $\mathcal{F}$. *Denote by* $\nu_p(\mathcal{F})$ *and* $\nu_p(\mathcal{B})$ *their algebraic multiplicities respectively. Then*

$$\nu_p(\mathcal{F}) = \nu_p(\mathcal{B}) - 1 + \xi_p(\mathcal{F}). \tag{2.6}$$

Therefore,

$$\nu_p(\mathcal{F}) = \nu_p(\mathcal{B}) - 1$$

if, and only if, $\mathcal{F}$ *is a foliation of second type.*

Take a primitive parametrization $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, p)$, $\gamma(t) = (x(t), y(t))$, of a formal irreducible curve $B : f(x, y) = 0$ at $(\mathbb{C}^2, p)$. Note that B is a separatrix of the foliation $\mathcal{F} : \omega = 0$ if and only if $\gamma^*(\omega) = 0$. If B is not an $\mathcal{F}$-invariant curve, we define the *tangency order* of $\mathcal{F}$ along B at p as

$$\mathsf{tang}_p(\mathcal{F}, B) = \mathsf{ord}_t a(t), \tag{2.7}$$

where $\gamma^*(\omega) = a(t)dt$. The tangency order does not depend on the choosen parametrization of B, since $\mathsf{tang}_p(\mathcal{F}, B) + \mu_p(B) = i_p(B, v(f))$ where v is from (2.2). The tangency index $i_p(B, v(f))$ was introduced in Brunella (2010, p. 22).

The behavior under blow-up of the tangency order, in the non-dicritical case, was studied in Cano et al. (2019, equality (4)). The dicritical case is similar. Indeed, if $\mathcal{F} : \omega = 0$ is a singular foliation at $(\mathbb{C}^2, p)$, $\tilde{\mathcal{F}} : \tilde{\omega} = 0$ is its strict transform by the blow-up σ at p and B is not an $\mathcal{F}$-invariant curve then we have

$$\tilde{\omega} = \begin{cases} x^{-\nu_p(\mathcal{F})}\sigma^*(\omega) & \text{if } \sigma \text{ is non-dicritical;} \\ x^{-(\nu_p(\mathcal{F})+1)}\sigma^*(\omega) & \text{if } \sigma \text{ is dicritical.} \end{cases}$$

Evaluating $\tilde{\omega}$ in a parametrization of the strict transform (by σ) $\tilde{B}$ of B and taking orders we get

$$\mathsf{tang}_p(\mathcal{F}, B) = \begin{cases} \nu_p(\mathcal{F})\nu_p(B) + \mathsf{tang}_q(\tilde{\mathcal{F}}, \tilde{B}) & \text{if } \sigma \text{ is non-dicritical;} \\ (\nu_p(\mathcal{F}) + 1)\nu_p(B) + \mathsf{tang}_q(\tilde{\mathcal{F}}, \tilde{B}) & \text{if } \sigma \text{ is dicritical,} \end{cases} \tag{2.8}$$

where $q \in \tilde{B} \cap \sigma^{-1}(p)$.

In Cano et al. (2019, Corollary 1) it was stablished that $i_p(\mathcal{B}, B) \leq \mathsf{tang}_p(\mathcal{F}, B)+1$ and the equality holds if and only if $\mathcal{F}$ is of second type. In Cabrera and Mol (2022, Lemma 4.2), the authors improved (Cano et al. 2019, Corollary 1) as follows:

$$i_p(\mathcal{B}, B) = \mathsf{tang}_p(\mathcal{F}, B) - \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(B)\xi_q(\mathcal{F}) + 1,$$

where $\mathcal{F}$ is a singular foliation at $(\mathbb{C}^2, p)$, $\mathcal{B}$ is a balanced divisor of separatrices for $\mathcal{F}$ and B is a branch which is not $\mathcal{F}$-invariant. A proof, similar to the one given in Cabrera and Mol (2022), holds for formal and dicritical foliations.

3 The χ-Number of a Foliation

For a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ we introduce a new number

$$\chi_p(\mathcal{F}) := \left(\sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F}) \right) - \xi_p(\mathcal{F}).$$

Observe that

$$\chi_p(\mathcal{F}) = \sum_{q \in \mathcal{I}_p(\mathcal{F})\setminus\{p\}} \nu_q(\mathcal{F})\xi_q(\mathcal{F}) + (\nu_p(\mathcal{F}) - 1)\xi_p(\mathcal{F}). \tag{3.1}$$

In Mol and Rosas (2019, Proposition 9.5) the authors prove that the tangency excess is a C^∞ invariant, and after Rosas (2010) the algebraic multiplicity of a holomorphic foliation is a C^1 invariant. Hence the χ-number of a holomorphic foliation is a C^1 invariant. This invariant has the following properties:

Proposition 3.1 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$, then we get:*

(1) $\chi_p(\mathcal{F}) \geq 0$;
(2) *if $\mathcal{F}$ is of second type then $\chi_p(\mathcal{F}) = 0$;*
(3) *if $\chi_p(\mathcal{F}) = 0$, then either $\mathcal{F}$ has algebraic multiplicity 1 at p or $\mathcal{F}$ is of second type;*
(4) *if $\nu_p(\mathcal{F}) > 1$, then $\chi_p(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is of second type.*

Proof By (3.1) we have $\chi_p(\mathcal{F}) = \beta + (\nu_p(\mathcal{F}) - 1)\xi_p(\mathcal{F})$, where $\beta = \sum_{q \in \mathcal{I}_p(\mathcal{F})\setminus\{p\}} \nu_q(\mathcal{F})\xi_q(\mathcal{F})$. Clearly, $\chi_p(\mathcal{F}) \geq 0$, since it is the sum of two nonnegative numbers. Now, if $\mathcal{F}$ is of second type, we get $\xi_q(\mathcal{F}) = 0$, for all $q \in \mathcal{I}_p(\mathcal{F})$, which implies that $\chi_p(\mathcal{F}) = 0$. On the other hand, if $\chi_p(\mathcal{F}) = 0$ then $\beta = 0$ and $(\nu_p(\mathcal{F}) - 1)\xi_p(\mathcal{F}) = 0$. This finishes the proof of (3). Item (4) is an immediate consequence of (2) and (3). □

Remark 3.2 Let $\omega = 4xydx + (y - 2x^2)dy$ be a 1-form. Observe that the foliation $\mathcal{F} : \omega = 0$ at $(\mathbb{C}^2, 0)$ is not of second type, its algebraic multiplicity is one but $\chi_0(\mathcal{F}) = 1 \neq 0$.

4 Polar Intersection, Polar Excess and Milnor Numbers

Let $\omega = P(x, y)dx + Q(x, y)dy$ be a 1-form, where $P(x, y), Q(x, y) \in \mathbb{C}[[x, y]]$. If $\mathcal{F} : \omega = 0$ is a singular (analytic or formal) foliation then *the polar curve* of $\mathcal{F}$ at $(\mathbb{C}^2, p)$ with respect to a point $(a : b)$ of the complex projective line $\mathbb{P}^1(\mathbb{C})$ is the (analytic or formal) curve $\mathcal{P}^{\mathcal{F}}_{(a:b)} : aP(x, y) + bQ(x, y) = 0$. Observe that when $\mathcal{F}$ is the hamiltonian foliation associated to a function f, the polar curve of $\mathcal{F}$ coincides with the classical polar curve of f in the direction $(a : b)$ studied by Teissier (1977) and others. According to the general results on equisingularity (see Zariski 1970; Teissier 1975), there exists a Zariski open U of the space $\mathbb{P}^1(\mathbb{C})$ of projection directions such that for $(a : b)$ the polar curves are all equisingular. Any element of this set is called *generic polar curve* of the foliation $\mathcal{F}$ and we will denote it by $\mathcal{P}^{\mathcal{F}}$.

We borrow from (Genzmer and Mol 2018, Sect. 4) the notion of polar curve of a meromorphic 1-form: let $\eta = \frac{\omega}{H(x,y)}$ be a meromorphic 1-form, where $\omega = P(x, y)dx + Q(x, y)dy$ with $P(x, y), Q(x, y), H(x, y) \in \mathbb{C}[[x, y]]$. The *polar curve* of η at $(\mathbb{C}^2, p)$ with respect to $(a : b) \in \mathbb{P}^1(\mathbb{C})$ is the divisor $\mathcal{P}^{\eta}_{(a:b)}$ with formal meromorphic equation

$$\frac{aP(x, y) + bQ(x, y)}{H} = 0.$$

A polar curve $\frac{aP(x,y)+bQ(x,y)}{H} = 0$ of a meromorphic 1-form $\frac{\omega}{H(x,y)}$ will be generic if the polar curve $aP(x, y) + bQ(x, y) = 0$ is a generic polar curve of the foliation defined by the 1-form ω.

Let $B : h(x, y) = 0$ be a separatrix of a singular foliation $\mathcal{F}$. The *polar intersection number* of $\mathcal{F}$ with respect to B is the intersection number $i_p(\mathcal{P}^{\mathcal{F}}, B)$.

Lemma 4.1 *Let $B : h(x, y) = 0$ be a separatrix of a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ and consider F_B and G_B two balanced divisors of separatrices for $\mathcal{F}$ adapted to B. Then*

$$i_p(\mathcal{P}^{dF_B}, B) = i_p(\mathcal{P}^{dG_B}, B).$$

Proof Put $F_B = \frac{h\cdot\hat{f}\cdot g_1\cdots g_l}{\phi_1\cdots\phi_m}$ and $G_B = \frac{h\cdot\hat{f}\cdot h_1\cdots h_s}{\psi_1\cdots\psi_r}$, where $g_i(x, y) = 0$, $h_i(x, y) = 0$, $\phi_i(x, y) = 0$ and $\psi_i(x, y) = 0$ are dicritical separatrices of $\mathcal{F}$, $\hat{f}(x, y) = 0$ defines the reduced curve which is the union of all isolated separatrices of $\mathcal{F}$ except perhaps $h(x, y) = 0$ when this is also isolated. We get

$$dF_B = \frac{\phi(\mathcal{P}\cdot h + \hat{f}\cdot g_1\cdots g_l\cdot dh) - h\cdot\hat{f}\cdot g_1\cdots g_l d(\phi)}{\phi^2}$$

and

$$dG_B = \frac{\psi(\mathcal{Q}\cdot h + \hat{f}\cdot h_1\cdots h_s\cdot dh) - h\cdot\hat{f}\cdot h_1\cdots h_s d(\psi)}{\psi^2},$$

where $\phi = \phi_1\cdots\phi_m$, $\psi = \psi_1\cdots\psi_r$, $\mathcal{P} = d(\hat{f}\cdot g_1\cdots g_l) =: \mathcal{P}_1 dx + \mathcal{P}_2 dy$ and $\mathcal{Q} = d(\hat{f}\cdot h_1\cdots h_s) =: \mathcal{Q}_1 dx + \mathcal{Q}_2 dy$. Put $u = \hat{f}\cdot g_1\cdots g_l$ and $v = \hat{f}\cdot h_1\cdots h_s$. Hence

$$\mathcal{P}^{d\hat{F}_B} = \frac{[\phi(\mathcal{P}_1\cdot h + u\cdot\partial_x h) - hu\cdot\partial_x\phi]a + [\phi(\mathcal{P}_2\cdot h + u\cdot\partial_y h) - hu\cdot\partial_y\phi]b}{\phi^2},$$

and

$$\mathcal{P}^{d\hat{G}_B} = \frac{[\psi(\mathcal{Q}_1\cdot h + v\cdot\partial_x h) - hv\cdot\partial_x\psi]a + [\psi(\mathcal{Q}_2\cdot h + v\cdot\partial_y h) - hv\cdot\partial_y\psi]b}{\psi^2}.$$

So

$$\begin{aligned} i_p(\mathcal{P}^{d\hat{F}_B}, B) &= i_p\left(\hat{f}\cdot g_1\cdots g_l\cdot\left(a\partial_x h + b\partial_y h\right), h\right) - i_p(\phi_1\cdots\phi_m, h) \\ &= i_p\left(\hat{f}\cdot\left(a\partial_x h + b\partial_y h\right), h\right) + i_p(g_1\cdots g_l, h) - i_p(\phi_1\cdots\phi_m, h), \end{aligned} \tag{4.1}$$

and

$$i_p(\mathcal{P}^{d\hat{G}_B}, B) = i_p\left(\hat{f}\cdot h_1\cdots h_s\cdot\left(a\partial_x h + b\partial_y h\right), h\right) - i_p(\psi_1\cdots\psi_r, h)$$
$$= i_p\left(\hat{f}\cdot\left(a\partial_x h + b\partial_y h\right), h\right) + i_p(h_1\cdots h_s, h) - i_p(\psi_1\cdots\psi_r, h). \tag{4.2}$$

We claim that $i_p(g_1\cdots g_l, h) - i_p(\phi_1\cdots\phi_m, h) = i_p(h_1\cdots h_s, h) - i_p(\psi_1\cdots\psi_r, h)$. Indeed, if every dicritical separatrix of $\mathcal{F}$ is smooth and transversal to any isolated separatrix, then using properties of the intersection multiplicity we have

$$\begin{aligned} i_p(g_1\cdots g_l, h) - i_p(\phi_1\cdots\phi_m, h) &= \nu_p(h)\left[\sum_{j=1}^{l}\nu_p(g_j) - \sum_{j=1}^{m}\nu_p(\phi_j)\right] \\ &= \nu_p(h)\left[\sum_{j=1}^{s}\nu_p(h_j) - \sum_{j=1}^{r}\nu_p(\psi_j)\right] \\ &= i_p(h_1\cdots h_s, h) - i_p(\psi_1\cdots\psi_r, h), \end{aligned} \tag{4.3}$$

where the equality (4.3) holds since F_B and G_B are two balanced divisors of separatrices for $\mathcal{F}$.

In the general case, after the reduction of singularities of the foliation we can suppose that the strict transform of every dicritical separatrix of $\mathcal{F}$ is smooth and transversal to any strict transform of every isolated separatrix. We finish the proof using Noether formula. □

Lemma 4.1 allows us to define the *polar excess number* of a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ with respect to a separatrix B of $\mathcal{F}$ as

$$\Delta_p(\mathcal{F}, B) := i_p(\mathcal{P}^{\mathcal{F}}, B) - i_p(\mathcal{P}^{dF_B}, B), \tag{4.4}$$

where F_B is any balanced divisor of separatrices for $\mathcal{F}$ adapted to B. On the other hand, if the foliation $\mathcal{F}$ is non-dicritical then it is enough to consider F_B as the total union of the separatrices of $\mathcal{F}$.

Using properties of the intersection number we extend the definitions of polar intersection and polar excess numbers to any divisor $\mathcal{B} := \sum_B a_B B$ of separatrices of $\mathcal{F}$ in the following way:

$$i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}) = \sum_B a_B i_p(\mathcal{P}^{\mathcal{F}}, B)$$

and

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B \Delta_p(\mathcal{F}, B) = \Delta_p(\mathcal{F}, \mathcal{B}_0) - \Delta_p(\mathcal{F}, \mathcal{B}_\infty). \tag{4.5}$$

If $\mathcal{B}$ is a primitive divisor then the difference $\Delta_p(\mathcal{F}, \mathcal{B}_0) - \Delta_p(\mathcal{F}, \mathcal{B}_\infty)$ is independent of the choosen primitive balanced divisor of separatrices for $\mathcal{F}$ (see Fernández-Pérez and Mol 2019, Sect. 3.6, p. 1123).

By Genzmer and Mol (2018, Proposition 4.6), $\Delta_p(\mathcal{F}, B)$ is a non-negative integer number for any irreducible component B of $\mathcal{B}_0$. As a consequence $\Delta_p(\mathcal{F}, \mathcal{B})$ is also a non-negative integer number for any effective divisor of separatrices $\mathcal{B}$.

The *Milnor number* $\mu_p(\mathcal{F})$ of the foliation $\mathcal{F}$ at p given by the 1-form $\omega = P(x, y)dx + Q(x, y)dy$ is defined by

$$\mu_p(\mathcal{F}) = i_p(P, Q).$$

Remember that we consider P and Q coprime, so $\mu_p(\mathcal{F})$ is a non negative integer. In Camachoet al. (1984, Theorem A) it was proved that the Milnor number of a foliation is a topological invariant.

On the other hand, the *Milnor number* $\mu_p(C)$ at p of a plane curve C (non necessary irreducible) with equation $f(x, y) = 0$ is

$$\mu_p(C) = i_p\left(\partial_x f, \partial_y f\right).$$

Observe that $\mu_p(C)$ is finite if and only if f has not multiple factors, that is, the curve C is reduced.

Generalized curve foliations have a property of minimization of Milnor numbers and are characterized, in the non-dicritical case, by several authors, see for instance Brunella (1997, Proposition 7) and Cavalier and Lehmann (2001, Théorème 3.3). Recently, in Genzmer and Mol (2018, Theorem A), the authors have characterized singular generalized curve foliations in terms of its polar excess number as follows: let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a balanced divisor of separatrices for $\mathcal{F}$, then $\Delta_p(\mathcal{F}, \mathcal{B}_0) = 0$ if and only if $\mathcal{F}$ is a generalized curve foliation.

The following proposition provides a formula to compute the polar excess number of a foliation with respect to the zero divisor of a reduced balanced divisor of separatrices.

Proposition 4.2 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a reduced balanced divisor of separatrices for $\mathcal{F}$. Then*

$$\Delta_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) - \mu_p(\mathcal{B}_0) - \nu_p(\mathcal{B}_0) + 1.$$

Moreover, $\mathcal{F}$ is a generalized curve foliation if and only if

$$i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) = \mu_p(\mathcal{B}_0) + \nu_p(\mathcal{B}_0) - i_p(\mathcal{B}_0, \mathcal{B}_\infty) - 1.$$

Proof Let $\omega = P(x, y)dx + Q(x, y)dy$ be a 1-form inducing $\mathcal{F}$ and $f(x, y) = 0$ and $g(x, y) = 0$ be the reduced equations of $\mathcal{B}_0$ and $\mathcal{B}_\infty$ respectively. Since $\mathcal{B}$ is a set of separatrices of $\mathcal{F}$ adapted to $\mathcal{B}_0$, by (4.4) and the definition of the polar curve of $d(f/g)$ we have

$$\begin{aligned}\Delta_p(\mathcal{F}, \mathcal{B}_0) &= i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) - i_p(\mathcal{P}^{d(f/g)}, \mathcal{B}_0)\\ &= i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) - i_p\left(\frac{(g\partial_x f - f\partial_x g)a + (g\partial_y f - f\partial_y g)b}{g^2}, f\right)\end{aligned}$$

$$= i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) - i_p\left(\frac{g(a\partial_x f + b\partial_y f) - f(a\partial_x g + b\partial_y g)}{g^2}, f\right)$$

where $(a : b) \in \mathbb{P}^1$. By applying properties on intersection numbers and Teissier's Proposition (Teissier 1973, Chap. II, Proposition 1.2), we get

$$\begin{aligned}\Delta_p(\mathcal{F}, \mathcal{B}_0) &= i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) - i_p(a\partial_x f + b\partial_y f, f) + i_p(g, f)\\ &= i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) - \mu_p(\mathcal{B}_0) - \nu_p(\mathcal{B}_0) + 1.\end{aligned}$$

The second part of the proposition follows from the first part and the characterization of generalized curve foliations given in Genzmer and Mol (2018, Theorem A). □

The second part of Proposition 4.2 generalizes (Cano et al. 2019, Proposition 2) as it does not have the restriction of non-dicriticality.

We give a numerical illustration of Proposition 4.2.

Example 4.3 Let $\mathcal{F}$ be the foliation at $(\mathbb{C}^2, 0)$ defined by $\omega = xdy - ydx$. Observe that $\mathcal{F}$ is a dicritical generalized curve foliation called the *radial foliation*. It has only one dicritical component whose valence is 0 in its reduction process of singularities. Thus $\mathcal{B} = (x) + (y) + (x - y) - (x + y)$ is a reduced balanced divisor of separatrices for $\mathcal{F}$, where $\mathcal{B}_0 : xy(x - y) = 0$ and $\mathcal{B}_\infty : x + y = 0$. We get $\mu_0(\mathcal{B}_0) = 4$, $\nu_0(\mathcal{B}_0) = 3$, $i_0(\mathcal{B}_0, \mathcal{B}_\infty) = 3$ and $i_0(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) = 3$. Observe that if we consider the reduced balanced divisor of separatrices $\mathcal{B} = (x) + (y)$ for $\mathcal{F}$ then $\mathcal{B}_\infty$ is a unit, so $i_p(\mathcal{B}_0, \mathcal{B}_\infty) = 0$ and now $\mu_0(\mathcal{B}_0) = 1$, $\nu_0(\mathcal{B}_0) = 2$. On the other hand, if we consider the foliation $\mathcal{F}$ with a saddle-node (so $\mathcal{F}$ is not a generalized curve foliation) and equation as in (2.4) we get again $\mathcal{B} = (x) + (y)$ but $i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) = k + 2 \neq 2$.

Lemma 4.4 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B}$ be a balanced divisor of separatrices for $\mathcal{F}$. Then*

$$i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}) = \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}) - \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F}),$$

where the summation runs over all infinitely near points of $\mathcal{F}$ at p.

Proof Let $\mathcal{P}^{\mathcal{F}}$ be a generic polar. Denote by $\Gamma(\mathcal{P}^{\mathcal{F}})$ the set of irreducible components of $\mathcal{P}^{\mathcal{F}}$. It follows from (Cabrera and Mol 2022, Lemma 4.2) that

$$\begin{aligned}i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}) &= \sum_{A \in \Gamma(\mathcal{P}^{\mathcal{F}})} i_p(A, \mathcal{B})\\ &= \sum_{A \in \Gamma(\mathcal{P}^{\mathcal{F}})} \left(\mathsf{tang}_p(\mathcal{F}, A) - \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(A)\xi_q(\mathcal{F}) + 1\right)\\ &= \sum_{A \in \Gamma(\mathcal{P}^{\mathcal{F}})} (\mathsf{tang}_p(\mathcal{F}, A) + 1) - \sum_{A \in \Gamma(\mathcal{P}^{\mathcal{F}})} \left(\sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(A)\xi_q(\mathcal{F})\right)\end{aligned}$$

$$= \sum_{A\in\Gamma(\mathcal{P}^{\mathcal{F}})} (\mathsf{tang}_p(\mathcal{F}, A) + 1) - \sum_{q\in\mathcal{I}_p(\mathcal{F})} \left(\sum_{A\in\Gamma(\mathcal{P}^{\mathcal{F}})} \nu_q(A) \right) \xi_q(\mathcal{F})$$
$$= \sum_{A\in\Gamma(\mathcal{P}^{\mathcal{F}})} (\mathsf{tang}_p(\mathcal{F}, A) + 1) - \sum_{q\in\mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{P}^{\mathcal{F}})\xi_q(\mathcal{F}). \tag{4.6}$$

According to the proof of Cano et al. (2019, Proposition 2), we have

$$\sum_{A\in\Gamma(\mathcal{P}^{\mathcal{F}})} (\mathsf{tang}_p(\mathcal{F}, A) + 1) = \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}),$$

and by Cano et al. (2019, Remark 1) $\nu_q(\mathcal{P}^{\mathcal{F}}) = \nu_q(\mathcal{F})$. Substituting these terms in the Eq. (5.5), we obtain

$$i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}) = \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}) - \sum_{q\in\mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F}).$$

□

Lemma 4.4 improves (Cano et al. 2019, Proposition 2) determining explicitly the difference between the polar intersection number with respect to a balanced divisor of separatrices $\mathcal{B}$ and the sum of the Milnor number and the algebraic multiplicity of the foliation $\mathcal{F}$. It also generalizes the result to dicritical foliations. On the other hand comparing Lemma 4.4 and Cabrera and Mol (2022, Proposition 4.3) (proved for complex analytic foliations, but it also holds for formal foliations) we conclude that

$$\sum_{q\in\mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F}) = \sum_{q\in\mathcal{I}_p(\mathcal{F})} \nu_q(B)\xi_q(\mathcal{F}),$$

for any B which is not an $\mathcal{F}$-invariant curve. Hence the sum $\sum_{q\in\mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F})$ coincides with the *tangency excess of* $\mathcal{F}$ *along any irreducible curve* which is not an $\mathcal{F}$-invariant curve, introduced in Cabrera and Mol (2022, equality (8)). In particular, after the definition of the χ-number, this tangency excess equals $\chi_p(\mathcal{F}) + \xi_p(\mathcal{F})$.

Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ induced by the vector field v and B be a separatrix of $\mathcal{F}$. Let $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, p)$ be a primitive parametrization of B, we can consider the *multiplicity of* $\mathcal{F}$ *along* B *at* p defined by

$$\mu_p(\mathcal{F}, B) = \mathsf{ord}_t\theta(t), \tag{4.7}$$

where $\theta(t)$ is the unique vector field at $(\mathbb{C}, 0)$ such that $\gamma_*\theta(t) = v \circ \gamma(t)$, see for instance Camacho et al. (1984, p. 159). If $\omega = P(x, y)dx + Q(x, y)dy$ is a 1-form inducing $\mathcal{F}$ and $\gamma(t) = (x(t), y(t))$, we get

$$\theta(t) = \begin{cases} -\frac{Q(\gamma(t))}{x'(t)} & \text{if } x(t) \neq 0 \\ \frac{P(\gamma(t))}{y'(t)} & \text{if } y(t) \neq 0. \end{cases} \tag{4.8}$$

Hence, by taking orders we obtain

$$\mu_p(\mathcal{F}, B) = \begin{cases} \mathrm{ord}_t\, Q(\gamma(t)) - \mathrm{ord}_t x(t) + 1 & \text{if } x(t) \neq 0; \\ \mathrm{ord}_t\, P(\gamma(t)) - \mathrm{ord}_t y(t) + 1 & \text{if } y(t) \neq 0. \end{cases} \tag{4.9}$$

The following proposition has been proved in Cano et al. (2019, Proposition 1) for non-dicritical foliations, but we may check that it is valid for dicritical foliations.

Proposition 4.5 *Consider a separatrix B of a singular (holomorphic or formal) foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$. We have*

$$i_p(\mathcal{P}^{\mathcal{F}}, B) = \mu_p(\mathcal{F}, B) + \nu_p(B) - 1.$$

Remark 4.6 Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C = \cup_{j=1}^{\ell} C_j$ is the total union of separatrices of $\mathcal{F}$. Applying Proposition 4.5 to $\mathcal{F}$ and df, where $C : f(x, y) = 0$, we get $\Delta(\mathcal{F}, C_j) = \mu_p(\mathcal{F}, C_j) - \mu_p(df, C_j)$, for $j = 1, \ldots, \ell$. Since $\Delta(\mathcal{F}, C_j) \geq 0$ we have $\mu_p(\mathcal{F}, C_j) \geq \mu_p(df, C_j)$ for any separatrix C_j of $\mathcal{F}$.

As a consequence of Lemma 4.4 and Proposition 4.5 we obtain a generalization of Cano et al. (2019, Corollary 2).

Proposition 4.7 *Let $\mathcal{F}$ be a singular (holomorphic or formal) foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor for separatrices of $\mathcal{F}$. We have*

$$\mu_p(\mathcal{F}) = \sum_B a_B \mu_p(\mathcal{F}, B) + \chi_p(\mathcal{F}) - \deg(\mathcal{B}) + 1.$$

Proof By summing up polar intersection numbers over all irreducible components of $\mathcal{B}$ and applying Proposition 4.5, we get

$$\begin{aligned} i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}) &= \sum_B a_B i_p(\mathcal{P}^{\mathcal{F}}, B) = \sum_B a_B (\mu_p(\mathcal{F}, B) + \nu_p(B) - 1) \\ &= \sum_B a_B \mu_p(\mathcal{F}, B) + \left(\sum_B a_B \nu_p(B) \right) - \deg(\mathcal{B}) \\ &= \sum_B a_B \mu_p(\mathcal{F}, B) + \nu_p(\mathcal{B}) - \deg(\mathcal{B}). \end{aligned}$$

From Lemma 4.4, Proposition 2.4 and the definition of the χ-number of $\mathcal{F}$ we get

$$\begin{aligned} \mu_p(\mathcal{F}) &= \sum_B a_B \mu_p(\mathcal{F}, B) + \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F}) \xi_q(\mathcal{F}) - \nu_p(\mathcal{F}) + \nu_p(\mathcal{B}) - \deg(\mathcal{B}) \\ &= \sum_B a_B \mu_p(\mathcal{F}, B) + \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F}) \xi_q(\mathcal{F}) - \xi_p(\mathcal{F}) - \deg(\mathcal{B}) + 1 \end{aligned}$$

$$= \sum_{B} a_B \mu_p(\mathcal{F}, B) + \chi_p(\mathcal{F}) - \deg(\mathcal{B}) + 1.$$

□

Let $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial, where the origin is an isolated singular point of the hypersurface $f^{-1}(0)$. The notion of the Milnor number $\mu(f)$ was introduced in Milnor (1969, Sect. 7) as the degree of the mapping $z \to \frac{\nabla(f)}{||\nabla(f)||}$, where ∇ denotes the gradient function. In particular, for complex plane curves, Milnor proved, using topological tools, the purely algebraic equality $\mu(f) = 2\delta(f) - r(f) + 1$, where $\delta(f)$ is the *number of double points* and $r(f)$ is the number of irreducible factors of f (see Milnor 1969, Theorem 10.5). The reader can find further formulae for the Milnor number of a plane curve in Wall (2004, Sect. 6.5). In particular in Wall (2004, Theorem 6.5.1) it was established the relationship between the Milnor number of a reduced plane curve and the Milnor numbers of its irreducible components. The ingredients of the proof of Wall are Milnor fibrations and the Euler characteristic. We give another proof of this relationship, using foliations:

Proposition 4.8 *Let $C : f(x, y) = 0$ be a germ of reduced singular curve at $(\mathbb{C}^2, p)$. Assume that $C = \cup_{j=1}^{\ell} C_j$ is the decomposition of C in irreducible components C_j : $f_j(x, y) = 0$, where $f(x, y) = f_1(x, y) \cdots f_\ell(x, y)$. Then*

$$\mu_p(C) + \ell - 1 = \sum_{j=1}^{\ell} \mu_p(C_j) + 2 \sum_{1 \le i < j \le \ell} i_p(C_i, C_j).$$

Proof Applying Proposition 4.7 to the foliation defined by $\omega = df$ and to the balanced divisor of separatrices $C = \sum_{j=1}^{\ell} C_j$ we have

$$\mu_p(C) + \ell - 1 = \sum_{j=1}^{\ell} \mu_p(df, C_j). \tag{4.10}$$

It follows from Proposition 4.5 that

$$\mu_p(df, C_j) = i_p(\mathcal{P}^{df}, C_j) - \nu_p(C_j) + 1, \quad \text{for } j = 1, \ldots, \ell.$$

Using properties on intersection numbers, we have

$$i_p(\mathcal{P}^{df}, C_j) = i_p(\mathcal{P}^{df_j}, C_j) + \sum_{i \ne j} i_p(C_i, C_j).$$

From Teissier's Proposition (Teissier 1973, Chap. II, Proposition 1.2), we get

$$i_p(\mathcal{P}^{df_j}, C_j) = \mu_p(C_j) + \nu_p(C_j) - 1.$$

Thus

$$\mu_p(df, C_j) = \mu_p(C_j) + \sum_{i \neq j} i_p(C_i, C_j). \tag{4.11}$$

The proof ends, by substituting (4.11) in (4.10). □

Theorem A *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then*

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}) - \sum_B a_B \mu_p(dF_B, B) + \deg(\mathcal{B}) - 1 - \chi_p(\mathcal{F}),$$

where F_B is a balanced divisor of separatrices for $\mathcal{F}$ adapted to B. Hence, if $\mathcal{F}$ is a foliation of second type, then

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}) - \sum_B a_B \mu_p(dF_B, B) + \deg(\mathcal{B}) - 1.$$

Proof By (4.5) and (4.4)

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B \Delta_p(\mathcal{F}, B) = \sum_B a_B \left(i_p(\mathcal{P}^{\mathcal{F}}, B) - i_p(\mathcal{P}^{dF_B}, B) \right)$$
$$= i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}) - \sum_B a_B i_p(\mathcal{P}^{dF_B}, B).$$

Hence, after Lemma 4.4 and Proposition 4.5 we have

$$\begin{aligned}
\Delta_p(\mathcal{F}, \mathcal{B}) &= \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}) - \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F}) \xi_q(\mathcal{F}) \\
&\quad - \sum_B a_B \left(\mu_p(dF_B, B) + \nu_p(B) - 1 \right) \\
&= \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}) - \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F}) \xi_q(\mathcal{F}) \\
&\quad - \left(\sum_B a_B \mu_p(dF_B, B) \right) - \nu_p(\mathcal{B}) + \deg(\mathcal{B}).
\end{aligned}$$

We finish the proof after Proposition 2.4 and the definition of the χ-number of $\mathcal{F}$. On the other hand if $\mathcal{F}$ is a foliation of second type then $\chi_p(\mathcal{F}) = 0$ and the second part of the theorem follows. □

From Theorem A and Proposition 4.7 we get:

Corollary 4.9 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then*

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B (\mu_p(\mathcal{F}, B) - \mu_p(dF_B, B)),$$

where F_B is a balanced divisor of separatrices for $\mathcal{F}$ adapted to B.

Corollary 4.9 restricted to non-dicritical singular foliations provides us a new characterization of non-dicritical generalized curve foliations.

Corollary 4.10 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C = \cup_{j=1}^{\ell} C_j$ is the total union of separatrices of $\mathcal{F}$. Then $\mathcal{F}$ is a generalized curve foliation if and only if*

$$\mu_p(\mathcal{F}, C_j) = \mu_p(df, C_j), \qquad \text{for all} \quad j = 1, \ldots, \ell,$$

where $f(x, y) = 0$ is a reduced equation of C at p.

Proof According to Genzmer and Mol (2018, Theorem A), $\mathcal{F}$ is a generalized curve foliation at $(\mathbb{C}^2, p)$ if and only if $\Delta_p(\mathcal{F}, C) = 0$, where C is the total union of separatrices of $\mathcal{F}$. It follows from Corollary 4.9 that $\Delta_p(\mathcal{F}, C) = \sum_{j=1}^{\ell}(\mu_p(\mathcal{F}, C_j) - \mu_p(df, C_j))$, where $f(x, y) = 0$ is a reduced equation of C at p. Thus, by Remark 4.6, $\Delta_p(\mathcal{F}, C) = 0$ if and only if $\mu_p(\mathcal{F}, C_j) = \mu_p(df, C_j)$ for all $j = 1, \ldots, \ell$. □

Remark 4.11 Observe that if, in Corollary 4.10, the curve $C : f(x, y) = 0$ is irreducible then we rediscover the classic characterization of generalized curve foliations, that is, $\mu_p(\mathcal{F}) = \mu_p(df) = \mu_p(C)$.

5 The Gómez-Mont–Seade–Verjovsky Index

Let $\mathcal{F} : \omega = 0$ be a singular foliation at $(\mathbb{C}^2, p)$. Let $C : f(x, y) = 0$ be an $\mathcal{F}$-invariant curve, where $f(x, y) \in \mathbb{C}[[x, y]]$ is reduced. Then, as in the convergent case, there are $g, h \in \mathbb{C}[[x, y]]$ (depending on f and ω), with f and g and f and h relatively prime and a 1-form η (see Suwa 1995, Lemma 1.1 and its proof) such that

$$g\omega = hdf + f\eta. \tag{5.1}$$

The *Gómez-Mont–Seade–Verjovsky index* of the foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ (GSV-index) with respect to an analytic $\mathcal{F}$-invariant curve C is

$$GSV_p(\mathcal{F}, C) = \frac{1}{2\pi i} \int_{\partial C} \frac{g}{h} d\left(\frac{h}{g}\right), \tag{5.2}$$

where $g, h \in \mathbb{C}\{x, y\}$ are from (5.1). This index was introduced in Gómez-Mont et al. (1991) but here we follow the presentation of Brunella (1997). If C is irreducible then equality (5.2) becomes

$$GSV_p(\mathcal{F}, C) = \text{ord}_t\left(\frac{h}{g} \circ \gamma\right)(t) = i_p(f, h) - i_p(f, g), \tag{5.3}$$

where $\gamma(t)$ is a Puiseux parametrization of C. The same formula appears in (Suwa 2014, Corollary 5.2). An interesting survey on indices and residues is Corrêa and Seade (2024). By Brunella (1997, p. 532), we get the adjunction formula

$$GSV_p(\mathcal{F}, C_1 \cup C_2) = GSV_p(\mathcal{F}, C_1) + GSV_p(\mathcal{F}, C_2) - 2i_p(C_1, C_2), \tag{5.4}$$

for any two analytic $\mathcal{F}$-invariant curves, C_1 and C_2, without common irreducible components.

The equality (5.3) allows us to extend the definition of the GSV-index to a purely formal (non-analytic) irreducible $\mathcal{F}$-invariant curve; and the equality (5.4) allows us to extend the definition of the GSV-index to $\mathcal{F}$-invariant curves containing purely formal branches.

The following lemma generalizes the equality (5.3) to any reduced $\mathcal{F}$-invariant curve (containing perhaps purely formal branches):

Lemma 5.1 *Let $C : f(x, y) = 0$ be any reduced invariant curve of a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$. Then*

$$GSV_p(\mathcal{F}, C) = i_p(f, h) - i_p(f, g),$$

where $g, h \in \mathbb{C}[[x, y]]$ are from (5.1).

Proof Suppose, without lost of generality, that $f(x, y) = f_1(x, y) f_2(x, y)$, where $f_1, f_2 \in \mathbb{C}[[x, y]]$ are irreducible and put $C_i : f_i(x, y) = 0$ for $1 \leq i \leq 2$. By (5.1) we get

$$g\omega = hf_2 df_1 + f_1(hdf_2 + f_2\eta),$$

for some $g, h \in \mathbb{C}[[x, y]]$ relative prime with f and a 1-form η. Hence, if $\gamma_1(t)$ is a Puiseux parametrization of C_1, then after (5.3) we have

$$\begin{aligned} GSV_p(\mathcal{F}, C_1) &= \mathrm{ord}_t \left(\frac{hf_2}{g} \circ \gamma_1 \right)(t) = \mathrm{ord}_t \left(\frac{h}{g} \circ \gamma_1 \right)(t) + \mathrm{ord}_t \left(f_2 \circ \gamma_1 \right)(t) \\ &= \mathrm{ord}_t \left(\frac{h}{g} \circ \gamma_1 \right)(t) + i_p(C_1, C_2). \end{aligned}$$

Similarly, if $\gamma_2(t)$ denotes a Puiseux parametrization of C_2 then we have

$$GSV_p(\mathcal{F}, C_2) = \mathrm{ord}_t \left(\frac{h}{g} \circ \gamma_2 \right)(t) + i_p(C_1, C_2).$$

The proof follows after equality (5.4) and properties of the intersection number. □

In this section, we will use the following result due to Genzmer–Mol (Genzmer 2007, Theorem B) that establishes a relationship between the GSV-index and the polar excess number of a foliation with respect to a set of separatrices.

Theorem 5.2 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Let C be a reduced curve of separatrices and $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a balanced divisor of separatrices for $\mathcal{F}$ adapted* *to C. Then*

$$GSV_p(\mathcal{F}, C) = \Delta_p(\mathcal{F}, C) + i_p(C, \mathcal{B}_0 \backslash C) - i_p(C, \mathcal{B}_\infty).$$

We note that the above theorem implies that if $\mathcal{B}$ is an effective balanced divisor of separatrices for $\mathcal{F}$, then $GSV_p(\mathcal{F}, \mathcal{B}) = \Delta_p(\mathcal{F}, \mathcal{B})$.

On the other hand, as a consequence of Proposition 4.2 and Theorem 5.2 we get

Corollary 5.3 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Let $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a reduced balanced divisor of separatrices for $\mathcal{F}$. Then*

$$GSV_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) - \mu_p(\mathcal{B}_0) - \nu_p(\mathcal{B}_0) + 1.$$

Proof By applying Theorem 5.2 to $C = \mathcal{B}_0$, we have

$$GSV_p(\mathcal{F}, \mathcal{B}_0) = \Delta_p(\mathcal{F}, \mathcal{B}_0) - i_p(\mathcal{B}_0, \mathcal{B}_\infty), \tag{5.5}$$

and it follows from Proposition 4.2 that

$$\Delta_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{P}^{\mathcal{F}}, \mathcal{B}_0) + i_p(\mathcal{B}_0, \mathcal{B}_\infty) - \mu_p(\mathcal{B}_0) - \nu_p(\mathcal{B}_0) + 1. \tag{5.6}$$

The proof ends by substituting (5.6) in (5.5). □

The following proposition holds for an arbitrary foliation and any subset of separatrices and is not restricted only to convergent separatrices as in Cano et al. (2019, Proposition 4). The proof is similar, and is written for the reader's understanding.

Proposition 5.4 *Let $C : f(x, y) = 0$ be any reduced invariant curve of a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$. Then*

$$GSV_p(\mathcal{F}, C) = i_p(\mathcal{P}^{\mathcal{F}}, C) - i_p(\mathcal{P}^{df}, C).$$

Proof Let $\omega = P(x, y)dx + Q(x, y)dy$ be a 1-form inducing $\mathcal{F}$. By equality (5.1) we get $g\omega = hdf + f\eta$, where η is a formal 1-form and $g, h \in \mathbb{C}[[x, y]]$ with g and f relatively prime and h and f also relatively prime. Let $(a : b) \in \mathbb{P}^1$ such that the polar curves $aP(x, y) + bQ(x, y) = 0$ and $a\partial_x f + b\partial_y f = 0$ of $\mathcal{F}$ and df respectively, are generic. We have $g \cdot (aP + bQ) = h \cdot (a\partial_x f + b\partial_y f) + fk$, for some $k \in \mathbb{C}[[x, y]]$. Then $i_p(f, g \cdot (aP + bQ)) = i_p(f, h \cdot (a\partial_x f + b\partial_y f) + fk) = i_p(f, h \cdot (a\partial_x f + b\partial_y f))$. So $i_p(f, g \cdot (aP + bQ)) = i_p(f, h) + i_p(f, a\partial_x f + b\partial_y f)$.
On the other hand $i_p(f, g \cdot (aP + bQ)) = i_p(f, g) + i_p(f, aP + bQ)$, hence

$$i_p(f, g) + i_p(f, aP + bQ) = i_p(f, h) + i_p(f, a\partial_x f + b\partial_y f). \tag{5.7}$$

We finish the proof using Lemma 5.1 and equality (5.7). □

Remark 5.5 If $\mathcal{F}$ is a non-dicritical singular foliation at $(\mathbb{C}^2, p)$, where C is the total union of separatrices of $\mathcal{F}$ then $i_p(\mathcal{P}^{\mathcal{F}}, f) - i_p(\mathcal{P}^{df}, f) = \Delta_p(\mathcal{F}, C)$, but in general these two values are different as the following example shows: consider the foliation $\mathcal{F}$ defined by $\omega = 2xdy - 3ydx$ and the curve $C : y^2 - x^3 = 0$. Note that $\mathcal{F}$ admits the meromorphic first integral y^2/x^3 and so that C is $\mathcal{F}$-invariant. We get $i_0(\mathcal{P}^{\mathcal{F}}, C) - i_0(\mathcal{P}^{df}, C) = -1$, and $\Delta_0(\mathcal{F}, C) = 0$, since $\mathcal{F}$ is a generalized curve foliation.

We obtain the following corollary.

Corollary 5.6 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and C is the total union of separatrices of $\mathcal{F}$. Then*

$$GSV_p(\mathcal{F}, C) = \mu_p(\mathcal{F}) - \mu_p(C) - \chi_p(\mathcal{F}).$$

Proof By Proposition 5.4 we have $GSV_p(\mathcal{F}, C) = i_p(\mathcal{P}^{\mathcal{F}}, C) - i_p(\mathcal{P}^{df}, C)$ and applying Lemma 4.4 to $\mathcal{F}$ and C, we get $i_p(\mathcal{P}^{\mathcal{F}}, C) = \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}) - \sum_{q\in\mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F})$. Teissier's Proposition (Teissier 1973, Chap. II, Proposition 1.2) implies that

$$i_p(\mathcal{P}^{df}, C) = \mu_p(C) + \nu_p(C) - 1.$$

Thus

$$\begin{aligned}
GSV_p(\mathcal{F}, C) &= i_p(\mathcal{P}^{\mathcal{F}}, C) - i_p(\mathcal{P}^{df}, C)\\
&= \mu_p(\mathcal{F}) + \nu_p(\mathcal{F}) - \sum_{q\in\mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F}) - (\mu_p(C) + \nu_p(C) - 1)\\
&= \mu_p(\mathcal{F}) - \mu_p(C) + \underbrace{(\nu_p(\mathcal{F}) - \nu_p(C) + 1)}_{\xi_p(\mathcal{F})} - \sum_{q\in\mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F})\\
&= \mu_p(\mathcal{F}) - \mu_p(C) - \left(\underbrace{\sum_{q\in\mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F})\xi_q(\mathcal{F}) - \xi_p(\mathcal{F})}_{\chi_p(\mathcal{F})}\right)\\
&= \mu_p(\mathcal{F}) - \mu_p(C) - \chi_p(\mathcal{F}).
\end{aligned}$$

□

To finish this section we state a relationship between the GSV-index and the multiplicity of $\mathcal{F}$ along a fixed separatrix.

Proposition 5.7 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and $B : f(x, y) = 0$ be a separatrix of $\mathcal{F}$. Then*

$$\mu_p(\mathcal{F}, B) = \begin{cases} GSV_p(\mathcal{F}, B) + \mathrm{ord}_t \partial_y f(\gamma(t)) - \mathrm{ord}_t x(t) + 1 & \text{if } x(t) \neq 0;\\ GSV_p(\mathcal{F}, B) + \mathrm{ord}_t \partial_x f(\gamma(t)) - \mathrm{ord}_t y(t) + 1 & \text{if } y(t) \neq 0, \end{cases} \quad (5.8)$$

where $\gamma(t) = (x(t), y(t))$ is a Puiseux parametrization of B. In particular, if B is a non-singular separatrix, then $\mu_p(\mathcal{F}, B) = GSV_p(\mathcal{F}, B)$.

Proof Let $\omega = P(x, y)dx + Q(x, y)dy$ be a 1-form inducing $\mathcal{F}$ and $f(x, y) = 0$ be a reduced equation of B. By equality (5.1) we get

$$g\omega = hdf + f\eta,$$

where η is a formal 1-form and $g, h \in \mathbb{C}[[x, y]]$, where g and f are relatively prime and h and f are relatively prime. From equality (4.8), we have that the unique vector field $\theta(t)$ such that $\gamma_*\theta(t) = v(\gamma(t))$, where $v = -Q(x, y)\frac{\partial}{\partial x} + P(x, y)\frac{\partial}{\partial y}$, is given by

$$\theta(t) = \begin{cases} \dfrac{-\left(\frac{h}{g}\right)(\gamma(t))\partial_y f(\gamma(t))}{x'(t)} & \text{if } x(t) \neq 0; \\ \dfrac{\left(\frac{h}{g}\right)(\gamma(t))\partial_x f(\gamma(t))}{y'(t)} & \text{if } y(t) \neq 0. \end{cases} \tag{5.9}$$

Therefore, the proof follows taking orders. □

6 Tjurina Number

Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ defined by the 1-form $\omega = P(x, y)dx + Q(x, y)dy$ and $C : f(x, y) = 0$ be a $\mathcal{F}$-invariant reduced curve. The *Tjurina number of $\mathcal{F}$ with respect to C* is

$$\tau_p(\mathcal{F}, C) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, P, Q).$$

The *Tjurina number* of any germ of reduced curve $C : f(x, y) = 0$, with $f(x, y) \in \mathbb{C}[[x, y]]$ is by definition

$$\tau_p(C) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, \partial_x f, \partial_y f).$$

In this section we will study the Tjurina number of a foliation with respect to a balanced divisor of separatrices. First of all we present a lemma on Commutative Algebra which we need in the sequel and we did not find it in the literature:

Lemma 6.1 *Let $f, g, p, q \in \mathbb{C}[[x, y]]$, where f and g are relatively prime. Then*

$$\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, gp, gq) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, p, q) + \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, g).$$

Proof Observe that $\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, r_1, \ldots, r_n) = \dim_{\mathbb{C}} \mathcal{O}/(r_1', \ldots, r_n')$, where $\mathcal{O} = \mathbb{C}[[x, y]]/(f)$ and $r_i' = r_i + (f)$ for any $i \in \{1, \ldots, n\}$ and any $r_i \in \mathbb{C}[[x, y]]$. We finish the proof using the following exact sequence:

$$0 \longrightarrow \mathcal{O}/(p', q') \stackrel{\sigma}{\longrightarrow} \mathcal{O}/(g'p', g'q') \stackrel{\delta}{\longrightarrow} \mathcal{O}/(g') \longrightarrow 0,$$

where $\sigma(z' + (p', q')) = g'z' + (g'p', g'q')$ and $\delta(z' + (g'p', g'q')) = z' + (g')$, for any $z' \in \mathcal{O}$. □

The following proposition has been proved by (Gómez-Mont 1998, Theorem 1) for a foliation with a set of convergent separatrices. We show that the same result holds in the formal context for effective reduced balanced divisor of separatrices.

Proposition 6.2 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and C be a reduced curve of separatrices of $\mathcal{F}$. Then*

$$\tau_p(\mathcal{F}, C) - \tau_p(C) = GSV_p(\mathcal{F}, C).$$

Proof Let $\omega = P(x, y)dx + Q(x, y)dy$ be a 1-form inducing $\mathcal{F}$ and $f(x, y) = 0$ be the reduced equation of C. By equality (5.1) we get $g\omega = hdf + f\eta$, where η is a formal 1-form and $g, h \in \mathbb{C}[[x, y]]$ with g and f relatively prime and h and f also relatively prime.

Hence $gPdx + gQdy = (h\partial_x f + f\eta_x)dx + (h\partial_y f + f\eta_y)dy$, where $\eta = \eta_x dx + \eta_y dy$. We get

$$gP = h\partial_x f + f\eta_x, \quad \text{and} \quad gQ = h\partial_y f + f\eta_y. \tag{6.1}$$

After equalities (6.1), properties of the intersection number and Lemma 6.1, we have

$$\begin{aligned}
\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, gP, gQ) &= \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h\partial_x f + f\eta_x, h\partial_y f + f\eta_y) \\
&= \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h\partial_x f, h\partial_y f) \\
&= \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h) + \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, \partial_x f, \partial_y f) \\
&= \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h) + \tau_p(C).
\end{aligned}$$

Again, by Lemma 6.1 we get

$$\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, P, Q) + \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, g) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h) + \tau_p(C).$$

Hence $\tau_p(\mathcal{F}, C) - \tau_p(C) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, h) - \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(f, g) = i_p(f, h) - i_p(f, g)$. The proof follows from Lemma 5.1. □

Example 6.3 Let $\mathcal{F}$ be the foliation defined by the formal normal form of a saddle-node see (2.4) $\omega = x^{k+1}dy - y(1 + \lambda x^k)dx, \quad k \geq 1, \quad \lambda \in \mathbb{C}$. The total union of separatrices of $\mathcal{F}$ is $C = C_1 \cup C_2$, where $C_1 : x = 0$ (strong separatrix) and $C_2 : y = 0$ (weak separatrix). An equality (5.1) for C_1 is given for $g = 1, h = -y(1 + \lambda x^k)$ and $\eta = x^k dy$, hence by Lemma 5.1 we get $GSV_0(\mathcal{F}, C_1) = i_0(x, h) - i_0(x, g) = 1$. Similarly, an equality (5.1) for C_2 is given for $g = 1, h = x^{k+1}$ and $\eta = -(1+\lambda x^k)dx$, thus $GSV_0(\mathcal{F}, C_2) = i_0(y, h) - i_0(y, g) = k + 1$. Therefore, one finds

$$GSV_0(\mathcal{F}, C) = GSV_0(\mathcal{F}, C_1) + GSV_0(\mathcal{F}, C_2) - 2i_0(C_1, C_2) = 1 + (k+1) - 2 = k.$$

On the other hand, we get $\tau_0(\mathcal{F}, C) - \tau_0(C) = (k + 1) - 1 = k = GSV_0(\mathcal{F}, C)$.

Corollary B *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and C is the total union of separatrices of $\mathcal{F}$. Then*

$$\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C) + \chi_p(\mathcal{F}).$$

Moreover, if $\mathcal{F}$ is of second type then $\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C)$.

Proof It is a consequence of Proposition 6.2 and Corollary 5.6. □

Now, we characterize generalized curve foliations at $(\mathbb{C}^2, p)$ in terms of the Tjurina numbers.

Corollary 6.4 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a reduced balanced divisor of separatrices for $\mathcal{F}$. Then $\mathcal{F}$ is a generalized curve foliation if and only if $\tau_p(\mathcal{B}_0) - \tau_p(\mathcal{F}, \mathcal{B}_0) = i_p(\mathcal{B}_0, \mathcal{B}_\infty)$.*

Proof It follows from (Genzmer and Mol 2018, Theorem A) that $\mathcal{F}$ is a generalized curve foliation if and only if $\Delta_p(\mathcal{F}, \mathcal{B}_0) = 0$. Applying Theorem 5.2 to $C = \mathcal{B}_0$, we get $GSV_p(\mathcal{F}, \mathcal{B}_0) = \Delta_p(\mathcal{F}, \mathcal{B}_0) - i_p(\mathcal{B}_0, \mathcal{B}_\infty)$. Hence $\mathcal{F}$ is a generalized curve foliation if and only if $GSV_p(\mathcal{F}, \mathcal{B}_0) = -i_p(\mathcal{B}_0, \mathcal{B}_\infty)$. The proof ends, by applying Proposition 6.2 to $C = \mathcal{B}_0$. □

If $\mathcal{B} = \sum_B a_B B$ is a divisor of separatrices for $\mathcal{F}$ then we put

$$T_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B \tau_p(\mathcal{F}, B).$$

The following theorem gives a relationship between the Milnor and Tjurina numbers and the χ-number, studied in Sect. 3.

Theorem C *Let $\mathcal{F}$ be be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then*

$$\begin{aligned}\mu_p(\mathcal{F}) - T_p(\mathcal{F}, \mathcal{B}) = &\sum_B a_B[\mu_p(dF_B, B) - \tau_p(B)] - \deg(\mathcal{B}) + 1 + \chi_p(\mathcal{F}) \\ &- \sum_B a_B[i_p(B, (F_B)_0 \backslash B) - i_p(B, (F_B)_\infty)],\end{aligned}$$

where F_B is a balanced divisor of separatrices for $\mathcal{F}$ adapted to B.

Proof By Proposition 6.2 we get $T_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B(GSV_p(\mathcal{F}, B) + \tau_p(B))$. Then $T_p(\mathcal{F}, \mathcal{B}) - \sum_B a_B \tau_p(B) = \sum_B a_B GSV_p(\mathcal{F}, B)$. From Theorem 5.2, we have

$$\begin{aligned}T_p(\mathcal{F}, \mathcal{B}) - \sum_B a_B \tau_p(B) &= \sum_B a_B[\Delta_p(\mathcal{F}, B) + i_p(B, (F_B)_0 \backslash B) - i_p(B, (F_B)_\infty)] \\ &= \Delta_p(\mathcal{F}, \mathcal{B}) + \sum_B a_B[i_p(B, (F_B)_0 \backslash B) - i_p(B, (F_B)_\infty)],\end{aligned}$$

where F_B is a balanced divisor of separatrices for $\mathcal{F}$ adapted to B. We finish the proof using Theorem A. □

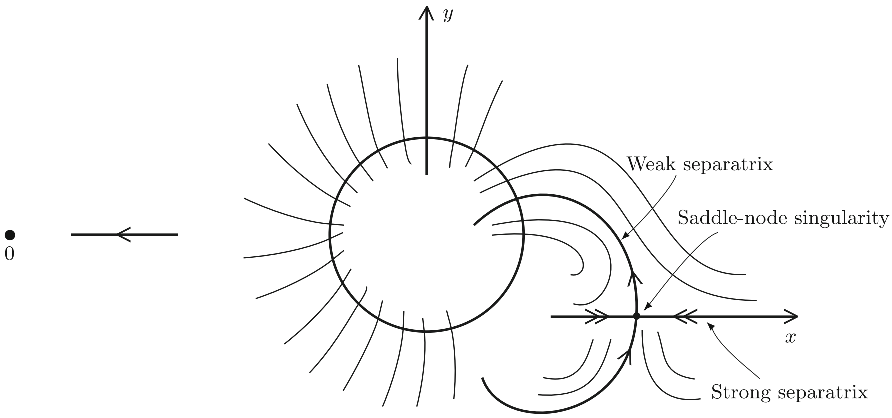

Fig. 1 Dicritical foliation $\mathcal{F}_k$

In order to illustrate Theorem C we present a family of dicritical foliations that are not of second type (Fig. 1).

Example 6.5 Let $\lambda \in \mathbb{C}$ and $k \geq 3$ integer. Let $\mathcal{F}_k$ be the singular foliation at $(\mathbb{C}^2, 0)$ defined by

$$\omega_k = y(2x^{2k-2} + 2(\lambda+1)x^2y^{k-2} - y^{k-1})dx + x(y^{k-1} - (\lambda+1)x^2y^{k-2} - x^{2k-2})dy.$$

The foliation $\mathcal{F}_k$ is dicritical.

After one blow-up, the foliation has a unique non-reduced singularity q. A further blow-up applied to q produces a reduction of singularities of the foliation with a dicritical component and a tangent saddle-node with strong separatrix transversal to exceptional divisor. Therefore $\mathcal{F}_k$ is not of second type. Let $B_1 : y = 0$ and $B_2 : x = 0$, then $\mathcal{B} = B_1 + B_2$ is an effective balanced divisor of separatrices for $\mathcal{F}_k$. A simple calculation leads to:

$$\nu_0(\mathcal{F}_k) = k, \quad \nu_q(\mathcal{F}_k) = k-1, \quad \xi_0(\mathcal{F}_k) = k-1, \quad \xi_q(\mathcal{F}_k) = k-1,$$

thus $\chi_p(\mathcal{F}_k) = 2(k-1)^2$. Moreover $\mu_0(\mathcal{F}_k) = (k-2)(2k-2)+5k-4$, $T_0(\mathcal{F}_k, \mathcal{B}) = 3k-1$, $\tau_0(B_1) = \tau_0(B_2) = 0$. Since $F(x, y) = xy$ defines a balanced divisor of separatrices for $\mathcal{F}_k$ adapted to B_1 and B_2, we get $\mu_0(dF, B_1) + \mu_0(dF, B_2) = 2$. Hence we have $T_0(\mathcal{F}_k, \mathcal{B}) = 3k-1$ and Theorem C is verified.

Corollary 6.6 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. If $\mathcal{F}$ is of second type, then*

$$\mu_p(\mathcal{F}) - T_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B[\mu_p(dF_B, B) - \tau_p(B)] - \deg(\mathcal{B}) + 1$$
$$- \sum_B a_B[i_p(B, (F_B)_0 \backslash B) - i_p(B, (F_B)_\infty)],$$

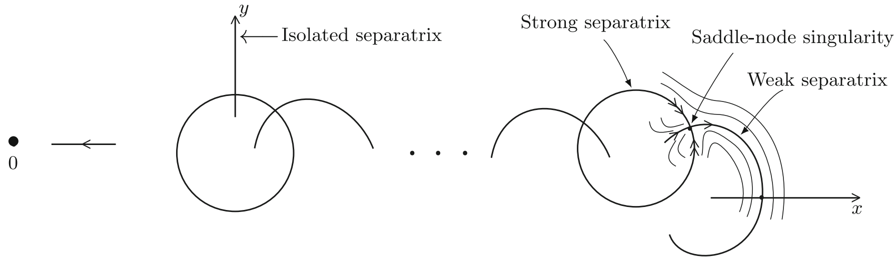

Fig. 2 Dulac foliation $\mathcal{F} : \omega = (ny + x^n)dx - xdy$

where F_B is a balanced divisor of separatrices for $\mathcal{F}$ adapted to B.

Proof By Lemma 3.1 item (2), $\chi_p(\mathcal{F}) = 0$ and the proof follows from Theorem C.

The following example shows that, in general, the reciprocal of Corollary 6.6 is not true (Fig. 2).

Example 6.7 *(Dulac's foliation)* The foliation $\mathcal{F}$ defined by the 1-form $\omega = (ny + x^n)dx - xdy$, with $n \geq 2$, admits a unique separatrix $C : x = 0$. Since $\nu_0(\mathcal{F}) = 1 \neq 0 = \nu_0(C) - 1$, so $\mathcal{F}$ is not of second type. Moreover $T_0(\mathcal{F}, C) = 1$, $\tau_0(C) = 0$, $\mu_0(\mathcal{F}) = 1$, $\mu_0(dx, C) = 0$. Hence, $\mathcal{F}$ verifies the equality of Corollary 6.6 but it is not a foliation of second type at $0 \in \mathbb{C}^2$.

Now, we apply Theorem C to non-dicritical singular foliations at $(\mathbb{C}^2, p)$.

Corollary D *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C = \cup_{j=1}^{\ell} C_j$ is the total union of separatrices of $\mathcal{F}$. Then*

$$\mu_p(\mathcal{F}) - T_p(\mathcal{F}, C) = \mu_p(C) - \sum_{j=1}^{\ell} \tau_p(C_j) + \chi_p(\mathcal{F}) - \sum_{j=1}^{\ell} i_p(C_j, C \backslash C_j).$$

Proof By taking the effective divisor $\mathcal{B} = (f)$, where $f(x, y) = 0$ is an equation of C, and applying Proposition 4.7 to the foliation df, we get

$$\mu_p(C) = \sum_{j=1}^{\ell} \mu_p(df, C_j) - \ell + 1. \tag{6.2}$$

The proof follows from Theorem C. □

Corollary 6.8 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$. Assume that $\mathcal{F}$ is non-dicritical and $C = \cup_{j=1}^{\ell} C_j$ is the total union of separatrices of $\mathcal{F}$. Then*

$$T_p(\mathcal{F}, C) - \tau_p(\mathcal{F}, C) = \sum_{j=1}^{\ell} \tau(C_j) - \tau(C) + 2 \sum_{1 \leq i < j \leq \ell} i_p(C_i, C_j).$$

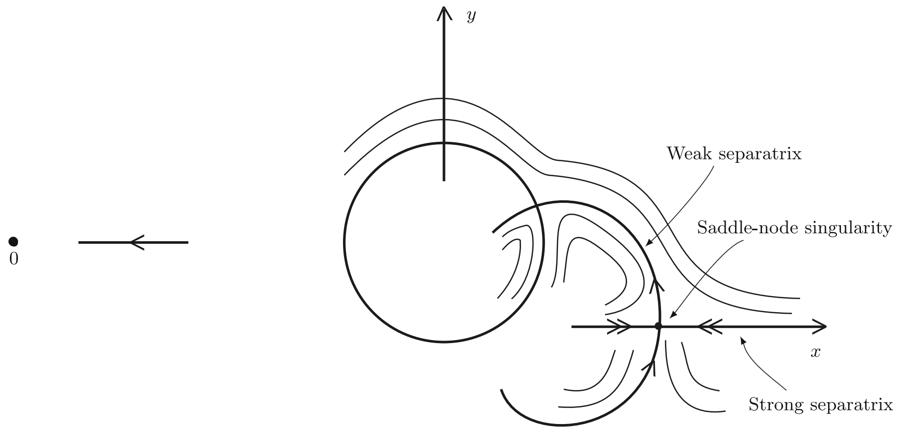

Fig. 3 Foliation $\mathcal{F} : \omega = 4xydx + (y - 2x^2)dy$

Here is an example to illustrate Corollary D (Fig. 3).

Example 6.9 Let $\omega = 4xydx + (y - 2x^2)dy$ be a 1-form defining a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$. The unique separatrix of $\mathcal{F}$ is the curve $C : y = 0$. Since $\nu_0(\mathcal{F}) = 1$ and $\nu_0(C) = 1$, the foliation $\mathcal{F}$ is not of second type at the origin (see Proposition 2.4). Moreover, we get $T_0(\mathcal{F}, C) = 2$, $\tau_0(C) = 0$, $\mu_0(\mathcal{F}) = 3$, $\mu_0(C) = 0$, and $\chi_0(\mathcal{F}) = 1$.

6.1 Milnor and Tjurina Numbers and Some Residue-Type Indices

We finish this section stating numerical relationships between some classic indices, such as the Baum–Bott, Camacho–Sad and variational indices, of a singular foliation at $(\mathbb{C}^2, p)$ and the Milnor and Tjurina numbers.

Let $\mathcal{F}$ be a singular foliation defined by a 1-form ω as in (2.2). Let $J(x, y)$ be the Jacobian matrix of $(Q(x, y), -P(x, y))$. The *Baum–Bott index* (see Baum and Bott 1972) of $\mathcal{F}$ at p is

$$\mathrm{BB}_p(\mathcal{F}) = \mathrm{Res}_p \left\{ \frac{(\mathrm{tr} J)^2}{-P \cdot Q} dx \wedge dy \right\},$$

where $\mathrm{tr} J$ denotes the trace of J.

The *Camacho–Sad index* of $\mathcal{F}$ (CS index) with respect to an analytic $\mathcal{F}$-invariant curve C is

$$\mathrm{CS}_p(\mathcal{F}, C) = -\frac{1}{2\pi i} \int_{\partial C} \frac{1}{h} \eta, \tag{6.3}$$

where g, h are from (5.1). The Camacho–Sad index was introduced by these authors in Camacho and Sad (1982) for a non-singular $\mathcal{F}$-invariant curve C. Later, Lins Neto (1986) and Suwa (1995) generalize this index to singular $\mathcal{F}$-invariant curves. If C is irreducible then equality (6.3) becomes

$$\mathrm{CS}_p(\mathcal{F}, C) = -\mathrm{Res}_{t=0} \left(\gamma^* \frac{1}{h} \eta \right), \tag{6.4}$$

where $\gamma(t)$ is a Puiseux parametrization of C. By (Brunella 2010, p. 38) (see also Suwa 1998) we get the adjunction formula

$$\mathrm{CS}_p(\mathcal{F}, C_1 \cup C_2) = \mathrm{CS}_p(\mathcal{F}, C_1) + \mathrm{CS}_p(\mathcal{F}, C_2) + 2i_p(C_1, C_2), \tag{6.5}$$

for any two analytic $\mathcal{F}$-invariant curves, C_1 and C_2, without common irreducible components. The equality (6.4) allows us to extend the definition of the CS index to a purely formal (non-analytic) irreducible $\mathcal{F}$-invariant curve; and the equality (6.5) allows us to extend the definition of the CS index to $\mathcal{F}$-invariant curves containing purely formal branches.

In a neighborhood of a non-singular point of the foliation $\mathcal{F}$, there is a 1-form α such that $d\omega = \alpha \wedge \omega$. If α' is other such 1-form, then α and α' coincide over every leaf of $\mathcal{F}$. Hence, in a neighborhood of 0 (away 0) there exists a holomorphic multi-valued 1-form α such that $d\omega = \alpha \wedge \omega$ and that its restriction to each leaf of $\mathcal{F}$ is single-valued. We say that α is an *exponent form* for ω. The *variational index* or *variation* of $\mathcal{F}$ relative to C at p is

$$\mathrm{Var}_p(\mathcal{F}, C) = \mathrm{Res}_{t=0}\left(\alpha\big|_C\right) = \frac{1}{2\pi i}\int_{\partial C} \alpha.$$

The variational index was introduced in Khanedani and Suwa (1997). It is additive:

$$\mathrm{Var}_p(\mathcal{F}, C_1 \cup C_2) = \mathrm{Var}_p(\mathcal{F}, C_1) + \mathrm{Var}_p(\mathcal{F}, C_2),$$

where C_1 and C_2 are $\mathcal{F}$-invariant curves without common factors. For any divisor $\mathcal{B} = \sum_B a_B B$ of separatrices for $\mathcal{F}$ we put

$$\mathrm{Var}_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B \mathrm{Var}_p(\mathcal{F}, B).$$

For any analytic $\mathcal{F}$-invariant curve C we have, after (Brunella 1997, Proposition 5),

$$\mathrm{Var}_p(\mathcal{F}, C) = \mathrm{CS}_p(\mathcal{F}, C) + GSV_p(\mathcal{F}, C). \tag{6.6}$$

Proposition 6.10 *Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, p)$ and let $\mathcal{B} = \sum_B a_B B$ be a balanced divisor of separatrices for $\mathcal{F}$. Then*

$$\mathrm{BB}_p(\mathcal{F}) = \mathrm{Var}_p(\mathcal{F}, \mathcal{B}) + \mu_p(\mathcal{F}) - \sum_B a_B \mu_p(dF_B, B) + \deg(\mathcal{B}) - 1 - \chi_p(\mathcal{F}) + \sum_{q \in \mathcal{I}_p(\mathcal{F})} \xi_q^2(\mathcal{F}),$$

where F_B is a balanced divisor of separatrices for $\mathcal{F}$ adapted to B.

Moreover, if $\mathcal{F}$ is non-dicritical and C is the total union of separatrices of $\mathcal{F}$ then

$$\mathrm{BB}_p(\mathcal{F}) = CS_p(\mathcal{F}, C) + \tau_p(\mathcal{F}, C) - \tau_p(C) + \mu_p(\mathcal{F}) - \mu_p(C) - \chi_p(\mathcal{F}) + \sum_{q \in \mathcal{I}_p(\mathcal{F})} \xi_q^2(\mathcal{F}).$$

Proof Suppose that $\mathcal{F}$ is a singular foliation. By (Fernández-Pérez and Mol 2019, Theorem 5.2) we get

$$\mathrm{BB}_p(\mathcal{F}) = \mathrm{Var}_p(\mathcal{F}, \mathcal{B}) + \Delta_p(\mathcal{F}, \mathcal{B}) + \sum_{q \in \mathcal{I}_p(\mathcal{F})} \xi_q^2(\mathcal{F}),$$

where the summation runs over all infinitely near points of $\mathcal{F}$ at p. The proof of the first equality follows applying Theorem A to $\Delta_p(\mathcal{F}, \mathcal{B})$.

The proof of the non-dicritical case follows from the first part of the proposition (for $\mathcal{B} = C$ and $F_{C_i} = f$, for any $1 \leq i \leq \ell$, where $f(x, y) = 0$ is an equation of C), equality (6.6), Proposition 6.2 and equality (6.2). □

7 Bound for the Milnor Sum of an Algebraic Curve

Let $\mathcal{F}$ be a holomorphic foliation on the complex projective plane $\mathbb{P}^2$. The degree of $\mathcal{F}$ is the number of tangencies between $\mathcal{F}$ and a generic line. It is well-known that $\mathcal{F}$ has a least one singular point an there is a lot of activity around the foliations of $\mathbb{P}^2$ having a unique singularity (for instance Cerveau and Déserti 2023 and Castorena et al. 2024). Let C be an algebraic $\mathcal{F}$-invariant curve in $\mathbb{P}^2$. We say that C is *non-dicritical* if every singular point of $\mathcal{F}$ on C is non-dicritical.

The classification of holomorphic foliations on $\mathbb{P}^2$ of degree d has been of great interest in Algebraic Geometry (see for example the recent paper Castorena et al. 2024). It is therefore interesting to obtain properties of the foliations in the projective plane.

As an application of our previous results, we have the following theorem:

Theorem 7.1 *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^2$ of degree d leaving invariant a non-dicritical algebraic curve C of degree d_0 such that for each $p \in \mathsf{Sing}(\mathcal{F}) \cap C$, all local branches of* $\mathrm{Sep}_p(\mathcal{F})$ *are contained in C. Then*

$$\sum_{p \in C} \mu_p(C) = \left(\sum_{p \in \mathsf{Sing}(\mathcal{F}) \cap C} [\mu_p(\mathcal{F}) - \chi_p(\mathcal{F})] \right) + d_0^2 - (d+2)d_0.$$

In particular,

$$\sum_{p \in C} \mu_p(C) \leq \left(\sum_{p \in \mathsf{Sing}(\mathcal{F}) \cap C} \mu_p(\mathcal{F}) \right) + d_0^2 - (d+2)d_0.$$

Moreover, if $\mathsf{Sing}(\mathcal{F}) \subset C$, then

$$\sum_{p \in C} \mu_p(C) \leq d^2 - d(d_0 - 1) + (d_0 - 1)^2.$$

Proof Fix an arbitrary point $p \in \mathsf{Sing}(\mathcal{F}) \cap C$. Then, by Corollary 5.6, we have

$$\mu_p(C) = \mu_p(\mathcal{F}) - \chi_p(\mathcal{F}) - GSV_p(\mathcal{F}, C).$$

Since for each $p \in \mathsf{Sing}(\mathcal{F}) \cap C$, all local branches of $\mathrm{Sep}_p(\mathcal{F})$ are contained in C, we get equality $\sum_{p\in\mathsf{Sing}(\mathcal{F})\cap C} GSV_p(\mathcal{F}, C) = (d+2)d_0 - d_0^2$ (cf. Brunella 1997, Proposition 4), and this implies

$$\sum_{p\in C} \mu_p(C) = \left(\sum_{p\in\mathsf{Sing}(\mathcal{F})\cap C} [\mu_p(\mathcal{F}) - \chi_p(\mathcal{F})] \right) + d_0^2 - (d+2)d_0.$$

Note that, in particular, we get the inequality

$$\sum_{p\in C} \mu_p(C) \le \left(\sum_{p\in\mathsf{Sing}(\mathcal{F})\cap C} \mu_p(\mathcal{F}) \right) + d_0^2 - (d+2)d_0,$$

since $\chi_p(\mathcal{F}) \ge 0$ by Proposition 3.1 item (1). On the other hand, if $\mathsf{Sing}(\mathcal{F}) \subset C$, we use the equality $\sum_{p\in\mathsf{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = d^2 + d + 1$ (cf. Brunella 2010, p. 28) to finish the proof. □

Remark 7.2 Observe that the hypothesis *all local branches of* $\mathrm{Sep}_p(\mathcal{F})$ *and all singularities of* $\mathcal{F}$ *are contained in* C in Theorem 7.1 implies, by (Fernández-Pérez and Mol 2019, Proposition 6.1), that $\deg C = \deg \mathcal{F} + 2$ and $\mathcal{F}$ is a logarithmic foliation.

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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