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Results in Mathematics

On the Equisingularity Class of the General Higher Order Polars of Plane Branches

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Abstract. In this paper we describe the factorization of the higher order polars of a generic branch in its equisingularity class. We generalize the results of Casas-Alvero and Hefez-Hernandes-Hernández to higher order polars.

Mathematics Subject Classification. Primary 32S05; Secondary 32S15.

Keywords. Irreducible plane curve, higher order polar, equisingularity class.

1. Introduction

Let $f(x,y) \in \mathbb{C}[[x,y]]$ be an irreducible formal power series and $C = \{f(x,y) = 0\}$ be the *branch* determined by $f(x,y) = 0$. The *multiplicity* of C is the order of f. When this multiplicity is $n > 1$ we say that C is *singular*. Otherwise C is a *smooth* branch. In this paper we will consider singular branches. After a change of coordinates, if necessary, we may assume that $x = 0$ is not tangent to the curve C at the origin. This is equivalent to $\mathrm{ord} f(0,y) = \mathrm{ord} f = n$. By Newton Theorem there is $\alpha(x) = \sum_{i \geq n} a_i x^{i/n} \in \mathbb{C}[[x^{1/n}]] \subset \mathbb{C}[[x]]^*$ such that $f(x, \alpha(x)) = 0$, where $\mathbb{C}[[x]]^*$ denotes the ring of Puiseux power series. The power series $\alpha(x)$ is called a *Newton-Puiseux root* of $f(x,y)$. It is well-known that the set of all Newton-Puiseux roots of $f(x,y)$ is $\mathrm{Zer} f := \{\alpha_\epsilon(x) = \sum_{i \geq n} a_i \epsilon^i x^{i/n} \ : \ \epsilon \in \mathbb{U}_n\}$, where $\mathbb{U}_n$ is the multiplicative group of nth complex roots of unity. By Puiseux Theorem

$$f(x,y) = u(x,y) \prod_{\epsilon \in \mathbb{U}_n} (y - \alpha_\epsilon(x)), \tag{1}$$

where $u(x,y)$ is a unit in $\mathbb{C}[[x,y]]$.

The *index* of $\alpha \in \mathbb{C}[[x]]^*$ is the smallest natural number m such that α belongs to $\mathbb{C}[[x^{1/m}]]$. To any $\alpha(x) = \sum_i a_i x^{i/n} \in \mathbb{C}[[x]]^*$ of positive order and index n we associate with two finite sequences $(e_i)_i$ and $(b_i)_i$ of natural numbers as follows: $e_0 = b_0 = n$; if $e_k > 1$ then $b_{k+1} := \min\{i \ : \ a_i \neq 0;\ \gcd(e_k, i) < e_k\}$ and $e_{k+1} := \gcd(e_k, b_{k+1})$. The sequence $(e_i)_i$ is strictly decreasing and for some $h \in \mathbb{N}$ we have $e_h = 1$. The sequence $(b_0, b_1, \dots, b_h)$ is called the *characteristic* of α. By [9, Lemma 6.8] we get

$$\operatorname{ord}(\alpha_\epsilon(x) - \alpha(x)) = \frac{b_j}{n} \text{ if and only if } \epsilon \in \mathbb{U}_{e_{j-1}} \setminus \mathbb{U}_{e_j}. \tag{2}$$

Let $\lambda_l(x)$ be the sum of all terms of $\alpha(x)$ of degree strictly less than $\frac{b_l}{b_0}$. We denote by $f_l(y)$ the minimal polynomial of $\lambda_l(x)$ in the ring $\mathbb{C}[[x]][y]$. The polynomial $f_l(y)$ does not depend on the choice of $\alpha(x) \in \operatorname{Zer} f$ and its degree is $\frac{n}{e_{l-1}}$.

Observe that the characteristic of α_ϵ equals the characteristic of α. The *characteristic* of an irreducible power series $f(x, y) \in \mathbb{C}[[x, y]]$ is the characteristic of any of its Newton-Puiseux roots. The set of *characteristic exponents* of f is $\operatorname{Char}(f) = \left\{\frac{b_i}{n} \ : \ i \in \{1, \dots, h\}\right\}$. After (2) the characteristic exponents of f are the orders of differences of any two of its distinct Newton-Puiseux roots.

Let $C = \{f(x, y) = 0\}$ and $D = \{g(x, y) = 0\}$ be two curves with $f, g \in \mathbb{C}[[x, y]]$. The *intersection multiplicity* of C and D is $i_0(C, D) = \dim \mathbb{C}[[x, y]]/(f, g)$ where $(\cdot, \cdot)$ denotes the ideal generated by two power series. Usually $i_0(C, D)$ is also denoted by $i_0(f, g)$.

If C and D are branches then the *contact* of C and D is

$$\operatorname{cont}(C, D) = \operatorname{cont}(f, g) = \max\{\operatorname{ord}(\alpha - \gamma) \ : \ \alpha \in \operatorname{Zer} f,\ \gamma \in \operatorname{Zer} g\}.$$

If α is a Puiseux series and $v \in \mathbb{C}[[x, y]]$ is irreducible then we put

$$\operatorname{cont}(\alpha, v) = \max\{\operatorname{ord}(\alpha - \gamma) \ : \ \gamma \in \operatorname{Zer} v\}.$$

We say that the branches C and D are *equisingular* if and only if they have the same characteristic. We will denote by $K(b_0, b_1, \dots, b_h)$ the coset of equisingular branches of characteristic $(b_0, b_1, \dots, b_h)$. If $C = \{f(x, y) = 0\}$ is a branch in $K(b_0, b_1, \dots, b_h)$, by abuse of language we will put $f \in K(b_0, b_1, \dots, b_h)$. Let $f(x, y) = \sum_{ij} a_{ij} x^i y^j \in K(b_0, b_1, \dots, b_h)$.

We say that $f \in K(b_0, b_1, \dots, b_h)$ is *generic* in its equisingularity class if within that class the coefficients of f satisfy a Zariski-open condition.

Let A be a nonempty subset of $\mathbb{N} \times \mathbb{N}$. The *Newton diagram* $\mathcal{N}(A)$ of the set A is the convex hull of $A + (\mathbb{R}_{\geq 0})^2$, where $+$ means the Minkowski sum. By definition, the *support* of any Newton diagram Δ is $\operatorname{supp}(\Delta) := \Delta \cap \mathbb{N}^2$. We say that $\mathcal{N}(A)$ is *convenient* if it intersects both coordinate axes. The *Newton polygon* of the Newton diagram Δ is the union of the compact edges of the boundary of Δ, and we will denote it by $\delta^*(\Delta)$. A convenient Newton diagram is *elementary* if its boundary has exactly one compact edge. In this case, following Teissier [12], we will denote by $\left\{\frac{m}{n}\right\}$ the elementary Newton

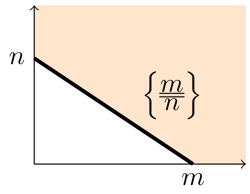

FIGURE 1. Elementary Newton diagram

diagram of $A = \{(m, 0), (0, n)\}$, for any positive natural numbers m, n (see Figure 1).

The *inclination* of the elementary Newton diagram $\left\{\frac{m}{n}\right\}$ (and of any of its translations) is m/n. Any convenient Newton diagram $\mathcal{N}$ can be written as a Minkowski sum of elementary Newton diagrams, where inclinations of successive elementary diagrams form a strictly decreasing sequence. This writing is called the *canonical representation* of $\mathcal{N}$. A convenient Newton diagram $\mathcal{N}$ can also be written as a sum of elementary Newton diagrams $\mathcal{N} = \sum_{i=1}^{r} \left\{\frac{m_i}{n_i}\right\}$ where $\gcd(m_i, n_i) = 1$ for any $i \in \{1, \ldots, r\}$ and $m_i/n_i \geq m_{i+1}/n_{i+1}$ for $i \in \{1, \ldots, r-1\}$. This new writing is called the *long canonical representation* of $\mathcal{N}$. The long canonical representation is unique.

Example 1.1. The long canonical representation of $\mathcal{N} = \left\{\frac{10}{4}\right\} + \left\{\frac{8}{6}\right\}$ is

$$\mathcal{N} = \left\{\frac{5}{2}\right\} + \left\{\frac{5}{2}\right\} + \left\{\frac{4}{3}\right\} + \left\{\frac{4}{3}\right\}.$$

Figure 2 illustrates both canonical representations.
If we drop the hypothesis of $\gcd(m_i, n_i) = 1$ in the definition of the long canonical representation we can express $\mathcal{N}$ in other ways that are not canonical, for example

$$\mathcal{N} = \left\{\frac{10}{4}\right\} + \left\{\frac{4}{3}\right\} + \left\{\frac{4}{3}\right\}.$$

The Newton diagram $\mathcal{N}(f)$ of a nonzero power series $f(x, y) = \sum_{i,j} a_{ij} x^i y^j$ is the Newton diagram $\mathcal{N}(\mathrm{supp}(f))$, where $\mathrm{supp}(f) := \{(i, j) \in \mathbb{N}^2 : a_{ij} \neq 0\}$ is the support of f. It is well-known (see [3, Lemme 8.4.2]) that if $\sum_{i=1}^{r} \left\{\frac{M_i}{N_i}\right\}$ is the canonical representation of $\mathcal{N}(f)$ then for any $i \in \{1, \ldots, r\}$ there are exactly N_i Newton-Puiseux roots of f of order $\frac{M_i}{N_i}$. Let S be a compact edge of $\mathcal{N}(f)$ of inclination p/q, where p and q are coprime integers. The initial part of $f(x, y)$ with respect to S is the quasi-homogeneous polynomial $f_S(x, y) = \sum_{ij} a_{ij} x^i y^j$ where the sum runs over all points in $S \cap \mathrm{supp}(f)$. Let $f_S(x, y) = a x^k y^l \prod_{j=1}^{r} (y^q - c_j x^p)^{s_j}$ be the factorization of f_S into irreducible factors, where k, l are non-negative integers and $a, c_j \in \mathbb{C}\backslash\{0\}$ with c_j

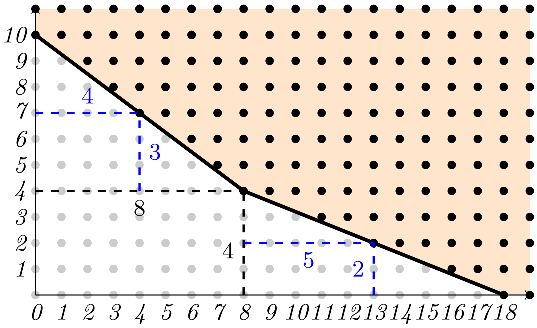

FIGURE 2. Canonical and long canonical representation of $\left\{\frac{10}{4}\right\}+\left\{\frac{8}{6}\right\}$

pairwise different. The power series $f(x,y)$ is *non-degenerate* (in the sense of Kouchnirenko [11]) on S if one of the following equivalent conditions holds:

(ND1) $s_j = 1$ for any $j \in \{1,\ldots,r\}$.
(ND2) All non-zero roots of $f_S(1,y)$ are simple.
(ND3) All Newton-Puiseux roots of f of order p/q have different initial coefficients.

Let Δ be a Newton diagram and k a nonnegative integer. The *symbolic kth derivative* $\Delta^{(k)}$ of Δ is the Newton diagram of the set $(\Delta-(0,k))\cap\mathbb{N}^2$.

Example 1.2. The symbolic first derivative of $\Delta=\left\{\frac{12}{5}\right\}$ is $\Delta^{(1)}=\left\{\frac{10}{4}\right\}$ and its symbolic second derivative is $\Delta^{(2)}=\left\{\frac{3}{1}\right\}+\left\{\frac{5}{2}\right\}$ (see Figure 3).

The main result of this paper is

Theorem 1.3. *Let $f\in\mathbb{C}[[x,y]]$ be a generic element of $K(b_0,\ldots,b_h)$. Put $e_i=\gcd(b_0,\ldots,b_i)$, $n_i=\frac{e_{i-1}}{e_i}$, $m_i=\frac{b_i}{e_i}$ and $\Delta_i=\left\{\frac{m_i}{n_i}\right\}$ for $i\in\{1,\ldots,h\}$. Fix $1\le k<b_0$ and let $\{1,\ldots,i_k\}=\{\,j\in\{1,\ldots,h\}:e_{j-1}>k\,\}$. Then $\frac{\partial^k f}{\partial y^k}$ admits the following factorization:*

$$\frac{\partial^k f}{\partial y^k}=\Gamma^{(1)}\cdots\Gamma^{(i_k)},$$

where, for any $\ell\in\{1,\ldots,i_k\}$, the power series $\Gamma^{(\ell)}$ is not necessarily irreducible, and it verifies:

(1) $\operatorname{cont}(f,v)=\frac{b_\ell}{b_0}$ for any irreducible factor v of $\Gamma^{(\ell)}$.
(2) Let t be the natural number such that $0<t\le n_\ell$ and $t\equiv k\ (\mathrm{mod}\ n_\ell)$. If $\sum_{j=1}^{r}\left\{\frac{M_j}{N_j}\right\}$ is the long canonical representation of $\Delta_\ell^{(t)}$ and $m=$

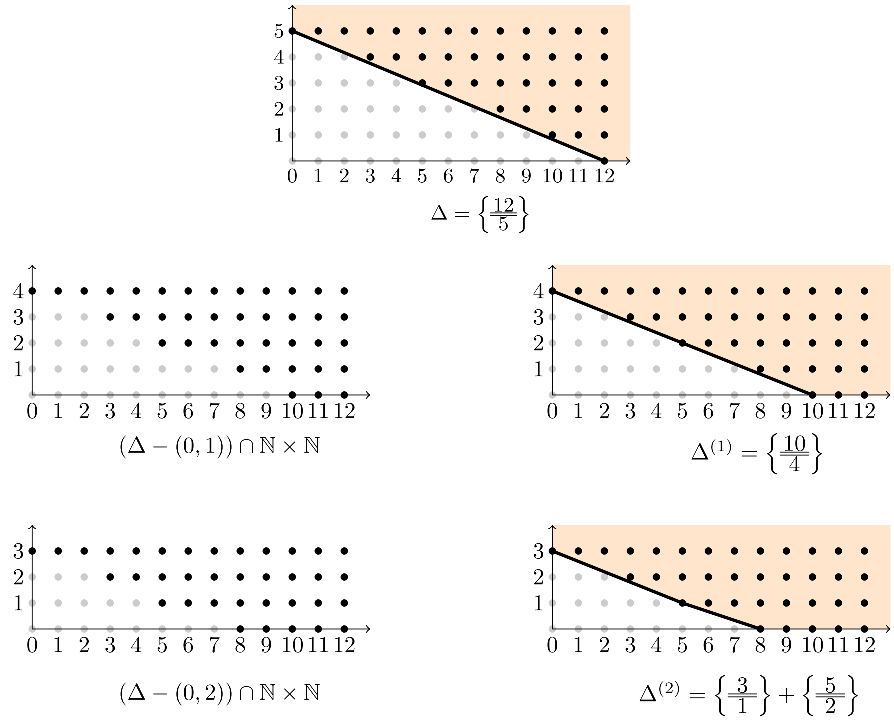

FIGURE 3. Symbolic derivatives

$\min\{e_\ell, k\} - \lceil \frac{k}{n_\ell} \rceil$ *then* $\Gamma^{(\ell)}$ *can be written as a product of irreducible factors*

$$\Gamma^{(\ell)} = \prod_{j=1}^{r} z_j^{(\ell)} \prod_{i=1}^{m} w_i^{(\ell)}$$

such that

(2a) for any power series $z_j^{(\ell)}$, $\mathrm{cont}(f_\ell, z_j^{(\ell)}) = \frac{M_j}{n_1 \cdots n_{\ell-1} N_j}$ *and*

$$Char(z_j^{(\ell)}) = \begin{cases} \left\{\frac{b_1}{b_0}, \ldots, \frac{b_{\ell-1}}{b_0}\right\} & \text{if } N_j = 1 \\ \left\{\frac{b_1}{b_0}, \ldots, \frac{b_{\ell-1}}{b_0}, \frac{M_j}{n_1 \cdots n_{\ell-1} N_j}\right\} & \text{if } N_j > 1. \end{cases}$$

(2b) for any power series $w_i^{(\ell)}$, $\mathrm{Char}(w_i^{(\ell)}) = \left\{\frac{b_1}{b_0}, \ldots, \frac{b_\ell}{b_0}\right\}$ *and the contact* $\mathrm{cont}(f_\ell, w_i^{(\ell)}) = \frac{b_\ell}{b_0}$.

(2c) $\mathrm{cont}(v_1, v_2) = \min\{\mathrm{cont}(f_l, v_1), \mathrm{cont}(f_l, v_2)\}$ *for any two different irreducible factors* v_1, v_2 *of* $\Gamma^{(\ell)}$.

The curve $\left\{\frac{\partial f}{\partial y}=0\right\}$ is called the *first polar* of $C=\{f(x,y)=0\}$ and $\left\{\frac{\partial^k f}{\partial y^k}=0\right\}_{k=2}^{b_0-1}$ are called *higher order polars* of C.

Theorem 1.3 will be proved in Section 4. It improves the results of Casas-Alvero (see [2]) and those of the first and second author (see [7]) since, under the hypothesis that f is generic in its equisingularity class, we fully describe the equisingularity class of the considered polar curve. Theorem 1.3 also generalizes the results of Casas-Alvero (see [1]) and Hefez-Hernandes-Hernández (see [10]) to higher order polars.

It is well known that the equisingularity class of the polar curves can vary within a family of equisingular branches. The motivation of this paper was to prove that it is fixed and independent of the analytical type of the branches, under the hypothesis that they are generic in their equisingularity class.

2. Symbolic Derivatives of a Newton Diagram

In this section we prove some properties of the symbolic derivatives of a Newton diagram.

Property 2.1. For any Newton diagram Δ and any nonnegative integers k, l we have $(\Delta^{(k)})^{(l)}=\Delta^{(k+l)}$.

Proof. Note that $\operatorname{supp}(\Delta^{(k+l)})=(\operatorname{supp}(\Delta)-(0,k+l))\cap\mathbb{N}^2=(\operatorname{supp}(\Delta^{(k)})-(0,l))\cap\mathbb{N}^2=\operatorname{supp}(\Delta^{(k)})^{(l)}$. $\square$

Let $\omega\in(\mathbb{R}_{>0})^2$ and Δ be any Newton diagram. The *ω-weigthed initial part* of Δ is

$$\operatorname{in}_\omega(\Delta):=\{d\in\Delta\ :\ \langle d,\omega\rangle=\min\{\langle e,\omega\rangle\ :\ e\in\Delta\}\},$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical scalar product in $\mathbb{R}^2$.

The Minkowski sum of Newton diagrams satisfies the following property (see [5, Theorem 1.5, Chapter IV]).

Property 2.2. Let Δ_1,Δ_2 be two Newton diagrams and $\omega\in(\mathbb{R}_{>0})^2$. Then

$$\operatorname{in}_\omega(\Delta_1+\Delta_2)=\operatorname{in}_\omega(\Delta_1)+\operatorname{in}_\omega(\Delta_2).$$

Remark 2.3. Let

$$\mathcal{N}=\sum_{i=1}^{r}\left\{\frac{M_i}{N_i}\right\} \tag{3}$$

be the canonical or the long canonical representation of $\mathcal{N}$. For any $0\le j\le r$ we put $A_j:=(a_j,b_j)$, where $a_j=\sum_{i=j+1}^{r}M_i$ and $b_j=\sum_{i=1}^{j}N_i$ (see Figure 4).

Note that the set $T:=\{A_j\ :\ 0\le j\le r\}$ is a subset of the Newton polygon of $\mathcal{N}$ containing the vertices of $\mathcal{N}$, with equality if and only if (3) is the canonical representation of $\mathcal{N}$. In fact, if we consider $\omega_j:=(N_j,M_j)$ then

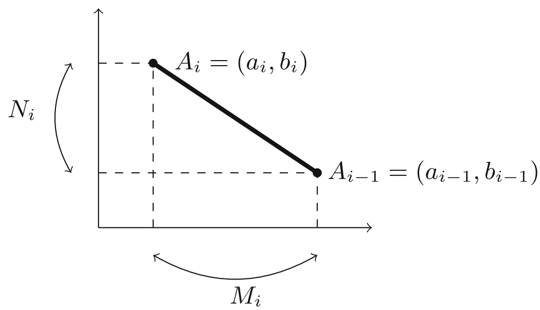

FIGURE 4. Points A_{i-1} and A_i

$(M_i, 0) \in \mathrm{in}_{\omega_j}\left(\left\{\frac{M_i}{N_i}\right\}\right)$ for any $i > j$ and $(0, N_i) \in \mathrm{in}_{\omega_j}\left(\left\{\frac{M_i}{N_i}\right\}\right)$ for any $i \le j$. By Property 2.2 we have $(a_j, b_j) \in \mathrm{in}_{\omega_j}(\Delta)$, so $A_j \in \delta^*(\Delta)$.

For any Newton polygon Δ let $\mathrm{trunc}(\Delta, k) := \mathcal{N}(\{(i,j) \in \Delta \cap \mathbb{N}^2 \ : \ j \ge k\})$. It follows directly from the definitions that

$$\mathrm{trunc}(\Delta, k) = \Delta^{(k)} + (0, k). \tag{4}$$

Proposition 2.4. *Let $\Delta = \sum_{i=1}^{r}\left\{\frac{M_i}{N_i}\right\}$ be the long canonical representation of the convenient Newton diagram Δ. Put $R = \sum_{i=1}^{s}\left\{\frac{M_i}{N_i}\right\}$, $L = \sum_{i=s+1}^{r}\left\{\frac{M_i}{N_i}\right\}$ and assume that $0 \le k \le \sum_{i=1}^{s} N_i$. Then*

$$\Delta^{(k)} = R^{(k)} + L. \tag{5}$$

Proof. By Remark 2.3 the points $A_j = (a_j, b_j) = (\sum_{i=j+1}^{r} M_i, \sum_{i=1}^{j} N_i)$ for $0 \le j \le r$ are lattice points of $\delta^*(\Delta)$.

Let $R_1 = \mathcal{N}(\{A_0, \ldots, A_s\})$. Since the points $A_0, \ldots, A_s$ are the lattice points of $\delta^*(R_1)$, we get by Remark 2.3 that $R_1 = (a_s, 0) + R$. The same argument applies $L_1 = \mathcal{N}(\{A_s, \ldots, A_r\})$ giving $L_1 = (0, b_s) + L$. Since $\Delta = L_1 \cup R_1$, we get $\mathrm{trunc}(\Delta, k) = L_1 \cup \mathrm{trunc}(R_1, k)$, and consequently $\mathrm{trunc}(\Delta, k) = L + \mathrm{trunc}(R, k)$. Hence equality (5) follows. $\square$

Corollary 2.5. *Let $\Delta = \sum_{i=1}^{r}\left\{\frac{M_i}{N_i}\right\}$ be the long canonical representation of a convenient Newton diagram Δ. Then*

$$\Delta^{(1)} = \left\{\frac{M_1}{N_1}\right\}^{(1)} + \sum_{i=2}^{r}\left\{\frac{M_i}{N_i}\right\}.$$

Recall the notion of continued fraction expansions of rational numbers.

Let $n, m \in \mathbb{N}$ with $0 < n < m$. Denote by $[h_0, h_1, \ldots, h_s]$ the **continued fraction expansion** of $\frac{m}{n}$, that is:

$$\frac{m}{n} = h_0 + \cfrac{1}{h_1 + \cfrac{1}{h_2 + \cfrac{1}{\ddots + \cfrac{1}{h_s}}}}. \tag{6}$$

Note that the expansion given in equation (6) is unique if we impose the condition that $h_s > 1$, that is, s is the minimal possible value. This is the classical definition of a continued fraction expansion. However, if $h_s > 1$, then $[h_0, h_1, \ldots, h_s] = [h_0, h_1, \ldots, h_s - 1, 1]$. Therefore, if necessary, we can always assume that s is even.

Given the expansion (6), we put $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = h_0$, $q_0 = 1$ and consider the irreducible fractions

$$\frac{p_i}{q_i} = [h_0, h_1, \ldots, h_i] = h_0 + \cfrac{1}{h_1 + \cfrac{1}{h_2 + \cfrac{1}{\ddots + \cfrac{1}{h_i}}}}$$

for $1 \leq i \leq s$. The next properties are well-known (see for example [13]).

Properties 2.6. With the above notations we have:

(1) $p_{i+1} = h_{i+1} p_i + p_{i-1}$ and $q_{i+1} = h_{i+1} q_i + q_{i-1}$, for $0 \leq i \leq s-1$.
(2) $p_i q_{i-1} - p_{i-1} q_i = (-1)^{i+1}$.
(3) $\gcd(p_i, q_i) = 1$.
(4) $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots \leq \frac{m}{n}$.
(5) $\frac{p_1}{q_1} > \frac{p_3}{q_3} > \cdots \geq \frac{m}{n}$.

Observe that $\frac{p_s}{q_s} = \frac{m}{n}$. If m and n are coprime then $m = p_s$ and $n = q_s$.

Proposition 2.7. *If* $\Delta = \left\{\frac{m}{n}\right\}$ *with* $n, m \in \mathbb{N}$ *coprime then*

$$\Delta^{(1)} = \begin{cases} \sum_{i=1}^{s/2} h_{2i} \left\{\frac{p_{2i-1}}{q_{2i-1}}\right\} & \text{if } s \text{ is even} \\ \sum_{i=1}^{(s-1)/2} h_{2i} \left\{\frac{p_{2i-1}}{q_{2i-1}}\right\} + \left\{\frac{p_s - p_{s-1}}{q_s - q_{s-1}}\right\} & \text{if } s \text{ is odd.} \end{cases}$$

In particular if $\Delta = \left\{\frac{m}{1}\right\}$ *then* $\Delta^{(1)} = \left\{\frac{0}{0}\right\}$, *that is the first quadrant.*

Proof. Suppose that s is even. Consider the Newton diagram

$$N := (0,1) + \sum_{i=1}^{s/2} h_{2i} \left\{\frac{p_{2i-1}}{q_{2i-1}}\right\} = (0,1) + \sum_{i=1}^{s/2} \left\{\frac{h_{2i} p_{2i-1}}{h_{2i} q_{2i-1}}\right\}.$$

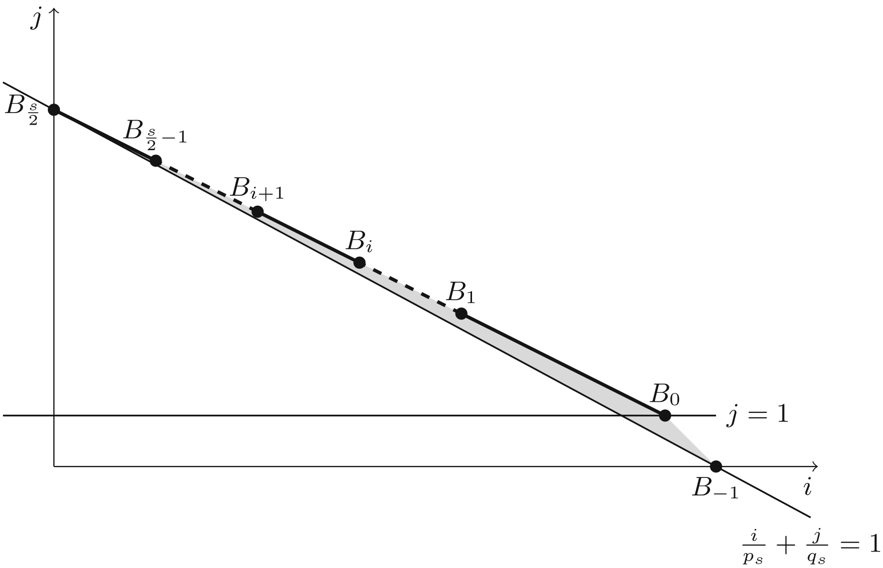

FIGURE 5. Points B_i

Since $p_1/q_1 > p_3/q_3 > \cdots > p_{s-1}/q_{s-1}$, we get by Remark 2.3 that the points

$$B_i := \Big(\sum_{j=i+1}^{s/2} h_{2j}p_{2j-1},\, 1+\sum_{j=1}^{i} h_{2j}q_{2j-1}\Big)$$

are the vertices of N for $i = 0, \ldots, s/2$. By the first item of Properties 2.6 we get

$$B_i = \left(\sum_{j=i+1}^{s/2}(p_{2j}-p_{2j-2}),\, 1+\sum_{j=1}^{i}(q_{2j}-q_{2j-2})\right) = (p_s - p_{2i}, q_{2i})$$

for $i = 0, \ldots, s/2$.

We claim that

$$N = \mathcal{N}(\{B_0, \ldots, B_{s/2}\}) = \operatorname{trunc}(\Delta, 1). \tag{7}$$

Consider the closed polygon $\mathcal{P}$ which vertices are $B_{-1} := (p_s, 0), B_0, \ldots, B_{s/2}$ (see Figure 5).

In order to prove equality (7) it is enough to show that there are no lattice points in the interior of $\mathcal{P}$. Let B denote the number of lattice points on the boundary of the polygon $\mathcal{P}$ and let I denote the number of lattice points in its interior.

By the third item of Properties 2.6 we get $B = 2+\sum_{i=0}^{s/2-1} h_{2i+2}$. By Pick's Formula [4, Theorem 13.51], we have $2\operatorname{Area}\mathcal{P} = 2I + B - 2$. On the other hand if $\triangle_i$ denotes the triangle of vertices O, B_{i-1}, B_i for $i = 0, \ldots, r$ then $2\mathrm{Area}\,\mathcal{P} = \sum_{i=0}^{s/2} 2\mathrm{Area}\,\triangle_i - p_s q_s$. We have $2\mathrm{Area}\,\triangle_0 = p_s$ and $2\mathrm{Area}\,\triangle_i = (p_s - p_{2i-2})q_{2i} - (p_s - p_{2i})q_{2i-2} = p_s q_{2i} - p_s q_{2i-2} + p_{2i}q_{2i-2} - p_{2i-2}q_{2i} = p_s q_{2i} - p_s q_{2i-2} + (h_{2i}p_{2i-1} + p_{2i-2})q_{2i-2} - p_{2i-2}(h_{2i}q_{2i-1} + q_{2i-2}) = p_s q_{2i} - p_s q_{2i-2} + h_{2i}(p_{2i-1}q_{2i-2} - p_{2i-2}q_{2i-1}) = p_s q_{2i} - p_s q_{2i-2} + h_{2i}$ for $i = 1, \ldots, s/2$. Hence

$$2\mathrm{Area}\,\mathcal{P} = p_s + \sum_{i=1}^{s/2}(p_s q_{2i} - p_s q_{2i-2} + h_{2i}) - p_s q_s = \sum_{i=1}^{s/2} h_{2i}.$$

Therefore $2\mathrm{Area}\,\mathcal{P} = B - 2$, and so $I = 0$.

Suppose now that s is odd. Note that $[h_0, \ldots, h_s]$ can be represented by the even continued fraction $[h_0, \ldots, h_s - 1, 1]$. We have $[h_0, \ldots, h_s - 1] = \frac{\tilde{p}_s}{\tilde{q}_s}$, where, by the first item of Properties 2.6, we get $\tilde{p}_s = (h_s - 1)p_{s-1} + p_{s-2} = p_s - p_{s-1}$ and $\tilde{q}_s = (h_s - 1)q_{s-1} + q_{s-2} = q_s - q_{s-1}$. Therefore the proof for the odd case follows from the statement for the even case. $\square$

3. Technical Tools

The *extreme right edge* of a Newton polygon is its compact edge of greatest inclination.

Lemma 3.1. *Let $\lambda(x) = \sum_{i=1}^{N} a_i x^{i/n}$ be a finite Puiseux power series of characteristic $(n, b_1, \ldots, b_l)$ and let $v \in \mathbb{C}[[x, y]]$ be an irreducible power series such that $v(x, \lambda(x)) \neq 0$. Let*

$$\sum_{i=1}^{r}\left\{\frac{M_i}{N_i}\right\}, \tag{8}$$

be the canonical representation of the Newton diagram of $\hat{v}(x, y) := v(x^n, y + \lambda(x^n))$. Then

(i) $M_i/N_i \leq N$ for all i such that $1 < i \leq r$,
(ii) $M_1/N_1 = n\,\mathrm{cont}(\lambda, v)$,
(iii) if $M_1/N_1 > N$ and $\hat{v}$ is non-degenerate on the extreme right edge of its Newton polygon, then M_1 and N_1 are coprime and

$$Char(v) = \begin{cases} \left(\frac{b_1}{n}, \ldots, \frac{b_\ell}{n}\right) & \text{if } N_1 = 1 \\ \left(\frac{b_1}{n}, \ldots, \frac{b_\ell}{n}, \frac{M_1}{nN_1}\right) & \text{if } N_1 > 1. \end{cases}$$

Proof. Let $\alpha_1, \ldots, \alpha_m$ be the Newton-Puiseux roots of v. Then the set of Newton-Puiseux roots of $\hat{v}$ equals $\{\alpha_i(x^n) - \lambda(x^n) \,:\, 1 \leq i \leq m\}$. Hence the set of inclinations of the edges of the Newton diagram of $\hat{v}$ is equal to $\{\,n\,\mathrm{ord}(\alpha_i(x) - \lambda(x)) \,:\, 1 \leq i \leq m\,\}$. In particular the biggest inclination M_1/N_1 of the Newton polygon of $\hat{v}(x, y)$ equals $n\,\mathrm{cont}(\lambda, v)$, which gives (ii).

If $M_1/N_1 \leq N$ then (i) is clearly true. Hence in what follows, assume that $n\,\mathrm{cont}(\lambda, v) = M_1/N_1 > N$. Then any Newton-Puiseux root α_i of v that realizes the contact with λ has the form $\alpha_i = \lambda + c_i x^{M_1/(nN_1)} + \cdots$ with some $c_i \neq 0$. Thus for any $1 \leq j \leq m$: either $\mathrm{ord}(\alpha_j - \lambda) = M_1/(nN_1)$ or $\mathrm{ord}(\alpha_j - \lambda) \leq N/n$. This proves (i).

Assume that $\hat{v}$ is non-degenerate on the compact edge S of (8) of inclination M_1/N_1 and suppose to the contrary that α_i has a characteristic exponent γ bigger than $M_1/(nN_1)$. Then there exists $k \neq i$ such that $\gamma = \mathrm{ord}(\alpha_k - \alpha_i)$. This implies that $c_i x^{M_1/N_1}$ is the initial term of both $\alpha_i(x^n) - \lambda(x^n)$ and $\alpha_k(x^n) - \lambda(x^n)$. Consequently after (ND3), $\hat{v}$ is degenerate on the edge S which is a contradiction. Thus all characteristic exponents of α_i are less than of equal to $M_1/(nN_1)$.

By (8) there are N_1 Newton-Puiseux roots of $\hat{v}$ of order $\frac{M_1}{N_1}$. Write $\frac{M_1}{nN_1} = \frac{m_{l+1}}{n \cdot n_{l+1}}$ with m_{l+1} and n_{l+1} coprime. According to (2) there are n_{l+1} Newton-Puiseux roots α_j of v such that $\mathrm{ord}(\alpha_j - \alpha_i) > \frac{b_l}{n}$. These Newton-Puiseux roots of v yield the Newton-Puiseux roots $\alpha_j(x^n) - \lambda(x^n)$ of $\hat{v}$ of order M_1/N_1. Hence $N_1 = n_{l+1}$. This proves (iii). $\square$

Corollary 3.2. *Let $\lambda = \sum_{i=1}^{N} a_i x^{i/n}$ be a finite Puiseux series of characteristic $(n, b_1, \ldots, b_l)$ with minimal polynomial $g \in \mathbb{C}[[x]][y]$. Let $v \in \mathbb{C}[[x, y]]$ be a power series coprime with g. Set $\hat{v}(x, y) = v(x^n, y + \lambda(x^n))$. Let*

$$\sum_{i=1}^{s} \Big\{ \frac{M_i}{N_i} \Big\}$$

be the long canonical representation of $\mathcal{N}(\hat{v})$. Assume that for some rational number $q \geq N$ the power series $\hat{v}$ is non-degenerate on all edges of inclination bigger than q. Let r be the number of elements of the set $\{\, i \in \{1, \ldots, s\} : M_i/N_i > q \,\}$. Then there exists a decomposition $v = \prod_{i=1}^{a} v_i$ into irreducible factors in $\mathbb{C}[[x, y]]$ such that:

(i) $\mathrm{cont}(v_i, g) > q/n$ *if and only if* $1 \leq i \leq r$,

(ii) for every $1 \leq i \leq r$; $\mathrm{cont}(v_i, g) = \frac{M_i}{n \cdot N_i}$ *and*

$$Char(v_i) = \begin{cases} \left(\frac{b_1}{n}, \ldots, \frac{b_\ell}{n}\right) & \text{if } N_i = 1 \\ \left(\frac{b_1}{n}, \ldots, \frac{b_\ell}{n}, \frac{M_i}{nN_i}\right) & \text{if } N_i > 1, \end{cases}$$

(iii) for every $1 \leq i < j \leq r$; $\mathrm{cont}(v_i, v_j) = \min\{\mathrm{cont}(v_i, g), \mathrm{cont}(v_j, g)\}$.

Proof. Let $v = \prod_{i=1}^{a} v_i$ be a decomposition of v into irreducible factors in $\mathbb{C}[[x, y]]$ such that $\mathrm{cont}(g, v_i) \geq \mathrm{cont}(g, v_{i+1})$, for $1 \leq i < a$. Choose $r' \in \{1, \ldots, a\}$ such that $\mathrm{cont}(v_i, g) > q/n$ for any $1 \leq i \leq r'$ and $\mathrm{cont}(v_i, g) \leq q/n$ for any $r' + 1 \leq i \leq a$. Then by Lemma 3.1, for $1 \leq i \leq r'$, the Newton diagram of $\hat{v}_i := v_i(x^n, y + \lambda(x^n))$ has one edge L of inclination $n\,\mathrm{cont}(v_i, g)$ and all other edges have inclinations not greater than q. Let $V_i = \hat{v}/\hat{v}_i$.

Then $\mathcal{N}(\hat{v}) = \mathcal{N}(V_i) + \mathcal{N}(\hat{v}_i)$. In particular $\mathcal{N}(\hat{v})$ has an edge S of inclination $n\,\mathrm{cont}(v_i, g)$. Since $\hat{v}$ is non-degenerate on S and the initial part of $\hat{v}_i$ with respect to L divides the initial part of $\hat{v}$ with respect to S, we get, by (ND1) that $\hat{v}_i$ is non-degenerate on L. By (ii) and (iii) of Lemma 3.1, the long canonical representation of $\mathcal{N}(\hat{v}_i)$ has only one elementary Newton diagram (corresponding to L) of inclination greater than q. For $r' + 1 \leq i \leq a$, (i) and (ii) of Lemma 1 imply that the inclinations of $\mathcal{N}(\hat{v}_i)$ are less than or equal to q.

From the identity

$$\mathcal{N}(\hat{v}) = \sum_{i=1}^{a} \mathcal{N}(\hat{v}_i)$$

we have $r = r'$ and the extreme right compact edges of $\mathcal{N}(\hat{v}_i)$ for $1 \leq i \leq r$ are in one-to-one correspondence with the set of elementary Newton diagrams $\left\{\left\{\frac{M_i}{N_i}\right\}\right\}_{i=1}^{r}$ of the long canonical representation of $\mathcal{N}(\hat{v})$.

Then, by (iii) of Lemma 3.1, (i) and (ii) hold true.

Suppose that there exists $1 \leq i < j \leq r$ such that the conclusion of (iii) does not hold. This is possible only if $\mathrm{cont}(v_i, g) = \mathrm{cont}(v_j, g) < \mathrm{cont}(v_i, v_j)$. Let α_{i_0} be a Newton-Puiseux root of v_i such that $\mathrm{ord}(\alpha_{i_0} - \lambda) = \mathrm{cont}(v_i, g)$ and let α_{j_0} be a Newton-Puiseux root of v_j such that $\mathrm{ord}(\alpha_{j_0} - \alpha_{i_0}) = \mathrm{cont}(v_i, v_j)$. Then the Puiseux series $\alpha_{i_0}(x^n) - \lambda(x^n)$, $\alpha_{j_0}(x^n) - \lambda(x^n)$ have the same initial term of order $n\,\mathrm{cont}(v_i, g)$. Hence, by (ND3), $\hat{v}$ is degenerate on the edge of inclination $n\,\mathrm{cont}(v_i, g)$. This contradiction gives (iii). $\square$

Remark 3.3. For any positive integers r, s we have the epimorphism of groups $\mathbb{U}_r \ni \epsilon \longrightarrow \epsilon^s \in \mathbb{U}_{r/\gcd(r,s)}$. This becomes an isomorphism when r, s are coprime.

Properties 3.4. Let $n \in \mathbb{N}$, $n > 1$. Consider the strictly decreasing sequence $n = e_0 > e_1 > \cdots > e_h = 1$ from page 1. Put $n_i = \frac{e_{i-1}}{e_i}$ for $1 \leq i \leq h$. Then for any $l \in \{1, \ldots, h\}$ we get:

(1) $\prod_{\varepsilon \in \mathbb{U}_{e_{l-1}}} (t - c\varepsilon^{b_l}) = (t^{n_l} - c^{n_l})^{e_l}$ for any $c \in \mathbb{C}$.

(2) $\prod_{\varepsilon \in \mathbb{U}_{e_{l-1}} \setminus \mathbb{U}_{e_l}} (1 - \varepsilon^{b_l}) = n_l^{e_l}$.

(3) $\sum_{\varepsilon \in \mathbb{U}_{n_l}} \varepsilon^i = \begin{cases} n_l, & \text{if } i \equiv 0 \pmod{n_l} \\ 0, & \text{otherwise.} \end{cases}$

Proof. By Remark 3.3 the map $U_{e_{l-1}} \ni \varepsilon \to \varepsilon^{b_l} \in U_{n_l}$ is a group epimorphism, so

$$\prod_{\varepsilon \in U_{e_{l-1}}} (t - c\varepsilon^{b_l}) = \prod_{\tau \in U_{n_l}} (t - c\tau)^{e_l} = (t^{n_l} - c^{n_l})^{e_l}.$$

In order to prove (2) consider $h(x) := \prod_{\tau \in U_{n_l} \setminus \{1\}} (x - \tau)$. We have $(x-1)h(x) = x^{n_l} - 1$, hence $h(x) + (x-1)h'(x) = \frac{d}{dx}(x^{n_l} - 1) = n_l x^{n_l - 1}$. Substituting $x = 1$

we get $h(1) = n_l$ which gives

$$\prod_{\varepsilon \in U_{e_{l-1}} \setminus U_{e_l}} (1 - \varepsilon^{b_l}) = \prod_{\tau \in U_{n_l} \setminus \{1\}} (1-\tau)^{e_l} = n_l^{e_l}.$$

Statement (3) follows from Remark 3.3. □

Let $f(x,y) = \sum_{ij} a_{ij} x^i y^j \in \mathbb{C}[[x,y]$ and $\omega = (\omega_1, \omega_2) \in \mathbb{Q}^2_{>0}$. The *$\omega$-weighted order of f* is $\mathrm{ord}_\omega(f) = \min\{\omega_1 i + \omega_2 j \ : \ a_{ij} \neq 0\}$ and the *ω-weighted initial form of f* is $\mathrm{in}_\omega(f) = \sum_{ij} a_{ij} x^i y^j$, where the sum runs over (i,j) such that $\omega_1 i + \omega_2 j = \mathrm{ord}_\omega(f)$.

Lemma 3.5. *Let $f \in K(n, b_1, \ldots, b_h)$ and $\alpha = \sum_{i \geq n} a_i x^{i/n}$ be a Newton-Puiseux root of f. Let $\lambda = \sum_{i=n}^{b_l - 1} a_i x^{i/n}$ and $\hat{f}(x,y) = f(x^{n/e_{l-1}}, y + \lambda(x^{n/e_{l-1}}))$. Let Δ be the Newton diagram of $\hat{f}$. Then, for $k < e_{l-1}$ we have $\Delta^{(k)} = R^{(t)} + L$ where $R = \left\{\frac{m_l}{n_l}\right\}$ and t is the remainder of the division of k by n_l. The inclination of every compact edge of L is smaller than or equal to m_l/n_l and the inclination of every compact edge of $R^{(t)}$ is bigger than m_l/n_l. Moreover if f is a generic member of $K(n, \ldots, b_h)$ then*

$$\mathcal{N}\left(\frac{\partial^k \hat{f}}{\partial y^k}\right) = \Delta^{(k)} \tag{9}$$

and $\frac{\partial^k \hat{f}}{\partial y^k}$ is non-degenerate on all edges of its Newton diagram with inclinations bigger than m_l/n_l.

Proof. Observe that $\lambda(x^{n/e_{l-1}}) \in \mathbb{C}[[x]]$, so $\hat{f}(x,y) = f(x^{n/e_{l-1}}, y + \lambda(x^{n/e_{l-1}}))$ is a formal power series in $\mathbb{C}[[x,y]]$. The set of Newton-Puiseux roots of $\hat{f}(x,y)$ is $\mathrm{Zer}\hat{f} = \{\alpha_\epsilon(x^{n/e_{l-1}}) - \lambda(x^{n/e_{l-1}}) \ : \ \epsilon \in \mathbb{U}_n\}$. Hence $\{\mathrm{ord}(\gamma) \ : \ \gamma \in \mathrm{Zer}\hat{f}\} = \left\{\frac{b_j}{e_{l-1}} \ : \ j = 1, \ldots, l\right\}$. In particular the biggest inclination of the Newton diagram $\mathcal{N}(\hat{f})$ equals $\frac{m_l}{n_l}$. Denote by S the compact edge of $\mathcal{N}(\hat{f})$ of this inclination. If g is the minimal polynomial of $\lambda(x)$ then g is a l-semiroot of f, that is, $g \in \mathbb{C}[[x]][y]$ is monic, irreducible, its y-degree equals n/e_{l-1} and the intersection multiplicity of f and g is $\bar{b}_l := b_l + \sum_{i=1}^{l-1} \left(\frac{e_{i-1} - e_i}{e_{l-1}}\right) b_i$ (see [15, Theorem 3.9 (a)]). Hence the vertex of S living on the horizontal axis is $(\bar{b}_l, 0)$ since $i_0(\hat{f}, y) = \mathrm{ord}(f(x^{n/e_{l-1}}, \lambda(x^{n/e_{l-1}})) = i_0(f,g)$. On the other hand the length of the vertical projection of L equals the cardinality of the set

$$\{\alpha_\epsilon \in \mathrm{Zer}f \ : \ \mathrm{ord}(\alpha_\epsilon(x^{n/e_{l-1}}) - \lambda(x^{n/e_{l-1}})) = \tfrac{m_l}{n_l}\} = \{\alpha_\epsilon \in \mathrm{Zer}f \ : \ \mathrm{ord}(\alpha_\epsilon - \alpha) \geq \tfrac{b_l}{n}\}$$

which is, after (2), equal to e_{l-1}.

Let $k = qn_l + t$ be the Euclidean division of k by n_l. Then $\Delta = q\left\{\frac{m_l}{n_l}\right\} + \left\{\frac{m_l}{n_l}\right\} + L$, for some Newton diagram L with inclinations less than or equal $\frac{m_l}{n_l}$.

Consequently $\Delta^{(k)} = \left(\Delta^{(qn_l)}\right)^{(t)} = \left(\left\{\frac{m_l}{n_l}\right\} + L\right)^{(t)} = \left\{\frac{m_l}{n_l}\right\}^{(t)} + L$ where the first equality follows from Property 2.1, the second one follows from Proposition 2.4 since $\left(q\left\{\frac{m_l}{n_l}\right\}\right)^{(qn_l)} = \left\{\frac{0}{0}\right\}$ and the third equality also follows from Proposition 2.4 .

Now we are going to prove the second part of the lemma.

Suppose first that f is a Weierstrass polynomial, that is f is as in (1) with $u(x,y) = 1$. Then

$$\hat{f}(x,y) = \prod_{\epsilon\in\mathbb{U}_n} (y - (\alpha_\epsilon(x^{n/e_{l-1}}) - \lambda(x^{n/e_{l-1}}))). \tag{10}$$

Fix $q \in \{1,\ldots,n_l-1\}$ and let $z_q := a_{b_l+qe_l}$ be a coefficient of α treated as indeterminate. Expand $\hat{f}$ as a polynomial in z_q

$$\hat{f} = \hat{f}_{q,0} + \hat{f}_{q,1}z_q + \cdots + \hat{f}_{q,n}z_q^n. \tag{11}$$

Consider $\omega := (1, m_l/n_l)$.

Claim 1. The $\omega-$weighted order of $\hat{f}$ is $\mathrm{ord}_\omega(\hat{f}) = \bar{b}_l$ and the $\omega-$weighted initial form of $\hat{f}$ is

$$\mathrm{in}_\omega\hat{f} = ax^b(y^{n_l} - a_{b_l}^{n_l}x^{m_l})^{e_l} \tag{12}$$

for some nonzero complex number a and a nonnegative integer b.

Indeed, after (10) $\mathrm{in}_\omega\hat{f} = \prod_{\epsilon\in\mathbb{U}_n}\mathrm{in}_\omega A_\epsilon$, where $A_\epsilon = y - (\alpha_\epsilon(x^{n/e_{l-1}}) - \lambda(x^{n/e_{l-1}}))$. Notice that

$$\mathrm{in}_\omega A_\epsilon = \begin{cases} (1-\epsilon^{b_j})a_{b_j}x^{b_j/e_{l-1}}, \text{ if } \epsilon \in \mathbb{U}_{e_{j-1}}\backslash\mathbb{U}_{e_j} \text{ for } 1 \le j \le l-1 \\ \\ y - a_{b_j}\epsilon^{b_l}x^{b_l/e_{l-1}} \qquad \text{if } \epsilon \in \mathbb{U}_{e_{l-1}}. \end{cases}$$

Hence

$$\mathrm{in}_\omega\hat{f} = \left(\prod_{j=1}^{l-1}\prod_{\epsilon\in\mathbb{U}_{e_{j-1}}\backslash\mathbb{U}_{e_j}}(1-\epsilon^{b_j})a_{b_j}x^{b_j/e_{l-1}}\right)\prod_{\epsilon\in\mathbb{U}_{e_{l-1}}}\left(y - a_{b_l}\epsilon^{b_l}x^{b_l/e_{l-1}}\right).$$

By Properties 3.4 we get

$$\mathrm{in}_\omega\hat{f} = \left(\prod_{j=1}^{l-1} n_j^{e_j}a_{b_j}^{e_{j-1}-e_j}x^{b_j(e_{j-1}-e_j)/e_{l-1}}\right)(y^{n_l} - a_{b_l}^{n_l}x^{b_l/e_l})^{e_l}.$$

Notice that $b_l/e_l = m_l$ and the proof of Claim 1 follows taking $a := \prod_{j=1}^{l-1} n_j^{e_j}a_{b_j}^{e_{j-1}-e_j}$ and $b := \mathrm{ord}_x\hat{f}(x,0) - b_l = \bar{b}_l - b_l \in \mathbb{N}$.

Claim 2. Let $q \in \{1, \dots, n_l - 1\}$ and $\hat{f}_{q,0}$, $\hat{f}_{q,1}$ be as in (11). Then $\mathrm{ord}_\omega(\hat{f}_{q,1}) = \bar{b}_l + \frac{q}{n_l}$ and

$$\mathrm{in}_\omega(\hat{f} - \hat{f}_{q,0}) = \mathrm{in}_\omega \hat{f}_{q,1} z_q = -e_{l-1} a a_{b_l}^{s-1} x^{b+(m_l s+q)/n_l} y^{n_l - s}(y^{n_l} - a_{b_l}^{n_l} x^{m_l})^{e_l - 1} z_q,$$

where $s \in \{1, \dots, n_l\}$ is the solution of the congruence $m_l s + q \equiv 0 \pmod{n_l}$.

Indeed by Leibnitz rule

$$\frac{d}{dz_q} \hat{f} = \hat{f} \sum_{\varepsilon \in \mathbb{U}_n} \frac{-\varepsilon^{b_l + q e_l} x^{(m_l+q)/n_l}}{A_\varepsilon}.$$

Hence by Remark 3.3 (for $r = e_{l-1}$ and $s = e_l$) we get

$$\begin{aligned} \mathrm{in}_\omega \frac{d}{dz_q} \hat{f} &= \mathrm{in}_\omega \hat{f} \cdot \left(\sum_{\varepsilon \in \mathbb{U}_{e_{l-1}}} \frac{-\varepsilon^{b_l + q e_l} x^{(m_l+q)/n_l}}{y - \varepsilon^{b_l} a_{b_l} x^{m_l/n_l}} \right) \\ &= \mathrm{in}_\omega \hat{f} \cdot \left(\sum_{\varepsilon \in \mathbb{U}_{e_{l-1}}} \frac{-(\varepsilon^{e_l})^{m_l+q} x^{(m_l+q)/n_l}}{y - (\varepsilon^{e_l})^{m_l} a_{b_l} x^{m_l/n_l}} \right) \\ &= -e_l x^{(m_l+q)/n_l} \mathrm{in}_\omega \hat{f} \cdot \sum_{\theta \in \mathbb{U}_{n_l}} \frac{\theta^{m_l+q}}{y - \theta^{m_l} a_{b_l} x^{m_l/n_l}}. \end{aligned} \tag{13}$$

Let q' be a solution of the congruence $m_l q' \equiv q \pmod{n_l}$. Then

$$\sum_{\theta \in \mathbb{U}_{n_l}} \frac{\theta^{m_l+q}}{y - \theta^{m_l} a_{b_l} x^{m_l/n_l}} = \sum_{\theta \in \mathbb{U}_{n_l}} \frac{(\theta^{m_l})^{1+q'}}{y - \theta^{m_l} a_{b_l} x^{m_l/n_l}} = \sum_{\varepsilon \in \mathbb{U}_{n_l}} \frac{\varepsilon^{1+q'}}{y - \varepsilon a_{b_l} x^{m_l/n_l}}, \tag{14}$$

where the last equality follows from Remark 3.3 for $r = n_l$ and $s = m_l$.

Using the equality

$$\frac{y^{n_l} - a_{b_l}^{n_l} x^{m_l}}{y - \varepsilon a_{b_l} x^{m_l/n_l}} = \sum_{j=0}^{n_l - 1} \varepsilon^j a_{b_l}^j x^{j m_l/n_l} y^{n_l - 1 - j}$$

for any $\varepsilon \in \mathbb{U}_{n_l}$ we have

$$\begin{aligned} (y^{n_l} - a_{b_l}^{n_l} x^{m_l}) \sum_{\varepsilon \in \mathbb{U}_{n_l}} \frac{\varepsilon^{1+q'}}{y - \varepsilon a_{b_l} x^{m_l/n_l}} &= \sum_{j=0}^{n_l-1} \sum_{\varepsilon \in \mathbb{U}_{n_l}} \varepsilon^{1+q'+j} a_{b_l}^j x^{j m_l/n_l} y^{n_l-1-j} \\ &= n_l a_{b_l}^{j_0} x^{j_0 m_l/n_l} y^{n_l - 1 - j_0} \end{aligned} \tag{15}$$

where the last equality follows from the third part of Properties 3.4 and $j_0 \in \{0, \dots, n_l - 1\}$ satisfies $1 + q' + j_0 \equiv 0 \pmod{n_l}$, that is j_0 is the solution of the congruence $m_l(j+1) + q \equiv 0 \pmod{n_l}$.

From (13), (14) and (15) it follows

$$\mathrm{in}_\omega \tfrac{d}{dz_q} \hat{f} = \frac{\mathrm{in}_\omega \hat{f}}{(y^{n_l} - a_{b_l}^{n_l} x^{m_l})} (-1) e_{l-1} a_{b_l}^{s-1} x^{(m_l s+q)/n_l} y^{n_l - s} \tag{16}$$

with $s = j_0 + 1$.

We see that $\mathrm{in}_\omega \frac{d}{dz_q}\hat{f}$ does not depend on z_q. Thus, in view of the equality $\frac{d}{dz_q}\hat{f} = \hat{f}_{q,1} + 2\hat{f}_{q,2}z_q + \cdots + n\hat{f}_{q,n}z_q^{n-1}$ we have $\mathrm{in}_\omega \frac{d}{dz_q}\hat{f} = \mathrm{in}_\omega \hat{f}_{q,1}$ and $\mathrm{ord}_\omega(\hat{f}_{q,1}) < \mathrm{ord}_\omega(\hat{f}_{q,j})$ for $j > 1$. Consequently $\mathrm{in}_\omega(\hat{f} - \hat{f}_{q,0}) = \mathrm{in}_\omega(\hat{f}_{q,1}z_q + \cdots + \hat{f}_{q,n}z_q^n) = \mathrm{in}_\omega \hat{f}_{q,1}z_q$. Claim 2 follows from Claim 1 and (16).

Claim 3. Let $q \in \{1, \ldots, n_l - 1\}$. Consider $u(x, y) \in \mathbb{C}[[x, y]]$, $u(0, 0) = 1$. Put $\hat{u} = u(x^{n/e_{l-1}}, y + \lambda(x^{n/e_{l-1}})$. Then $\hat{u}\hat{f}$ is a polynomial in z_q equal to $\hat{u}\hat{f}_{q,0} + \hat{u}\hat{f}_{q,1}z_q + \cdots + \hat{u}\hat{f}_{q,n}z_q^n$, where $\hat{f}_{q,i}$ is as in (11). Moreover $\mathrm{in}_\omega \hat{u}\hat{f} = \mathrm{in}_\omega \hat{f}$ and $\mathrm{in}_\omega(\hat{u}\hat{f} - \hat{u}\hat{f}_{q,0}) = \mathrm{in}_\omega(\hat{f} - \hat{f}_{q,0})$.

Since $z_q = a_{b_l + qe_l}$ is not a coefficient of $\lambda(x)$ then $\hat{u}$ is independent of z_q. The first part of the claim follows. The second part also follows since the weighted initial part of a product is the product of the weighted initial parts of the factors, and $\mathrm{in}_\omega(\hat{u}) = 1$.

Consider now the truncation $\mathrm{trunc}(\Delta, k)$ and the lines $L_q : i + \frac{m_l}{n_l}j = \bar{b}_l + \frac{q}{n_l}$ where q is a natural number satisfying $0 \leq q \leq n_l$.

Claim 4. The lattice points on the compact edges of $\mathrm{trunc}(\Delta, k)$ with inclinations strictly bigger than $\frac{m_l}{n_l}$ belong to the lines L_q with $0 \leq q \leq n_l - 1$.

Indeed, consider $D := \{(i, j) \in \mathbb{R}^2 \ : \ \bar{b}_l \leq i + \frac{m_l}{n_l}j < \bar{b}_l + 1\} \cap \{(i, j) \in \mathbb{R}^2 \ : \ 0 \leq j \leq e_{l-1}\}$ (see Figure 6). Observe that any lattice point (i_0, j_0) in D belongs to $\bigcup_{q=0}^{n_l-1} L_q$ since the rational number $i_0 + \frac{m_l}{n_l}j_0$ belonging to the interval $[\bar{b}_l, \bar{b}_l + 1)$ has the form $\bar{b}_l + \frac{q}{n_l}$ for some $q \in \{0, \ldots, n_l - 1\}$. Let $k < e_{l-1}$ and consider $d := \min\{i \in \mathbb{N} \ : \ i + k\frac{m_l}{n_l} \geq \bar{b}_l\}$.

Let $\mathcal{B}$ be the intersection of the compact edges of $\mathrm{trunc}(\Delta, k)$ and the strip $\mathbb{R} \times [k, e_{l-1}]$. Since $\mathrm{trunc}(\Delta, k)$ is contained in Δ then $\mathcal{B}$ also. The set $\mathcal{B}$ is the graph of a piecewise linear, convex, decreasing function, contained in $L_0^+ := \{(i, j) \in \mathbb{R}^2 \ : \ i + \frac{m_l}{n_l}j \geq \bar{b}_l\}$. The endpoints of $\mathcal{B}$ are (b, e_{l-1}), (d, k). By convexity, $\mathcal{B}$ is contained in $L_{n_l}^- := \{(i, j) \in \mathbb{R}^2 \ : \ i + \frac{m_l}{n_l}j < \bar{b}_l + 1\}$ so $\mathcal{B} \subseteq D$ and Claim 4 follows.

Fix $q \in \{1, \ldots, n_l - 1\}$. The lattice points of $L_q \cap D$ are the solutions of the linear Diophantine equation $n_l i + m_l j = n_l \bar{b}_l + q$ for $0 \leq j \leq e_{l-1}$. Reducing this Diophantine equation modulo n_l we realize that there is no solution for $j = 0$ or $j = e_{l-1}$. Hence the number of these lattice points is $e_l - 1$ since $e_{l-1} = n_l e_l$. Under the assumptions of Claim 3, the polynomial $\mathrm{in}_\omega(\hat{u}\hat{f}_{q,1})$ has $e_l - 1$ monomials of ω-weigthed order $\bar{b}_l + \frac{q}{n_l}$ and y-degree strictly less than e_{l-1}. Consequently these lattice points are in the support of $\mathrm{in}_\omega(\hat{u}\hat{f}_{q,1})$.

Let f be a generic member of $K(n, b_1, \ldots, b_h)$. As the multiplication by a nonzero constant does not affect the statement of the lemma, we may assume that $f = uf^*$ where $f^* \in \mathbb{C}[[x]][y]$ is a Weierstrass polynomial and $u(x, y) \in \mathbb{C}[[x, y]]$ with $u(0, 0) = 1$.

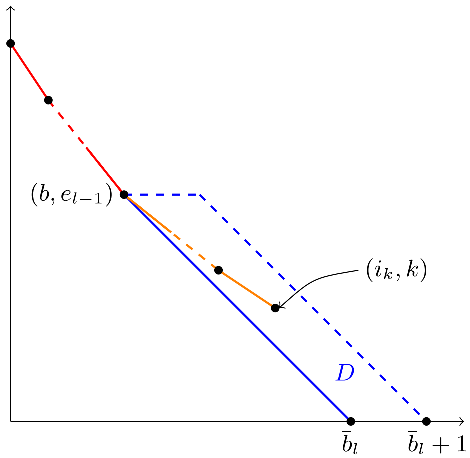

FIGURE 6. The set D

Note that equality (9) is equivalent to equality $\mathcal{N}\left(y^k \frac{\partial^k \hat{f}}{\partial y^k}\right) = \mathrm{trunc}(\Delta, k)$. Moreover it follows from (ND2) that $\frac{\partial^k \hat{f}}{\partial y^k}$ is non-degenerate on the edge S of its Newton diagram if and only if $y^k \frac{\partial^k \hat{f}}{\partial y^k}$ is also non-degenerate on the edge $S + (0, k)$ of its Newton diagram.

Let $\{(i_r, j_r)\}_{r=0}^s$ be the set of lattice points belonging to the compact edges of $\mathcal{B}$ with inclinations strictly bigger than $\frac{m_l}{n_l}$, ordered by the first coordinate, that is $i_r < i_{r+1}$ for any $r \in \{0, \dots, s-1\}$. Note that $(i_s, j_s) = (d, k)$. We have $\bar{b}_l = \mathrm{ord}_\omega(x^{i_0} y^{j_0}) < \mathrm{ord}_\omega(x^{i_1} y^{j_1}) < \cdots < \mathrm{ord}_\omega(x^{i_s} y^{j_s}) < \bar{b}_l + 1$. For any $r \in \{1, \dots, s\}$ there exists $q_r \in \{1, \dots, n_l - 1\}$ such that $\mathrm{ord}_\omega(x^{i_r} y^{j_r}) = \bar{b}_l + \frac{q_r}{n_l}$.

Set $y^k \frac{\partial^k \hat{f}}{\partial y^k} = \sum c_{ij} x^i y^j$.

By Claims 3 and 1, for any $r \in \{1, \dots, s\}$, we have

$$c_{i_r j_r} = W_r(z_1, \dots, z_{q_r - 1}) + \gamma_r z_{q_r}, \tag{17}$$

where $\gamma_r \in \mathbb{C} \backslash \{0\}$, $W_r \in \mathbb{C}[z_1, \dots, z_{q_r - 1}]$ and $c_{i_0 j_0}$ is a nonzero constant polynomial in $\mathbb{C}[z_1, \dots, z_{q_s}]$. The map

$$\begin{aligned} \Phi : \mathbb{C}^s &\longrightarrow \mathbb{C}^s \\ (z_{q_1}, \dots, z_{q_s}) &\longrightarrow \Phi(z_{q_1}, \dots, z_{q_s}) = (c_{i_1 j_1}(z_{q_1}, \dots, z_{q_s}), \dots, c_{i_s j_s}(z_{q_1}, \dots, z_{q_s})) \end{aligned}$$

is surjective after the triangular form of its components given by (17). The equality $\mathcal{N}\left(y^k \frac{\partial^k \hat{f}}{\partial y^k}\right) = \mathrm{trunc}(\Delta, k)$ is equivalent to the non-vanishing of all coefficients c_{ij} where $(i, j) \in \{(i_r, j_r)\}_{r=1}^s$ is a vertex of $\mathcal{B}$.

Assume for a moment that the equality $\mathcal{N}\left(y^k \frac{\partial^k \hat{f}}{\partial y^k}\right) = \mathrm{trunc}(\Delta, k)$ holds. Let R be a compact edge of $\mathrm{trunc}(\Delta, k)$ of inclination bigger than $\frac{m_l}{n_l}$. Denote by α_R the maximum natural number i such that y^i divides the initial form g_R

of $g := y^k \frac{\partial^k \hat{f}}{\partial y^k}$ with respect to R. The non-degeneracy of $y^k \frac{\partial^k \hat{f}}{\partial y^k}$ on the compact edge R is equivalent to the non-vanishing of the discriminant of the polynomial $y^{-\alpha_R} g_R(1, y)$. Denote by H_R this discriminant. Since the coefficients of $y^{-\alpha_R} g_R(1, y)$ are in the set $\{c_{i_{q_\ell} j_{q_\ell}}\}_{\ell=0}^{s}$ then $H_R \in \mathbb{C}[c_{i_{q_1} j_{q_1}}, \ldots, c_{i_{q_s} j_{q_s}}] \backslash \{0\}$.

Consider

$$\mathcal{A}_1 := \{c_{ij} \ : \ (i, j) \in \{(i_r, j_r)\}_{r=1}^{s} \text{ is a vertex of } \mathcal{B}\}$$

and

$$\mathcal{A}_2 := \left\{H_R \ : \ R \text{ is a compact edge of } \operatorname{trunc}(\Delta, k) \text{ of inclination bigger than } \frac{m_l}{n_l}\right\}.$$

The complement of the solutions of the polynomial defined as the product of all elements of $\mathcal{A}_1 \cup \mathcal{A}_2$ is a non-empty open Zariski set in the target of Φ and its preimage by Φ is a non-empty open Zariski set in the source of Φ. Hence there is a non-empty open Zariski set in the space of coefficients of the Puiseux root $\alpha(x)$ of $f \in K(n, b_1, \ldots, b_g)$ such that

$$\mathcal{N}\left(\frac{\partial^k \hat{f}}{\partial y^k}\right) = \operatorname{trunc}(\Delta, k)$$

and $\frac{\partial^k \hat{f}}{\partial y^k}$ is non-degenerate on all edges of its Newton diagram which inclinations are bigger than m_l/n_l. This last non-empty open Zariski is the complement of the solutions of a polynomial depending on a finite number of coefficients of α, let us say $a_{s_1}, \ldots, a_{s_\ell}$; and we denote this polynomial by $G(a_{s_1}, \ldots, a_{s_\ell})$. Consider now the polynomial $\overline{G} = \prod_{\epsilon \in \mathbb{U}_n} G(\epsilon^{s_1} a_{s_1}, \ldots, \epsilon^{s_\ell} a_{s_\ell})$. By [8, Theorem 3], there exists a finite set of coefficients of f, let us say $a_{u_1 v_1}, \ldots, a_{u_I v_I}$ and a polynomial $W \in \mathbb{C}[T_1, \ldots, T_I]$ such that $W(a_{u_1 v_1}, \ldots, a_{u_I v_I}) = 0$ if and only if $\overline{G}(a_{s_1}, \ldots, a_{s_\ell}) = 0$. We conclude that if f is a generic element in $K(n, b_1, \ldots, b_h)$, that is $W(a_{u_1 v_1}, \ldots, a_{u_I v_I}) \neq 0$, then $G(a_{s_1}, \ldots, a_{s_\ell}) \neq 0$ and the lemma follows. □

4. Proof of the Main Theorem

In this section we will prove Theorem 1.3. Let f be a generic member of $K(b_0, \ldots, b_h)$. Remember that $e_i = \gcd(b_0, \ldots, b_i)$, for $0 \leq i \leq h$ and $n_i = \frac{e_{i-1}}{e_i}$, $m_i = \frac{b_i}{e_i}$, $\Delta_i = \left\{\frac{m_i}{n_i}\right\}$ for $1 \leq i \leq h$. Fix $1 \leq k < b_0$ and let $\ell \in \{1, \ldots, h\}$ be such that $e_{\ell-1} > k$. Let α be any Newton-Puiseux root of f.

Denote the sum of all terms of α of degree strictly less than $\frac{b_\ell}{b_0}$ by λ_ℓ and let $f_\ell(y)$ be the minimal polynomial of λ_ℓ in $\mathbb{C}[[x]][y]$. The degree of $f_\ell(y)$ equals $n_1 \cdots n_{\ell-1}$. Let $\frac{\partial^k f}{\partial y^k} = g_1 \cdots g_r$ be the factorization into irreducible factors of the kth derivative of f. Put $\Gamma^{(\ell)} := \prod_j g_j$ where the product runs over the factors g_j such that $\operatorname{cont}(g_j, f) = \frac{b_\ell}{b_0}$. According to [7, Theorem 6.2] we have that $\frac{\partial^k f}{\partial y^k} = \Gamma^{(1)} \cdots \Gamma^{(i_k)}$ which proves item (1) of the theorem.

We can write $\Gamma^{(\ell)} = \Gamma_1^{(\ell)}\Gamma_2^{(\ell)}$ verifying $\mathrm{cont}(g, f_\ell) > \frac{b_\ell}{b_0}$ for any irreducible factor g of $\Gamma_1^{(\ell)}$ and $\mathrm{cont}(g, f_\ell) = \frac{b_\ell}{b_0}$ for any irreducible factor g of $\Gamma_2^{(\ell)}$. Remark that the factors $\Gamma_1^{(\ell)}$ and $\Gamma_2^{(\ell)}$ coincide with those given in [7, Theorem 6.2].

After [7, Theorem 6.2 (v), (ii)] $\Gamma_2^{(\ell)} = \prod_{i=1}^m w_i^{(\ell)}$ where $m = \min\{e_\ell, k\} - \lceil \frac{k}{n_\ell} \rceil$ and the set of characteristic exponents of its irreducible factors $w_i^{(\ell)}$ is $\left\{\frac{b_1}{b_0}, \dots, \frac{b_\ell}{b_0}\right\}$.

Since $\frac{b_\ell}{b_0}$ is not in the support of λ_ℓ we get $\mathrm{cont}(f_\ell, w_i^{(\ell)}) = b_\ell/b_0$ for $1 \le i \le m$, and statement $(2b)$ follows.

On the other hand we get $\widehat{\frac{\partial^k f}{\partial y^k}} = \frac{\partial^k \hat{f}}{\partial y^k}$, so by Lemma 3.5 $\mathcal{N}\left(\frac{\partial^k \hat{f}}{\partial y^k}\right) = \left\{\frac{m_\ell}{n_\ell}\right\}^{(t)} + L$, where the inclinations of the compact edges of L are less than or equal to $\frac{m_\ell}{n_\ell}$. Moreover $\widehat{\frac{\partial^k f}{\partial y^k}}$ is non-degenerate on all edges of its Newton diagram which inclinations are bigger than m_l/n_l.

Now applying Corollary 3.2 to $\lambda = \lambda_\ell$, which characteristic is $\left(\frac{b_0}{e_{\ell-1}}, \frac{b_1}{e_{\ell-1}}, \dots, \frac{b_{\ell-1}}{e_{\ell-1}}\right)$, $g = f_l$, $v = \frac{\partial^k f}{\partial y^k}$ and $q = \frac{m_\ell}{n_\ell}$, we get that $\Gamma_1^{(\ell)}$ can be written as $\prod_{j=1}^r z_j^{(\ell)}$ with $z_j^{(\ell)}$ irreducible verifying statements $(2a)$ and $(2c)$ of the theorem.

In order to prove the statement $(2c)$ in full generality it is sufficient to show that

$$\mathrm{cont}(w_i^{(\ell)}, w_j^{(\ell)}) = \frac{b_\ell}{b_0} \quad \text{for } 1 \le i < j \le m.$$

Suppose that $\mathrm{cont}(w_i^{(\ell)}, w_j^{(\ell)}) > \frac{b_\ell}{b_0}$ for some $i, j \in \{1, \dots, m\}$, $i \neq j$. Then there is a nonzero complex number u and a Newton-Puiseux root γ_d of $w_d^{(\ell)}$ such that $\gamma_d = \lambda_\ell + ux^{b_\ell/b_0} + \cdots$, for $d = i, j$. We claim that u is not a root of the univariate polynomial $y^{n_\ell} - a_{b_\ell}^{n_\ell}$. Indeed suppose that $u = \tau^{n_\ell} a_{b_\ell}$ for some n_ℓ-th root of unity τ. Let ε be an $e_{\ell-1}$-th root of the unity such that $\tau = \varepsilon^{b_\ell}$. Then the Newton-Puiseux root α_ε of f has the form $\alpha_\varepsilon = \lambda_\ell + \varepsilon^{b_\ell} a_{b_\ell} x^{b_\ell/b_0} + \cdots = \lambda_\ell + ux^{b_\ell/b_0} + \cdots$, hence $\mathrm{ord}(\alpha_\varepsilon - w_d^{(\ell)}) > b_l/b_0$ which is a contradiction since $\mathrm{cont}(f, w_d^{(\ell)}) = b_l/b_0$ and we finished the proof of the claim.

Observe that $\tilde{\gamma}_d := \gamma_d(x^{n/e_{\ell-1}}) - \lambda_\ell(x^{n/e_{\ell-1}}) = ux^{m_\ell/n_l} + \cdots$ are Newton-Puiseux roots of $\frac{\partial^k \hat{f}}{\partial y^k}$, for $d = i, j$.

Let $F(y) := \mathrm{in}_\omega \hat{f}(x, y)|_{x=1}$ (see (12)). Hence we get $\frac{d^k F}{dy^k} = \mathrm{in}_\omega \left(\frac{\partial^k \hat{f}(x,y)}{\partial y^k}\right)_{|x=1}$. Given that $(y - ux^{m_\ell/n_\ell})^2$ is a factor of $\mathrm{in}_\omega\left(\frac{\partial^k \hat{f}(x,y)}{\partial y^k}\right)$ then $(y - u)^2$ is a factor of $\frac{d^k F}{dy^k}$ which is a contradiction since $\frac{d^k F}{dy^k}$ has no multiple complex roots except 0 and the roots of $y^{n_\ell} - a_{b_\ell}^{n_\ell}$(see [7, Corollary 5.4]).The proof of Theorem 1.3 is finished.

Example 4.1. Consider a generic element f of $K(12, 16, 31)$. Then

$$\frac{\partial f}{\partial y} = \Gamma^{(1)}\Gamma^{(2)}$$

where

$$\mathrm{cont}(f, v) = \begin{cases} \frac{4}{3} & \text{for any irreducible factor } v \text{ of } \Gamma^{(1)} \\ \frac{31}{12} & \text{for any irreducible factor } v \text{ of } \Gamma^{(2)}. \end{cases}$$

We have $(n_1, m_1) = (3, 4)$ and $(n_2, m_2) = (4, 31)$. The first symbolic derivatives of Newton diagrams $\Delta_1 = \left\{\frac{4}{3}\right\}$, $\Delta_2 = \left\{\frac{31}{4}\right\}$ are $\Delta_1^{(1)} = \left\{\frac{3}{2}\right\}$, $\Delta_2^{(1)} = 3\left\{\frac{8}{1}\right\}$. Hence $\Gamma^{(1)} = z_1^{(1)}$ and $\Gamma^{(2)} = \prod_{j=1}^{3} z_j^{(2)}$ where

- $\mathrm{cont}(f_1, z_1^{(1)}) = \frac{3}{2}$ and $\mathrm{Char}(z_1^{(1)}) = \{\frac{3}{2}\}$,
- $\mathrm{cont}(f_2, z_j^{(2)}) = \frac{8}{3}$ and $\mathrm{Char}(z_j^{(2)}) = \{\frac{4}{3}\}$, for $j \in \{1, 2, 3\}$.

For the second polar we have $\frac{\partial^2 f}{\partial y^2} = \Gamma^{(1)}\Gamma^{(2)}$ where as before

$$\mathrm{cont}(f, v) = \begin{cases} \frac{4}{3} & \text{for any irreducible factor } v \text{ of } \Gamma^{(1)} \\ \frac{31}{12} & \text{for any irreducible factor } v \text{ of } \Gamma^{(2)}. \end{cases}$$

Since $\Delta_1^{(2)} = \left\{\frac{2}{1}\right\}$, $\Delta_2^{(2)} = 2\left\{\frac{8}{1}\right\}$, we get in this case that $\Gamma^{(1)} = z_1^{(1)} w_1^{(1)}$ and $\Gamma^{(2)} = z_1^{(2)} z_2^{(2)}$ where

- $\mathrm{cont}(f_1, z_1^{(1)}) = \frac{2}{1}$ and $\mathrm{Char}(z_1^{(1)}) = \emptyset$, that is, $z_1^{(1)}$ is smooth,
- $\mathrm{cont}(f_1, w_1^{(1)}) = \frac{4}{3}$ and $\mathrm{Char}(w_1^{(1)}) = \{\frac{4}{3}\}$,
- $\mathrm{cont}(f_2, z_j^{(2)}) = \frac{8}{3}$ and $\mathrm{Char}(z_j^{(2)}) = \{\frac{4}{3}\}$, for $j \in \{1, 2\}$.

Consider now $g(x, y) \in K(12, 16, 31)$ which admits $\alpha(x) = x^{4/3} + x^2 + x^{31/12}$ as a Newton-Puiseux root. Applying a symbolic computation program **Maxima** we get $g(x, y) = y^{12} - 12x^2y^{11} + 66x^4y^{10} + h(x, y)$, where $\deg_y h(x, y) = 9$. Hence $\frac{\partial^{10} g}{\partial y^{10}} = 6 \cdot 11!(y - x^2)^2$. However, after Theorem 1.3, for a generic element $f \in K(12, 16, 31)$ we get, $\frac{\partial^{10} f}{\partial y^{10}} = \Gamma^{(1)} = z_1^{(1)}$, with $\mathrm{Char}(z_1^{(1)}) = \{\frac{3}{2}\}$, $\mathrm{cont}(f, z_1^{(1)}) = \frac{4}{3}$ and $\mathrm{cont}(f_1, z_1^{(1)}) = \frac{3}{2}$. We conclude that g is not a generic element of $K(12, 16, 31)$ in the sense of Theorem 1.3.,

Example 4.2. Consider a generic element f of $K(10, 14, 15)$. We have $\Delta_1 = \left\{\frac{m_1}{n_1}\right\} = \left\{\frac{7}{5}\right\}$ and $\Delta_2 = \left\{\frac{m_2}{n_2}\right\} = \left\{\frac{15}{2}\right\}$. By Proposition 2.7 the first symbolic derivatives of these Newton diagrams are $\Delta_1^{(1)} = 2\left\{\frac{3}{2}\right\}$ and $\Delta_2^{(1)} = \left\{\frac{8}{1}\right\}$.

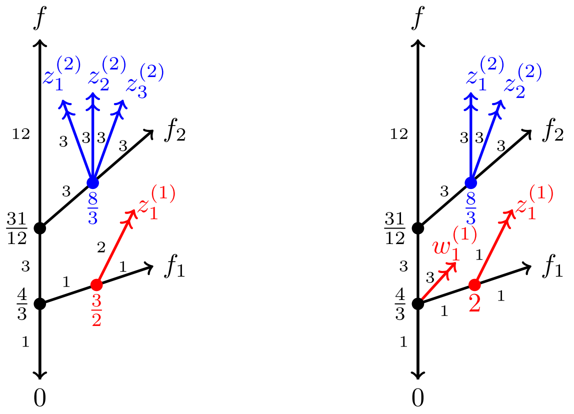

FIGURE 7. Eggers-Wall trees of Example 4.1: on the left $\Theta(ff_1f_2\frac{\partial f}{\partial y})$ and on the right $\Theta(ff_1f_2\frac{\partial^2 f}{\partial y^2})$

We get

$$\frac{\partial f}{\partial y} = \Gamma^{(1)}\Gamma^{(2)},$$

where

$$\operatorname{cont}(f,v) = \begin{cases} \frac{7}{5} \text{ for any irreducible factor } v \text{ of } \Gamma^{(1)} \\ \\ \frac{3}{2} \text{ for any irreducible factor } v \text{ of } \Gamma^{(2)}. \end{cases}$$

Moreover $\Gamma^{(1)} = z_1^{(1)}z_2^{(1)}$ and $\Gamma^{(2)} = z_1^{(2)}$ where

- $\operatorname{cont}(f_1, z_j^{(1)}) = \frac{3}{2}$ and $\operatorname{Char}(z_j^{(1)}) = \{\frac{3}{2}\}$, for $j \in \{1,2\}$;
- $\operatorname{cont}(f_2, z_1^{(2)}) = \frac{8}{5}$ and $\operatorname{Char}(z_1^{(2)}) = \{\frac{7}{5}\}$.

For the second polar we have $\frac{\partial^2 f}{\partial y^2} = \Gamma^{(1)}$ where $\operatorname{cont}(f,v) = \frac{7}{5}$ for any irreducible factor v of $\Gamma^{(1)}$.

In this case $\Delta_1^{(2)} = \left\{\frac{2}{1}\right\} + \left\{\frac{3}{2}\right\}$. Hence $\Gamma^{(1)} = z_1^{(1)}z_2^{(1)}w_1^{(1)}$ where

- $\operatorname{cont}(f_1, z_1^{(1)}) = \frac{2}{1}$ and $z_1^{(1)}$ is smooth,
- $\operatorname{cont}(f_1, z_2^{(1)}) = \frac{3}{2}$ and $\operatorname{Char}(z_2^{(1)}) = \{\frac{3}{2}\}$,
- $\operatorname{cont}(f_1, w_1^{(1)}) = \frac{7}{5}$ and $\operatorname{Char}(w_1^{(1)}) = \{\frac{7}{5}\}$.

Remark 4.3. Figures 7 and 8 illustrate Examples 4.1 and 4.2 using Eggers-Wall trees. Recall that the Eggers-Wall tree $\Theta(h)$ of a reduced power series $h(x,y)$ is a rooted tree with leaves corresponding to irreducible factors of h. For any two irreducible factors h_1, h_2 of h the last common vertex of the paths from the root of $\Theta(h)$ to h_1 and from the root to h_2 is labelled by the contact $\operatorname{cont}(h_1,h_2)$. The Eggers-Wall tree $\Theta(h)$ equipped with some additional

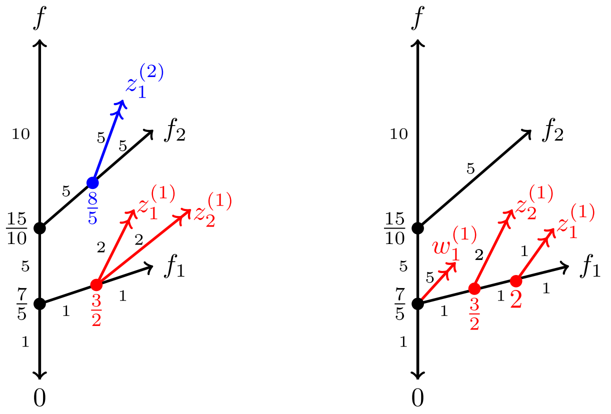

FIGURE 8. Eggers-Wall trees of Example 4.2: on the left $\Theta(ff_1f_2\frac{\partial f}{\partial y})$ and on the right $\Theta(ff_1f_2\frac{\partial^2 f}{\partial y^2})$

information (weights of edges) characterizes the equisingularity class of $h(x, y)$ (see [14] and [6]).

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Declarations

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