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An upper bound for the GSV-index of a foliation

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Abstract

Let $\mathcal{F}$ be a holomorphic foliation at $p \in \mathbb{C}^2$, and let B be a separatrix of $\mathcal{F}$. We prove the following upper bound $GSV_p(\mathcal{F}, B) \leq 4\tau_p(\mathcal{F}, B) - 3\mu_p(\mathcal{F}, B)$, where $GSV_p(\mathcal{F}, B)$ is the *Gómez-Mont-Seade-Verjovsky index* of the foliation $\mathcal{F}$ with respect to B, $\mu_p(\mathcal{F}, B)$ is the multiplicity of $\mathcal{F}$ along B and $\tau_p(\mathcal{F}, B)$ is the dimension of the quotient of $\mathbb{C}\{x, y\}$ by the ideal generated by the components of any 1-form defining $\mathcal{F}$ and any equation of B.

Keywords Holomorphic foliations · Gómez-Mont-Seade-Verjovsky index · Tjurina number · Multiplicity of a foliation along a divisor of separatrices

Mathematics Subject Classification Primary 32S65 · 32M25

1 Introduction

Let $\mathcal{F}$ be a germ of singular holomorphic foliation at $(\mathbb{C}^2, p)$ given by the 1-form $\omega := P(x, y)dx + Q(x, y)dy$, where $P(x, y), Q(x, y) \in \mathbb{C}\{x, y\}$, and B be a separatrix of $\mathcal{F}$. Several numerical invariants can be studied for the pair $(\mathcal{F}, B)$, such as the *Gómez-Mont–Seade–Verjovsky index* (see [10]) of $\mathcal{F}$ with respect to B, denoted by $GSV_p(\mathcal{F}, B)$, the *multiplicity of* $\mathcal{F}$*along* B, $\mu_p(\mathcal{F}, B)$ (see Sect. 2 for definition), and the *Tjurina number* of $\mathcal{F}$ with respect to $B : f(x, y) = 0$, defined by

$$\tau_p(\mathcal{F}, B) := \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(P, Q, f).$$

In this paper, inspired by the blow-up techniques developed by Wang [14], we establish an optimal upper bound for $GSV_P(\mathcal{F}, B)$ in terms of $\mu_p(\mathcal{F}, B)$ and $\tau_p(\mathcal{F}, B)$ as follows:

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Theorem A *Let $\mathcal{F}$ be a germ of a singular holomorphic foliation on $(\mathbb{C}^2, p)$, and B be a separatrix of $\mathcal{F}$ at p. Then*

$$GSV_p(\mathcal{F}, B) \leq 4\tau_p(\mathcal{F}, B) - 3\mu_p(\mathcal{F}, B) \tag{1}$$

where $GSV_p(\mathcal{F}, B)$ denotes the GSV-index of $\mathcal{F}$ with respect to B. Moreover, the equality holds if and only if B is smooth.

Note that if $B = \{f = 0\}$ is an irreducible plane curve germ, then by applying Theorem A to the foliation $\mathcal{F} : df = 0$, we obtain the bound stated in [6, Question 4.2] and proved simultaneously in [1], [8] and [14], since $\mu_p(\mathcal{F}, B) = \mu(B)$, $\tau_p(\mathcal{F}, B) = \tau(B)$, and $GSV_p(\mathcal{F}, B) = 0$ in this case. Here, $\mu(B)$ and $\tau(B)$ denote the classical Milnor and Tjurina numbers of B. A complete answer to this question was given by Almirón [2]. Moreover, the author proposed a broader perspective of the study of the difference between Milnor and Tjurina numbers for isolated complete intersection singularities.

2 Multiplicity of a foliation along a divisor of separatrices

Throughout this note $\mathcal{F}$ denotes a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$, given in local coordinates (x, y) centered at p by a 1-form $\omega := P(x, y)dx + Q(x, y)dy$, where $P(x, y), Q(x, y) \in \mathbb{C}\{x, y\}$ are coprime; or by its dual vector field

$$v := -Q(x, y)\frac{\partial}{\partial x} + P(x, y)\frac{\partial}{\partial y}.$$

The *algebraic multiplicity* $\nu_p(\mathcal{F})$ of $\mathcal{F}$ is the minimum of the orders $\text{ord}_p(P)$ and $\text{ord}_p(Q)$. Remember that a plane curve germ $f(x, y) = 0$ is an *$\mathcal{F}-$invariant curve* if $\omega \wedge df = (f.h)dx \wedge dy$, for some $h(x, y) \in \mathbb{C}\{x, y\}$ and a *separatrix* of $\mathcal{F}$ is an irreducible $\mathcal{F}$-invariant curve.

Let $\mathcal{F}$ be a germ of a singular foliation at $(\mathbb{C}^2, p)$ induced by the vector field v and B be a separatrix of $\mathcal{F}$. Let $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, p)$ be a primitive parametrization of B, the *multiplicity of $\mathcal{F}$ along B at p* is by definition

$$\mu_p(\mathcal{F}, B) = \text{ord}_t \theta(t), \tag{2}$$

where $\theta(t)$ is the unique vector field at $(\mathbb{C}, 0)$ such that $\gamma_* \theta(t) = v \circ \gamma(t)$, see for instance [4, p. 159]. If $\omega = P(x, y)dx + Q(x, y)dy$ is a 1-form inducing $\mathcal{F}$ and $\gamma(t) = (x(t), y(t))$, then

$$\theta(t) = \begin{cases} -\frac{Q(\gamma(t))}{x'(t)} & \text{if } x(t) \neq 0 \\ \frac{P(\gamma(t))}{y'(t)} & \text{if } y(t) \neq 0. \end{cases} \tag{3}$$

We extend the notion of multiplicity of $\mathcal{F}$ along any nonempty divisor $\mathcal{B} = \sum_B a_B B$ of separatrices of $\mathcal{F}$ at p in the following way:

$$\mu_p(\mathcal{F}, \mathcal{B}) = \left(\sum_B a_B \mu_p(\mathcal{F}, B) \right) - \deg(\mathcal{B}) + 1. \tag{4}$$

By convention we put $\mu_p(\mathcal{F}, \mathcal{B}) = 1$ for any empty divisor $\mathcal{B}$. In particular, when $\mathcal{B} = B_1 + \cdots + B_r$ is an effective divisor of separatrices of the singular foliation $\mathcal{F}$ at p, we rediscover [11, Equation (2.2), p. 329] (for the reduced plane curve $C := \cup_{i=1}^r B_i$).

Hence, if we write $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ where $\mathcal{B}_0$ and $\mathcal{B}_\infty$ are effective divisors we get

$$\mu_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}, \mathcal{B}_0) - \mu_p(\mathcal{F}, \mathcal{B}_\infty) + 1.$$

Since $\mu_p(\mathcal{F}, \mathcal{B}_\infty) \geq 1$ we have

$$\mu_p(\mathcal{F}, \mathcal{B}_0) \geq \mu_p(\mathcal{F}, \mathcal{B}).$$

Denote by $GSV_p(\mathcal{F}, \mathcal{B})$ the *Gómez-Mont-Seade-Verjovsky index* of the foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ (GSV-index) with respect to the effective divisor $\mathcal{B}$ of separatrices of $\mathcal{F}$. See [10] for details.

Remark 2.1 Let $B : f(x, y) = 0$ be a separatrix of $\mathcal{F}$ and let $\mathcal{G}_f$ be the hamiltonian foliation defined by $df = 0$. From (2) and (3), we have

$$\mu_p(\mathcal{G}_f, B) = \begin{cases} \text{ord}_t \partial_y f(\gamma(t)) - \text{ord}_t x(t) + 1 & \text{if } x(t) \neq 0 \\ \text{ord}_t \partial_x f(\gamma(t)) - \text{ord}_t y(t) + 1 & \text{if } y(t) \neq 0, \end{cases}$$

where $\gamma(t) = (x(t), y(t))$ is a Puiseux parametrization of B. By [7, Proposition 5.7], we get $\mu_p(\mathcal{F}, B) = GSV_p(\mathcal{F}, B) + \mu_p(\mathcal{G}_f, B)$. But, since B is irreducible, after [7, Proposition 4.7], $\mu_p(\mathcal{G}_f, B) = \mu_p(B)$ and so that $\mu_p(\mathcal{F}, B) = GSV_p(\mathcal{F}, B) + \mu_p(B)$. Hence, according to [7, Proposition 6.2], we obtain

$$\mu_p(\mathcal{F}, B) - \tau_p(\mathcal{F}, B) = \mu_p(B) - \tau_p(B). \tag{5}$$

Consequently when B is quasihomogeneous by ([12], Satz p.123 a) and d)] and (5) or by [15, Theorem 4], B is analytically equivalent to $y^n + x^m = 0$ for some natural coprime numbers $n, m > 1$, in this case, we get $\mu_p(\mathcal{F}, B) = \tau_p(\mathcal{F}, B)$.

3 Proof of Theorem A

Let C be an irreducible plane curve of multiplicity $\nu_p(C) = n$. Denote by $\tilde{C}$ the strict transform of C by a blow-up and $\bar{C}$ the normalization of C. We have natural morphisms

$$\pi : \bar{C} \xrightarrow{\rho} \tilde{C} \xrightarrow{\sigma} C. \tag{6}$$

Consider $(C, p) \xrightarrow{i} (\mathcal{C}, x_0) \xrightarrow{\phi} (T, t_0)$ the miniversal deformation of C. We extend to $\mathcal{C}$ the morphisms in (6):

$$\pi : \bar{\mathcal{C}} \xrightarrow{\rho} \tilde{\mathcal{C}} \xrightarrow{\sigma} \mathcal{C}.$$

We have two other natural morphisms $\tilde{C} \xrightarrow{\tilde{\phi}} T$ and $\bar{C} \xrightarrow{\bar{\phi}} T$ and its respectively extensions to the deformation, that is $\tilde{\mathcal{C}} \xrightarrow{\tilde{\phi}} T$ and $\bar{\mathcal{C}} \xrightarrow{\bar{\phi}} T$.

Let $\Omega_{\mathcal{C}/T}$ (respectively, $\Omega_{\tilde{\mathcal{C}}/T}$, respectively, $\Omega_{\bar{\mathcal{C}}/T}$) denote the $\mathcal{O}_{\mathcal{C}}$-module of relative Kähler differentials of $\mathcal{C}$ over T (respectively, the $\mathcal{O}_{\tilde{\mathcal{C}}}$-module of relative Kähler differentials of $\tilde{\mathcal{C}}$ over T, respectively the $\mathcal{O}_{\bar{\mathcal{C}}}$-module of relative Kähler differentials of $\bar{\mathcal{C}}$ over T). Put $G := \rho^*\Omega_{\tilde{\mathcal{C}}/T}/\pi^*\Omega_{\mathcal{C}/T}$. If C is singular then G is not zero and $\bar{\phi}_* G$ is a finite $\mathcal{O}_T$-module and after Wang ([14]) we put $\mathcal{D}(t) = \text{length}_{\mathcal{O}_{T,t}}((\bar{\phi}_* G)_t)$ and $\alpha := \min_{t \in T} \mathcal{D}(t)$. By [14, Claim 4.4] we have

$$\alpha > \frac{\nu_p(C)(\nu_p(C) - 1)}{4}, \quad \text{for} \quad \nu_p(C) \geq 2. \tag{7}$$

We will need the following lemma.

Lemma 3.1 *Let $\mathcal{F}$ be a germ of a singular foliation on $(\mathbb{C}^2, p)$, and B be a separatrix of $\mathcal{F}$ at p. If $\tilde{\mathcal{F}}$ (respectively $\tilde{B}$) is the strict transform of $\mathcal{F}$ (respectively of B) by σ at the point q, then*

$$3\mu_q(\tilde{\mathcal{F}}, \tilde{B}) - 4\tau_q(\tilde{\mathcal{F}}, \tilde{B}) + GSV_q(\tilde{\mathcal{F}}, \tilde{B}) \geq 3\mu_p(\mathcal{F}, B) - 4\tau_p(\mathcal{F}, B) + GSV_p(\mathcal{F}, B). \tag{8}$$

Moreover, the equality holds if and only if B is smooth.

Proof Suppose first $\nu_p(B) \geq 2$. By [5, Equality (1.2), p. 291] we have

$$\mu_q(\tilde{\mathcal{F}}, \tilde{B}) = \mu_p(\mathcal{F}, B) - \nu_p(B)(m_p(\mathcal{F}) - 1), \tag{9}$$

where

$$m_p(\mathcal{F}) = \begin{cases} \nu_p(\mathcal{F}) + 1 & \text{if } \sigma \text{ is dicritical} \\ \nu_p(\mathcal{F}) & \text{if } \sigma \text{ is nondicritical.} \end{cases}$$

On the other hand by the behavior under blow-up of the GSV index [3, p. 30] we get

$$GSV_q(\tilde{\mathcal{F}}, \tilde{B}) = GSV_p(\mathcal{F}, B) + \nu_p(B)(\nu_p(B) - m_p(\mathcal{F})), \tag{10}$$

and after [7, Proposition 6.2] we have

$$GSV(\mathcal{F}, B) = \tau_p(\mathcal{F}, B) - \tau_p(B). \tag{11}$$

From [14, Equation (4)] we obtain

$$\tau_p(B) - \tau_q(\tilde{B}) \geq \frac{\nu_p(B)(\nu_p(B) - 1)}{2} + \alpha, \tag{12}$$

where α was defined in (7). From (11), (10), and (12) we deduce

$$\tau_q(\tilde{\mathcal{F}}, \tilde{B}) \leq \tau_p(\mathcal{F}, B) + \frac{\nu_p(B)[\nu_p(B) + 1 - 2m_p(\mathcal{F})]}{2} - \alpha. \tag{13}$$

After (9), (10) and (13) we get

$$\begin{aligned} 3\mu_q(\tilde{\mathcal{F}}, \tilde{B}) - 4\tau_q(\tilde{\mathcal{F}}, \tilde{B}) + GSV_q(\tilde{\mathcal{F}}, \tilde{B}) &\geq 3\mu_p(\mathcal{F}, B) - 4\tau_p(\mathcal{F}, B) + GSV_p(\mathcal{F}, B) \\ &\quad + 4\alpha - \nu_p(B)(\nu_p(B) - 1) \\ &> 3\mu_p(\mathcal{F}, B) - 4\tau_p(\mathcal{F}, B) + GSV_p(\mathcal{F}, B), \end{aligned} \tag{14}$$

where the last inequality follows by (7), since $\nu_p(B) \geq 2$.

Suppose now that B is a non-singular curve. By a change of coordinates, if necessary, we can suppose that B is given by $x = 0$. Since B is a separatrix of $\mathcal{F} : \omega = Pdx + Qdy$ then $\omega \wedge dx = Qdx \wedge dy$. Hence $Q = xh$ for some convergent power series $h(x, y) \in \mathbb{C}\{x, y\}$ such that $h(0, y) \neq 0$ and $\mathcal{F} : \omega = P(x, y)dx + xh(x, y)dy$. Since $\gamma(t) = (0, t)$ is a parametrization of B, then $\mu_p(\mathcal{F}, B) = \text{ord}_t P(0, t)$ and

$$\tau_p(\mathcal{F}, B) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(P, xh, x) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(P, x) = \mu_p(\mathcal{F}, B). \tag{15}$$

Moreover by (11) we have the equality $GSV_p(\mathcal{F}, B) = \tau_p(\mathcal{F}, B)$. Thus, if $\nu_p(B) = 1$ then $\nu_p(\tilde{B}) = 1$, $\mu_p(\mathcal{F}, B) = \tau_p(\mathcal{F}, B) = GSV_p(\mathcal{F}, B)$ and $\mu_p(\tilde{\mathcal{F}}, \tilde{B}) = \tau_p(\tilde{\mathcal{F}}, \tilde{B}) = GSV_p(\tilde{\mathcal{F}}, \tilde{B})$, so (8) becomes an equality. Moreover, if (8) is an equality, then B is necessarily smooth. Indeed, if B were singular, by (7) the number $-\nu_p(B)(\nu_p(B) - 1) + 4\alpha$ is positive and from the equality (14) we have that (8) is a strict inequality that is a contradiction. Hence the lemma follows. □

Remark 3.2 By definition, for any branch B we have $\tau_p(B) \leq \mu_p(B)$, so after [7, Proposition 6.2] and [7, Propositon 5.7] we get $\tau_p(\mathcal{F}, B) - GSV_p(\mathcal{F}, B) \leq \mu_p(\mathcal{F}, B) - GSV_p(\mathcal{F}, B)$ and

$$\tau_p(\mathcal{F}, B) \leq \mu(\mathcal{F}, B). \tag{16}$$

In addition, by (15), if B is smooth then $\tau_p(\mathcal{F}, B) = \mu(\mathcal{F}, B)$.

We will now prove the main theorem of this paper.

3.1 Proof of Theorem A

Suppose that $\mathcal{F}$ is a holomorphic foliation at $p \in \mathbb{C}^2$, B is a separatrix of $\mathcal{F}$ at p. First, assume that B is singular, that is, $\nu_p(B) \geq 2$. Then, there is a sequence of blow-ups of $\mathcal{F}$ at p (cf. Seidenberg [13]):

$$(\mathcal{F}^{(N)}, B^{(N)}) \longrightarrow \cdots \longrightarrow (\mathcal{F}^{(2)}, B^{(2)}) \longrightarrow (\mathcal{F}^{(1)}, B^{(1)}) \longrightarrow (\mathcal{F}^{(0)}, B^{(0)}) = (\mathcal{F}, B), \tag{17}$$

where $B^{(i+1)}$ denotes the strict transform of $B^{(i)}$ thought $q^{(i)}$ for each $i = 0, \ldots, N-1$, $B^{(N)}$ is smooth, and $\mathcal{F}^{(N)}$ is a foliation with a simple singularity at $q^{(N)} \in B^{(N)}$, or $\mathcal{F}^{(N)}$ is a regular foliation at $q^{(N)}$, and $B^{(N)}$ is a dicritical separatrix (cf. [9, p. 1423]). Put $\mu^{(i)} := \mu_{q^{(i)}}(\mathcal{F}^{(i)}, B^{(i)})$ and $\tau^{(i)} := \tau_{q^{(i)}}(\mathcal{F}^{(i)}, B^{(i)})$ for any $i \in \{1, \ldots, N\}$.

Applying Lemma 3.1 to each blow-up $(\mathcal{F}^{(i+1)}, B^{(i+1)}) \longrightarrow (\mathcal{F}^{(i)}, B^{(i)})$, we get

$$3\mu^{(i)} - 4\tau^{(i)} + GSV_{q^{(i)}}(\mathcal{F}^{(i)}, B^{(i)}) \leq 3\mu^{(i+1)} - 4\tau^{(i+1)} + GSV_{q^{(i+1)}}(\mathcal{F}^{(i+1)}, B^{(i+1)}).$$

In particular

$$3\mu_p(\mathcal{F}, B) - 4\tau_p(\mathcal{F}, B) + GSV_p(\mathcal{F}, B) < 3\mu^{(N)} - 4\tau^{(N)} + GSV_{q^{(N)}}(\mathcal{F}^{(N)}, B^{(N)}) = 0,$$

where the last equality holds, since $B^{(N)}$ is a smooth curve.

If B is a smooth curve then, by Remarks 3.2, and 2.1, $GSV_p(\mathcal{F}, B) = \mu_p(\mathcal{F}, B) = \tau_p(\mathcal{F}, B)$, then the inequality (1) is an equality. Reciprocally, if (1) is an equality, then B is necessarily smooth. Indeed, if B were singular, by Lemma 3.1, $3\mu_p(\mathcal{F}, B) - 4\tau_p(\mathcal{F}, B) + GSV_p(\mathcal{F}, B) < 0$, this is a contradiction. Hence, the theorem follows. □

4 Examples

In this section, we provide two examples of holomorphic foliations and separatrices that illustrate Theorem A.

Example 4.1 Let $\mathcal{G}_{m,n}$ be the germ of holomorphic foliation at $(\mathbb{C}^2, 0)$ defined by

$$\eta_{m,n} = mxdy - nydx,$$

where $1 \leq m \leq n$ are integers. The curve $B : y^m - x^n = 0$ is a separatrix of $\mathcal{G}_{m,n}$ at $0 \in \mathbb{C}^2$ and

$$GSV_0(\mathcal{G}_{m,n}, B) = m + n - mn,$$
$$\mu_0(\mathcal{G}_{m,n}, B) = 1,$$
$$\tau_0(\mathcal{G}_{m,n}, B) = 1.$$

We get

$$3\mu_0(\mathcal{G}_{m,n}, B) - 4\tau_0(\mathcal{G}_{m,n}, B) + GSV_0(\mathcal{G}_{m,n}, B) = m + n - mn - 1 = -(m-1)(n-1) \leq 0.$$

Thus, Theorem A is verified.

Example 4.2 Let $\mathcal{F}_m$ be the germ of holomorphic foliation at $(\mathbb{C}^2, 0)$ defined by

$$\omega_m = \left((2m+1)yx^{m+1} + my^{m+2}\right)dx + \left((1-m)xy^{m+1} - 2mx^{m+2}\right)dy,$$

where $m \in \mathbb{Z}_{>0}$. The curve $B : x^{2m+1} + x^m y^{m+1} + y^{2m} = 0$ is a separatrix of $\mathcal{F}$ at $0 \in \mathbb{C}^2$. We get

$$GSV_0(\mathcal{F}_m, B) = \begin{cases} 3 & \text{if } m = 1 \\ -2m^2 + 4m + 1 & \text{if } m > 1, \end{cases}$$

$$\mu_0(B) = 2m(2m-1) \qquad m \geq 1,$$

$$\tau_0(B) = \begin{cases} 2 & \text{if } m = 1 \\ 3m^2 & \text{if } m > 1, \end{cases}$$

$$\mu_0(\mathcal{F}_m, B) = \begin{cases} 5 & \text{if } m = 1 \\ 2m^2 + 2m + 1 & \text{if } m > 1 \end{cases}$$

and

$$\tau_0(\mathcal{F}_m, B) = \begin{cases} 5 & \text{if } m = 1 \\ m^2 + 4m + 1 & \text{if } m > 1. \end{cases}$$

A straightforward calculation reveals that

$$3\mu_0(\mathcal{F}_m, B) - 4\tau_0(\mathcal{F}_m, B) + GSV_0(\mathcal{F}_m, B) = \begin{cases} -2 & \text{if } m = 1 \\ -6m & \text{if } m > 1. \end{cases}$$

Therefore, Theorem A is verified.

5 Applications

We define the sum $T_p(\mathcal{F}, C) := \sum_{B \subset C} \tau_p(\mathcal{F}, B)$ of Tjurina numbers of $\mathcal{F}$ along the irreducible components of C, then we get the following inequality for foliations:

Corollary 5.1 *Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$. Let $\mathcal{B}$ be an effective primitive balanced divisor of separatrices for $\mathcal{F}$ at p. Then*

$$GSV_p(\mathcal{F}, \mathcal{B}) < 4T_p(\mathcal{F}, \mathcal{B}) - 3\mu_p(\mathcal{F}, \mathcal{B}) \tag{18}$$

Proof Suppose that $\mathcal{B} = \sum_{j=1}^{\ell} B_j$, where each B_j is a separatrix for $\mathcal{F}$. It follows from Theorem A that

$$\sum_{j=1}^{\ell} \left[3\mu_p(\mathcal{F}, B_j) - 4\tau_p(\mathcal{F}, B_j) + GSV_p(\mathcal{F}, B_j)\right] \leq -\ell.$$

Thus

$$3\sum_{j=1}^{\ell} \mu_p(\mathcal{F}, B_j) - 4T_p(\mathcal{F}, \mathcal{B}) + \sum_{j=1}^{\ell} GSV_p(\mathcal{F}, B_j) \leq -\ell$$

and, using (4) together with the GSV-index addition formula ([3, p. 29]), we obtain

$$3\left(\mu_p(\mathcal{F},\mathcal{B})+\ell-1\right)-4T_p(\mathcal{F},\mathcal{B})+GSV_p(\mathcal{F},\mathcal{B})+2\sum_{i\neq k} i_p(B_i,B_k)\leq -\ell,$$

so

$$3\mu_p(\mathcal{F},\mathcal{B})-4T_p(\mathcal{F},\mathcal{B})+GSV_p(\mathcal{F},\mathcal{B})\leq -4\ell+3-2\sum_{i\neq k} i_p(B_i,B_k).$$

Since $\ell\geq 1$, we get

$$3\mu_p(\mathcal{F},\mathcal{B})-4T_p(\mathcal{F},\mathcal{B})+GSV_p(\mathcal{F},\mathcal{B})<0.$$

□

The following example illustrates Corollary 5.1.

Example 5.2 Let $\mathcal{F}:\omega=(2x^7+5y^5)dx-xy^2(5y^2+3x^5)dy$, whose leaves are contained in the connected components of the curves $C_{\lambda,\zeta}:\zeta(y^5-x^7+x^5y^3)-\lambda x^5=0$, where $(\zeta,\lambda)\neq(0,0)$. Let $C:x=0$, and consider the effective primitive balanced divisor $\mathcal{B}=C+C_{1,1}$ of separatrices for $\mathcal{F}$ at $0\in\mathbb{C}^2$. We have $GSV_0(\mathcal{F},C)=GSV_0(\mathcal{F},C_{1,1})=5$, so

$$GSV_0(\mathcal{F},\mathcal{B})=GSV_0(\mathcal{F},C)+GSV_0(\mathcal{F},C_{1,1})-2i_0(C,C_{1,1})=5+5-2\cdot 5=0,$$

$\mu_0(\mathcal{F},\mathcal{B})=\mu_0(\mathcal{F},C)+\mu_0(\mathcal{F},C_{1,1})-deg(\mathcal{B})+1=5+21-2+1=25$, and $T_0(\mathcal{F},\mathcal{B})=\tau_0(\mathcal{F},C)+\tau_0(\mathcal{F},C_{1,1})=5+21=26$. Hence

$$GSV_p(\mathcal{F},\mathcal{B})=0<4T_p(\mathcal{F},\mathcal{B})-3\mu_p(\mathcal{F},\mathcal{B})=29.$$

We conclude the paper with a question:
Question: Is the inequality (18) true if instead of $T_p(\mathcal{F},\mathcal{B})$ we write $\tau_p(\mathcal{F},\mathcal{B})$?

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Declarations

Conflict of interest The authors of this work declare that they have no Conflict of interest.

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