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The topology of the generic polar curve and the Zariski invariant for branches of genus one

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Abstract

We study, for plane complex branches of genus one, the topological type of its generic polar curve, as a function of the semigroup of values and the Zariski invariant of the branch. We improve some results given by Casas-Alvero in 2023, since we filter the topological type fixed for the branch by the possible values of Zariski invariants.

1 Introduction

Let $C_f : f(x, y) = 0$ be a plane curve with $f(x, y) \in \mathbb{C}\{x, y\}$. The curve C_f is called a branch when f is irreducible. The multiplicity of the curve C_f, denoted by $n := \text{mult }(C_f)$, is by definition the order of f. The curve C_f is singular if $n > 1$, otherwise we say that C_f is smooth. After a change of coordinates, if necessary, we may assume that $x = 0$ is not tangent to C_f at 0. By Newton Theorem there is $\alpha(x) = \sum_{i \geq n} a_i x^{i/n} \in \mathbb{C}\{x^{1/n}\}$, such that $f(x, \alpha(x)) = 0$. The power series $\alpha(x)$ is called a Newton-Puiseux root of $f(x, y)$ and it determines a Puiseux parametrization of C_f as follows:

$$(x(t), y(t)) = \left(t^n, \sum_{i \geq n} a_i t^i \right).$$

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Now, if C_f is a branch then the set of all Newton-Puiseux roots of $f(x,y)$ is $\{\alpha_\epsilon(x)=\sum_{i\geq n} a_i\epsilon^i x^{i/n}\ :\ \epsilon\in\mathbb{U}_n\}$, where $\mathbb{U}_n$ is the multiplicative group of the nth complex roots of unity. Let $\alpha_\epsilon(x)=\sum_{i\geq n} a_i\epsilon^i x^{i/n}$ be a Newton-Puiseux root of C_f. We associate with it two sequences $(e_i)_i$ and $(\beta_i)_i$ of natural numbers as follows: $e_0:=\beta_0:=n$; if $e_k>1$ then $\beta_{k+1}:=\min\{i\ :\ a_i\neq 0, \gcd(e_k,i)<e_k\}$ and $e_{k+1}:=\gcd(e_k,\beta_{k+1})$. The sequence $(e_i)_i$ is strictly decreasing and for some $g\in\mathbb{N}$ we have $e_g=1$. The number g is the genus of C_f. The sequence $(\beta_0,\beta_1,\ldots,\beta_g)$ is called the sequence of characteristic exponents of $\alpha(x)$ and coincides with the sequence of characteristic exponents of any other Newton-Puiseux root of C_f, hence this sequence is also called the sequence of characteristic exponents of the branch C_f.

Two plane curves C_f and C_h are topologically equivalent (also called equisingular) if and only if there is a homeomorphism $\Psi: U\longrightarrow V$ where U and V are neighborhoods of the origin at $\mathbb{C}^2$ such that f (respectively h) is convergent in U (respectively in V) and $\Psi(C_f\cap U)=C_h\cap V$. It is well known (see [8]) that two branches C_f and C_h are topologically equivalent if and only if they have the same characteristic exponents and in such a case we will write $C_f\equiv C_h$. If the map Ψ is an analytic isomorphism we say that C_f and C_h are analytic equivalent. We will denote by $K(n,\beta_1,\beta_2,\ldots,\beta_g)$ the set of equisingular branches of characteristic exponents $(n,\beta_1,\ldots,\beta_g)$. If $C_f: f(x,y)=0$ is a branch in $K(n,\beta_1,\ldots,\beta_g)$, then we will put $f\in K(n,\beta_1,\ldots,\beta_g)$.

We associate to the branch C_f the set

$$\Gamma_f:=\{\mathrm{I}(f,h)\ :\ h\in\mathbb{C}\{x,y\},\ f\,\text{does not divide}\ h\}$$

where $\mathrm{I}(f,h)=\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/(f,h)$ is the intersection multiplicity of the curves C_f and C_h at the origin. It is well-known that Γ_f is a numerical semigroup called the semigroup of values of C_f. If $f\in K(n,\beta_1,\ldots,\beta_g)$ then the semigroup Γ_f is finitely generated by $g+1$ natural numbers $v_0<v_1<\cdots<v_g$ and there is a relationship between the sequence $\{v_i\}_{i=0}^g$ and the characteristic exponents $(n=\beta_0,\beta_1,\ldots,\beta_g)$ of C_f as follows:

1. $v_0=n=\beta_0$, $v_1=\beta_1$,
2. $v_j=n_{j-1}v_{j-1}+\beta_j-\beta_{j-1}$ for $2\leq j\leq g$, where $n_{j-1}:=\frac{e_{j-2}}{e_{j-1}}$.

We can then write $\Gamma_f=\langle v_0,\ldots,v_g\rangle:=\mathbb{N}v_0+\mathbb{N}v_1+\cdots+\mathbb{N}v_g$.

Since the minimal system of generators of Γ_f is equivalent to the set of characteristic exponents of the branch C_f, two branches C_f and C_h are topologically equivalent if and only if $\Gamma_f=\Gamma_h$.

Let us now consider two reduced curves C_f and C_h where $f=f_1\cdots f_r$ (respectively $h=h_1\cdots h_s$) is the decomposition of f (respectively of h) into irreducible factors. The curves C_f and C_h are topologically equivalent if and only if $r = s$ and there is a bijection $\sigma : \{C_{f_i}\}_{i=1}^r \longrightarrow \{C_{h_i}\}_{i=1}^r$ such that $\sigma(C_{f_i}) = C_{h_i}$ (at the cost of renumbering the branches of C_h), for all $i \in \{1, \dots, r\}$ the branches C_{f_i} and C_{h_i} are topologically equivalent and for all i, j with $1 \leq i, j \leq r$, $\mathrm{I}(f_i, f_j) = \mathrm{I}(h_i, h_j)$.

The polar curve of C_f with respect to a point $(a : b)$ of the complex projective line $\mathbb{P}^1(\mathbb{C})$ is the curve $P_{(a:b)}(f) : af_x + bf_y = 0$. There exists an open Zariski set U of $\mathbb{P}^1(\mathbb{C})$ such that $\{P_{(a:b)}(f) \ : \ (a : b) \in U\}$ is a family of topologically equivalent plane curves. Any element of this set is called generic polar curve of C_f and we will denote it by $P(f)$. It is well known that the topological type of $P(f)$ depends on the analytical type of C_f (see [7, Exemple 3]).

In [9], Zariski introduces an analytical invariant for C_f called the Zariski invariant (see Section 3). The aim of this work is to study the behaviour of the topological type of the generic polar curve with respect to the Zariski invariants for branches of genus one.

Casas-Alvero, in [1], presents results considering some analytic invariants of C_f that influence on the topological type of the generic polar curve $P(f)$ for a plane curve C_f in $K(n, m)$. In Section 2 we recall the concept of a non-degenerate curve. In Section 3 we study the so-called Zariski invariant. In Section 4 we study the topological type of the generic polar curves of plane branches in $\mathcal{L}(n, m, \lambda)$ that is branches in $K(n, m)$ with a fixed Zariski invariant λ. This provides a finner approach to understanding the behaviour of the topological type of $P(f)$ in terms of the Zariski invariant of C_f.

2 Topological type for a non-degenerate curve

Let us remember the notion of the Newton polygon of a curve. Let $S \subseteq \mathbb{N}^2$. The Newton diagram of S, denoted by $\mathcal{ND}(S)$, is by definition the convex hull of $S + (\mathbb{R}_{\geq 0})^2$, where $+$ denotes the Minkowski sum. The Newton polygon $\mathcal{NP}(S)$ of S is the compact polygonal boundary of $\mathcal{ND}(S)$.

The support of any power series $f(x, y) = \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{C}\{x, y\} \setminus \{0\}$ is supp $(f) := \{(i, j) \in \mathbb{N}^2 \ : \ a_{i,j} \neq 0\}$. The Newton diagram of f, denoted by $\mathcal{ND}(f)$, is $\mathcal{ND}(\text{supp } (f))$. The Newton polygon $\mathcal{NP}(f)$ of f is the Newton polygon of supp (f). The Newton diagram of f depends on coordinates but $\mathcal{ND}(uf) = \mathcal{ND}(f)$ for any unit $u \in \mathbb{C}\{x, y\}$. Hence the Newton diagram of a curve is by definition the Newton diagram of any of its equations.

Let L be a compact edge of $\mathcal{NP}(f)$. Denote by $|L|_1$ (respectively $|L|_2$) the length of the projection of L over the horizontal (respectively vertical) axis. The inclination of L is $i_L := \frac{|L|_1}{|L|_2}$. By [2, Lemme 8.4.2], if L is a compact edge of $\mathcal{NP}(f)$ then the curve C_f has $|L|_2$ NewtonPuiseux roots of order i_L.

On the other hand, if $f(x,y) = \sum_{i,j} a_{i,j}x^iy^j \in \mathbb{C}\{x,y\} \setminus \{0\}$ without multiple irreducible factors, then we associate with any compact edge L of $\mathcal{NP}(f)$ the polynomial $f_L(x,y) = \sum_{(i,j)\in L\cap \text{supp }(f)} a_{i,j}x^iy^j \in \mathbb{C}[x,y]$. We say that f is Newton non-degenerate with respect to the coordinates (x,y) if for any compact edge L of $\mathcal{NP}(f)$, the polynomial $f_L(x,y)$ has no critical points outside the axes $x=0$ and $y=0$, which is equivalent to the non-zero roots of the y-polynomial $f_L(1,y)$ are simple. According to Oka, we have

Proposition 2.1. *([5, Proposition 4.7]) Let C_f be a plane curve without multiple irreducible components and consider a coordinate system (x,y) such that $x=0$ is not tangent to C_f. If C_f is non-degenerate with respect to (x,y), then associated with any compact edge L of $\mathcal{NP}(f)$ there are $d_L = \gcd(|L|_1, |L|_2)$ branches $\{C_{f_i^{(L)}}\}_{i=1}^{d_L}$ of C_f with $\text{mult }(C_{f_i^{(L)}}) = \frac{|L|_2}{d_L}$ for any i and if $C_{f_i^{(L)}}$ is singular then its semigroup of values is $\Gamma_{f_i^{(L)}} = \left\langle \frac{|L|_2}{d_L}, \frac{|L|_1}{d_L} \right\rangle$. Moreover for any two compact edges L, L' (not necessarily different) of $\mathcal{NP}(f)$ and $f_i^{(L)} \neq f_j^{(L')}$ we get $\mathrm{I}(f_i^{(L)}, f_j^{(L')}) = \min\left\{\frac{|L|_1|L'|_2}{d_Ld_{L'}}, \frac{|L|_2|L'|_1}{d_Ld_{L'}}\right\}$.*

3 The Zariski invariant

Let $f(x,y) \in \mathbb{C}\{x,y\}$ irreducible and consider the plane branch $C : f(x,y) = 0$. Suppose that $f \in K(n,m)$. In particular, the semigroup associated to C_f is $\Gamma_f = \langle n,m\rangle$.

Up to an analytic change of coordinates we can suppose that $f \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial with $\deg_y(f) = \text{mult }(C_f) = n$.

Zariski proved in [9, pages 785-786] and [10, Chapitre III.1] that if $f \in K(n,m)$ then there exists a change of coordinates such that C_f is analytically equivalent to a plane branch with Puiseux parametrization (t^n, t^m) or

$$\left(t^n, t^m + t^{\lambda_f} + \sum_{i>\lambda_f} a_it^i\right),$$

where $\lambda_f + n \notin \Gamma_f$ is an analytical invariant called the Zariski invariant of C_f. In particular $\lambda_f > m$. If C_f is analytically equivalent to $y^n - x^m = 0$ then we put $\lambda_f = \infty$.

Any branch C_f such that $n=2$ or $f \in K(3,m)$ with $m \in \{4,5\}$ has Zariski invariant equal to $\lambda_f = \infty$ (see [4, Remark 1.2.10]). So, in what follows we consider branches C_f with multiplicity $n > 3$ or $f \in K(3,m)$ with $m \geq 7$.

According to the definition we get that the set of possible finite Zariski invariants of branches in $K(n,m)$ is

$$\mathcal{Z}(n,m) := \{\lambda \in \mathbb{N} \,:\, \lambda > m \text{ and } \lambda + n \notin \langle n,m\rangle\}.$$

Lemma 3.1. *Any* $\lambda \in \mathcal{Z}(n,m)$ *can be uniquely expressed by*

$$\lambda = im - jn \quad \textit{for} \quad 2 \leq \ i \leq n-1 \ , \ 2 \leq \ j < \frac{(i-1)m}{n}, \ \textit{and} \ (i,j) \in \mathbb{N}^2.$$

Proof. Recall that any $z \in \mathbb{Z}$ can be uniquely represented as

$$z = s_0 n + s_1 m \ \text{ with } \ 0 \leq s_1 < n \ \text{ and } \ s_0 \in \mathbb{Z}.$$

Moreover, $z = s_0 n + s_1 m \in \langle n,m \rangle$ if and only if $s_0 \geq 0$.

Since $\lambda + n \notin \langle n,m \rangle$ we get $\lambda \notin \langle n,m \rangle$. In this way, there are $s_0, s_1 \in \mathbb{Z}$ such that $\lambda = s_0 n + s_1 m$ with $s_0 < 0$ and $0 \leq s_1 \leq n-1$. As, $\lambda = s_0 n + s_1 m > m$ then $s_1 \geq 2$ and $\frac{(1-s_1)m}{n} < s_0$. Moreover $s_0 \leq -2$ (otherwise $\lambda + n$ belongs to $\langle n,m \rangle$). □

Let $T \subseteq \mathbb{R}^2$ be the triangle determined by the lines

$$y = n-1, \quad x = m-1 \ \text{ and } \ my + nx = nm,$$

that is, the triangle with vertices $(m-1, n-1), (m-1, \frac{n}{m})$ and $(\frac{m}{n}, n-1)$ (see the upper part of Figure 1). Denote by $\overset{o}{T}$ the set of lattice points in the interior of T.

The following proposition gives the relation between $\mathcal{Z}(n,m)$ and $\overset{o}{T}$.

Proposition 3.2. *The map*

$$\begin{array}{rcl} \Phi : \mathcal{Z}(n,m) & \longrightarrow & \overset{o}{T} \\ \lambda = \alpha m - \beta n & \longrightarrow & \Phi(\alpha m - \beta n) = (m-\beta, \alpha - 1), \end{array} \tag{1}$$

where $2 \leq \alpha \leq n-1$ *and* $2 \leq \beta < \frac{(\alpha-1)m}{n}$ *is a bijection between the set* $\mathcal{Z}(n,m)$ *and the lattice points in the interior of* T.

Proof. By Lemma 3.1 we have that Φ is an injective map.

On the other hand, if $(i,j) \in \overset{o}{T}$ then $in + jm = (n-1)m + \lambda$ for some $\lambda > m$, $2 \leq i \leq m-2$, $1 \leq j \leq n-2$, so $\lambda = m(j+1) - n(m-i)$. Therefore if $m - i = \beta$ and $j + 1 = \alpha$, then $2 \leq \alpha \leq n-1$ and $2 \leq \beta < \frac{(\alpha-1)m}{n}$, consequently, $\lambda \in \mathcal{Z}(n,m)$. Hence, Φ is a bijection. □

In the lower part of Figure 1 we illustrate the image of bijection Φ introduced in Proposition 3.2 for the case $\mathcal{Z}(5,12)$.

Let us consider in $\overset{o}{T}$ the following (n,m)-weighted ordering $\prec$:

If (x_0, y_0), $(x_1, y_1) \in \overset{o}{T}$ then we put

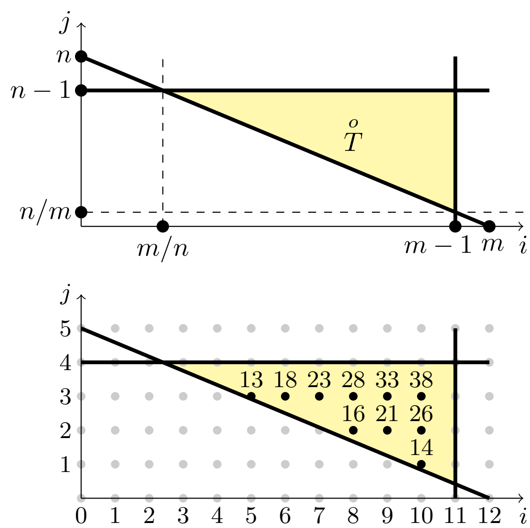

Figure 1: The triangle T and the elements of $\mathcal{Z}(5,12)$.

$$(x_0,y_0) \prec (x_1,y_1) \text{ if and only if } \; x_0 n + y_0 m < x_1 n + y_1 m.$$

In particular, the bijection Φ preserves orders, that is

$$\alpha_0 m - \beta_0 n < \alpha_1 m - \beta_1 n \text{ if and only if } \Phi(\alpha_0 m - \beta_0 n) \prec \Phi(\alpha_1 m - \beta_1 n).$$

The triangle T was spotted by Zariski in [10, page 107].

Let $\lambda \in \mathcal{Z}(n,m)$ and $\Phi(\lambda) = (p,q)$. We define

$$\begin{aligned} \mathcal{I}_\lambda : \;&=\; \{(i,j) \in \mathring{T} \;:\; (i,j) \succ (p,q)\} \qquad (2)\\ &=\; \{(i,j) \in \mathbb{N}^2 \;:\; in + jm > pn + qm,\; 0 \le i \le m-2,\; 0 \le j \le n-2\}. \end{aligned}$$

In Figure 2 we illustrate the set $\mathcal{I}_\lambda$.

In [6, Lemma 1.4] Peraire stablished a bijection between $\mathring{T}$ and the set $\{s := z - m \;:\; z \in \mathcal{Z}(n,m)\}$. Moreover, in [6, Theorem 1.5] Peraire proved that if $f \in K(n,m)$ with Zariski invariant $\lambda_f = m + s$ with $s > 0$, then there are analytic coordinates (x,y) such that

$$f(x,y) = y^n - x^m + x^p y^q + \sum_{(i,j)\in\mathcal{J}} a_{i,j} x^i y^j, \quad \text{for some} \;\; a_{i,j} \in \mathbb{C} \qquad (3)$$

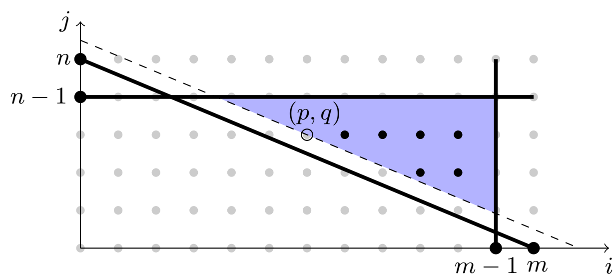

Figure 2: Points $(i,j) \in \overset{o}{T}$ with $in + jm > pn + qm$.

where $\mathcal{J} := \{(i,j) \in \mathbb{N}^2 \ : \ in + jm > mn + s, \ 0 \leq i \leq m-2, \ 0 \leq j \leq n-2\}$ and $np + qm = mn + s$.

Notice that $np+mq = mn+\lambda_f - m$, that is, $\lambda_f = (q+1)m-(m-p)n$. Since $\lambda_f \in \mathcal{Z}(n,m)$, by (1), we get $\Phi(\lambda_f) = (p,q) \in \overset{o}{T}$. Moreover, the sets $\mathcal{J}$ and $\mathcal{I}_{\lambda_f}$ are equal. In this way, we can rewrite (3) as

$$f(x,y) = y^n - x^m + x^p y^q + \sum_{(i,j)\in \mathcal{I}_\lambda} a_{i,j} x^i y^j. \tag{4}$$

Remark 3.3. *Since f is irreducible with order n, in (4) we get:*

$$n < p+q < m \ \text{ if } \ \lambda_f < 2m-n.$$

Indeed, we have $\Phi(\lambda_f) = (p,q)$ and $np+mq = (n-1)m+\lambda_f$, so if $\lambda_f < 2m-n$ then we get $np+mq < (n+1)m-n$ and $p+q < \frac{nm+(m-n)(p+1)}{m}$. The inequality follows because $p \leq m-2$ and $n < m$.

4 Topological type of generic polar curves of branches in $\mathcal{L}(n,m,\lambda)$

Let n and m be two coprime integers such that $2 < n < m$. Fix $\lambda \in \mathcal{Z}(n,m) \cup \{\infty\}$. Denote by $\mathcal{L}(n,m,\lambda)$ the set of branches in $K(n,m)$ with Zariski invariant equals λ.

Any branch in $\mathcal{L}(n,m,\infty)$ is analytically equivalent to the branch $y^n - x^m = 0$. On the other hand, if $\lambda \in \mathcal{Z}(n,m)$ then any branch $C_f \in \mathcal{L}(n,m,\lambda)$ admits, up to change of analytic coordinates, an equation f as (4).

Since we are interesting to study the topological type of generic polar curves of $C \in \mathcal{L}(n,m,\lambda)$ and, as noted in the introduction, it depends on the analytical type of C, from now on, for abuse of language, we will write $f \in \mathcal{L}(n,m,\lambda)$ and when we do so we will assume that f is as in (4).

By Lemma 3.1 and Proposition 3.2, if $\lambda \in \mathcal{Z}(n,m)$ then there are $\alpha, \beta \in \mathbb{N}$ with $2 \leq \alpha < n$ and $2 \leq \beta < \frac{(\alpha-1)m}{n}$ such that $\lambda = \alpha m - \beta n$ and $\Phi(\lambda) = (m-\beta, \alpha-1) =: (p,q)$, where Φ was defined in (1).

By definition, the point $(p,q) \in \overset{o}{T}$ corresponding to the invariant λ verifies the condition $np + qm = (n-1)m + \lambda > mn$.

We will study, via its Newton diagram, the topological type of the generic polar curve of a branch in $\mathcal{L}(n,m,\lambda)$.

4.1 The set T_λ

Given $\lambda \in \mathcal{Z}(n,m)$ and $f \in \mathcal{L}(n,m,\lambda)$ as in (4). Let us consider the generic polar curve $P(f) : af_x + bf_y = 0$ of f. If

$$\Theta_f := \{(p-1,q)\} \cup \{(i-1,j),(i,j-1) \in \mathbb{N}^2 : \ (i,j) \in \mathcal{I}_\lambda \cap \text{supp }(f)\}$$

and

$$E_\lambda := \{(0,n-1),(m-1,0),(p,q-1)\}$$

then the Newton polygon of the generic polar curve $P(f)$ equals the Newton polygon of $E_\lambda \cup \Theta_f$.

Put $N := (0,n-1), M := (m-1,0)$, $A := (p,q)$ and $B := (p,q-1)$.

If $F,G \in \mathbb{R}^2$, then we denote by $l_{F,G}$ the real line passing by F and G. Notice that $l_{N,M} : (n-1)x+(m-1)y = (n-1)(m-1)$ and $l_{N,B} : (n-q)x+py = (n-1)p$ whose slope is $\frac{q-n}{p} > -1$ because $n < p+q$ (see Remark 3.3).

We say that a point $(\alpha_1,\alpha_2) \in \mathbb{R}^2$ is above (respectively below) the line $\ell : ax+by = c$ if $a\alpha_1 + b\alpha_2 > c$ (respectively $a\alpha_1 + b\alpha_2 < c$).

Given $H = (h_1,h_2) \in \mathbb{R}^2$ we denote by $l_{\sigma,H} : nx + my = h_1 n + h_2 m$ the line with slope $\sigma := -\frac{n}{m}$ and passing through the point H. By (2), it follows that any $(i,j) \in \mathcal{I}_\lambda$ is above $l_{\sigma,A} : nx + my = pn + qm$.

Lemma 4.1. *Let $f \in \mathcal{L}(n,m,\lambda)$ with $\Phi(\lambda) = (p,q)$. Any point $(\alpha_1,\alpha_2) \in \Theta_f$ with $q-1 < \alpha_2 < n-1$ is above the line $l_{N,B}$.*

Proof. Since $\frac{q-n}{p} > -1$ (see Remark 3.3), we deduce from the equation of $l_{N,B}$ that $(p-1,q)$ is above this line. On the other hand any point $(i,j) \in \mathcal{I}_\lambda$ is above $l_{\sigma,A}$, then $(i-1,j)$ and $(i,j-1)$ are above $l_{\sigma,B}$ for every $(i,j) \in \mathcal{I}_\lambda$. Since the slope $\frac{q-n}{p}$ of $l_{N,B}$ is greater than the slope of $l_{\sigma,B}$, we have that $(i-1,j)$ and $(i,j-1)$ are above $l_{N,B}$ for every $(i,j) \in \mathcal{I}_\lambda$ with $q-1 < j < n-1$. □

Notice that by Lemma 4.1 the Newton polygon of the generic polar curve $P(f)$ for any $f \in \mathcal{L}(n,m,\lambda)$ is the Newton polygon of the set

$$E_\lambda \cup \{(\alpha_1,\alpha_2) \in \Theta_f \ : \ 0 \le \alpha_2 \le q-2\}. \tag{5}$$

Observe that the points on (5) arise from the following points of supp (f):

$$\{(0,n),(m,0),(p,q)\} \cup \{(i,j) \in \mathcal{I}_\lambda \cap \text{supp } (f) : \ 0 \le j \le q-1.$$

In order to study the topological type of the generic polar of an element $f \in \mathcal{L}(n,m,\lambda)$ we will use the sets E_λ and T_λ defined in the sequel.

Put $Q := (m-1,1)$. Denote by T_λ the set of points $(\alpha_1,\alpha_2) \in \mathbb{N}^2$ with $p < \alpha_1 < m-1$, $0 < \alpha_2 < q$ such that (α_1,α_2) is above $l_{\sigma,A}$ and below or on the line $l_{A,Q} : (m-1-p)(y-q)+(q-1)(x-p) = 0$. From the equations of the lines $l_{\sigma,A}$ and $l_{A,Q}$ we get

$$T_\lambda = \left\{ \begin{array}{ll} (\alpha_1,\alpha_2) \in \mathbb{N}^2 : & p < \alpha_1 < m-1, \ \ 0 < \alpha_2 < q \\ & \text{and} \ \ \frac{q-1}{m-1-p} \le \frac{q-\alpha_2}{\alpha_1 - p} < \frac{n}{m} \end{array} \right\}. \tag{6}$$

Figure 3 illustrates T_λ for $\lambda = 13 \in \mathcal{Z}(5,12)$, in this case $T_\lambda = \{(10,1),(8,2)\}$.

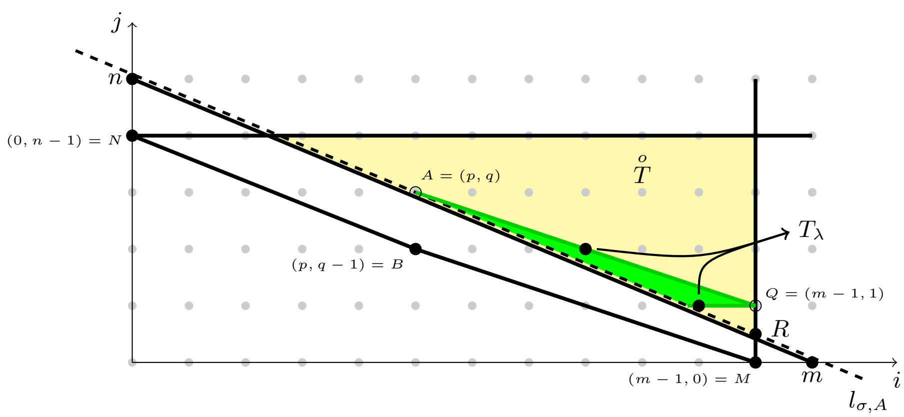

Figure 3: Points in T_λ for $\lambda = 13 \in \mathcal{Z}(5,12)$.

Now we highlight some cases where $T_\lambda = \emptyset$.

Lemma 4.2. *Let $\lambda \in \mathcal{Z}(n,m)$ with $\Phi(\lambda) = (p,q)$. We get $T_\lambda = \emptyset$ in the following cases:*

1. $\lambda > 2m - n$.
2. $\Phi(\lambda) = (p, 1)$, *that is* $q = 1$.
3. *For any* $\lambda \in K(3, m)$ *with* $m \geq 7$.
4. *For any* $\lambda \in K(n, n+1)$.

Proof. Since the intersection point of $l_{\sigma,A}$ with $x = m-1$ is $R = \left(m-1, \frac{\lambda+n-m}{m}\right)$ it follows that $\lambda > 2m - n$ if and only if $R = (m-1, \delta)$ with $\delta > 1$. But the intersection point of $l_{A,Q}$ with $x = m-1$ is $Q = (m-1, 1)$. In particular, if $\lambda > 2m - n$ we get $T_\lambda = \emptyset$ and the item 1) follows.

The item 2) follows directly from the definition of T_λ (see (6)).

For the third statement, recall that any $\lambda \in \mathcal{Z}(3, m)$ with $m \geq 7$ is expressed, according to Lemma 3.1, as $\lambda = 2m - 3j$ for some $2 \leq j < \frac{m}{3}$, so $\Phi(\lambda) = (m-j, 1)$ and the item 2) gives the result.

For the item 4), remark that $n+2 \notin \mathcal{Z}(n, n+1)$ since $n+2+n = 2(n+1) \in \langle n, n+1 \rangle$, so for any $\lambda \in \mathcal{Z}(n, n+1)$ we get $\lambda > n+2 = 2(n+1) - n$, that is we have the condition of item 1) and the lemma follows. $\square$

In the following subsection we will show how the set T_λ can be used to determine the Newton polygon of the generic polar curve of $f \in \mathcal{L}(n, m, \lambda)$.

4.2 Topological type of $P(f)$ for $f \in \mathcal{L}(n, m, \lambda)$

We recall that if C_f is such that $\lambda_f = \infty$ then, according to [9, pages 785-786], the plane branch is analytically equivalent to $y^n - x^m = 0$ with $\gcd(n, m) = 1$ and in this case, the topological type of the generic polar curve $P(f)$ is the same of the topological type of $y^{n-1} - x^{m-1} = 0$. Indeed by Proposition 2.1 that is, if $d := \gcd(n-1, m-1)$ then $P(f)$ have d irreducible equisingular components $\{P_i\}_i^d$ such that $\Gamma_{P_i} = \left\langle \frac{n-1}{d}, \frac{m-1}{d} \right\rangle$ for any $i \in \{1, \ldots, d\}$ and $\mathrm{I}(P_i, P_j) = \frac{(n-1)(m-1)}{d^2}$ for any $i \neq j$.

The following theorem highlights the relevance of the region T_λ to describe the Newton polygon of $P(f)$ of a plane branch $f \in \mathcal{L}(n, m, \lambda)$.

Theorem 4.3. *For any* $f = y^n - x^m + x^p y^q + \sum_{(i,j)\in \mathcal{I}_\lambda} a_{i,j} x^i y^j \in \mathcal{L}(n, m, \lambda)$ *with* $\lambda \in \mathcal{Z}(n, m)$ *such that* $\Phi(\lambda) = (p, q)$, *the Newton polygon of* $P(f)$ *is equal to the Newton polygon of the set*

$$E_\lambda \ \cup \ \{(i, j-1), (i-1, j) : \ (i, j) \in T_\lambda \cap \operatorname{supp}(f)\}. \tag{7}$$

In particular, the Newton polygon of $P(f)$ *is equal to the Newton polygon of the generic polar of* $y^n - x^m + x^p y^q + \sum_{(i,j)} a_{i,j} x^i y^j$, *where the sum runs on* $T_\lambda \cap \operatorname{supp}(f)$ *and, consequently if* $T_\lambda = \emptyset$ *then the Newton polygon of* $P(f)$ *is* $\mathcal{NP}(E_\lambda)$.

Proof. As a consequence of Lemma 4.1, the Newton polygon of the generic polar curve $P(f)$ for any $f = y^n - x^m + x^p y^q + \sum_{(i,j)\in \mathcal{I}_\lambda} a_{i,j}x^i y^j \in \mathcal{L}(n,m,\lambda)$ is equal to the Newton polygon of the set given in (5).

Notice that any point $(i,j) \in \mathcal{I}_\lambda$ is above the line $l_{\sigma,A}$, in particular those points $(i,j) \in \mathcal{I}_\lambda$ with $0 \le j < q$.

Remember that $l_{A,Q}$ is the line determined by the points $A = (p,q)$ and $Q = (m-1,1)$. Since $l_{A,Q}$ is parallel to the line $l_{B,M}$ passing by the points $B = (p, q-1)$ and $M = (m-1, 0)$, any point $(i,j) \in \text{supp}\ (f)$ above $l_{A,Q}$ with $1 \le j \le q-1$ give us the points $(i, j-1), (i-1, j) \in \Theta_f$ that are above the line $l_{B,M}$, in particular, $(i,j-1),(i-1,j) \in \mathcal{ND}(E_\lambda) \setminus \mathcal{NP}(E_\lambda)$. Since

$$\mathcal{ND}(E_\lambda) \subseteq \mathcal{ND}(P(f)),$$

the points $(i, j-1)$ and $(i-1, j)$ in Θ_f and above the line $l_{B,M}$ do not affect the Newton diagram of $P(f)$.

In this way, for any $f \in \mathcal{L}(n,m,\lambda)$ the Newton polygon of the generic polar $P(f)$ coincides with the Newton polygon of the set E_λ and the points $(i-1,j)$ and $(i,j-1)$ such that $(i,j) \in \text{supp}\ (f)$ with $1 \le j \le q-1$ that are above the line $l_{\sigma,A}$ and below or on the line $l_{A,Q}$, that is, the Newton polygon of $P(f)$ is equal to the Newton polygon of the set

$$E_\lambda \ \cup \ \{(i,j-1),(i-1,j) : \ (i,j) \in T_\lambda \cap \text{supp}\ (f)\}\,.$$

In particular, if $T_\lambda = \emptyset$, then the Newton polygon of $P(f)$ is $\mathcal{NP}(E_\lambda)$. $\square$

In [1, Section 3], Casas-Alvero considers the triangle $\hat{\Lambda}$ with vertices $(0,n)$, $(m-1,1)$ and $(m - \frac{m}{n}, 1)$. He proves that the Newton polygon of $P(f)$ for any element $f \in K(n,m)$ coincides with the Newton polygon of the generic polar of $y^n - x^m + \sum_{(i,j)\in\hat{\Lambda}} a_{i,j}x^i y^j$.

For $f \in \mathcal{L}(n,m,\lambda)$ with $\lambda > 2m-n$, the Newton polygon of the generic polar curve equals to the Newton polygon of E_λ (see Lemma 4.2 and Theorem 4.3). In Theorem 4.7 we will describe the topology of the polar in this case.

On the other hand, it follows, by Remark 3.3, that for any $\lambda < 2m-n$ such that $\Phi(\lambda) = (p,q)$ we get $(p,q) \in \hat{\Lambda}$ and $T_\lambda \subsetneq \hat{\Lambda}$. In this way, the set T_λ provides a refined set defining the Newton polygon of the generic polar curve $P(f)$ for any $f \in \mathcal{L}(n,m,\lambda)$ with $\lambda < 2m-n$.

Recall that any curve in $\mathcal{L}(n,m,\lambda)$ is analytically equivalent to a curve given by (4), that is, $f(x,y) = y^n - x^m + x^p y^q + \sum_{(i,j)\in\mathcal{I}_\lambda} a_{ij}x^i y^j$ with $\Phi(\lambda) = (p,q)$. We say that f is *generic* in $\mathcal{L}(n,m,\lambda)$ if the coefficients a_{ij} of f with $(i,j) \in \mathcal{I}_\lambda$ belong to an open Zariski set.

In the following proposition, we will describe a set of points whose Newton polygon coincides with $\mathcal{NP}(P(f))$ for a generic element $f \in \mathcal{L}(n,m,\lambda)$.

Proposition 4.4. *The Newton polygon of the generic polar curve $P(f)$ for a generic element $f \in \mathcal{L}(n,m,\lambda)$ equals the Newton polygon of the set $E_\lambda \cup S$ where $S = \emptyset$ if $\Phi(\lambda) = (p,1)$ or*

$$S = \left\{ \begin{array}{ll} \left(p + \left\lceil \frac{(q-j-1)m}{n} \right\rceil, j\right) : & 0 \le j \le q-2, \quad \text{and} \\ & \left\lceil \frac{(q-j)m}{n} \right\rceil \le \frac{(q-j)(m-p-1)}{q-1} \end{array} \right\}$$

if $\Phi(\lambda) = (p,q)$ with $q > 1$.

Proof. By Theorem 4.3, for any $f \in \mathcal{L}(n,m,\lambda)$ (generic or not) the Newton polygon $\mathcal{NP}(P(f))$ of $P(f)$ coincides with the Newton polygon of the set given in (7).

If $q = 1$ then it follows, by item 2 of Lemma 4.2, that $T_\lambda = \emptyset$ and consequently, $\mathcal{NP}(P(f)) = \mathcal{NP}(E_\lambda)$ for any $f \in \mathcal{L}(n,m,\lambda)$ generic or not.

Let us suppose that $q > 1$ and consider a generic $f \in \mathcal{L}(n,m,\lambda)$ given as in (4) whose coefficients a_{ij} live in the open Zariski set $U := \{a_{ij} \neq 0 : (i,j) \in T_\lambda\}$, that is for a generic $f \in \mathcal{L}(n,m,\lambda)$, we get $T_\lambda \cap \text{supp}\,(f) = T_\lambda$, so the Newton polygon of $P(f)$ coincides with the Newton polygon of the set

$$E_\lambda \ \cup \ \{(i,j-1),(i-1,j) : \ (i,j) \in T_\lambda\}\,. \tag{8}$$

Notice that the point $(i,j-1)$ and $(i-1,j)$ in (8) arise from the point $(i,j) \in T_\lambda \subset \mathcal{I}_\lambda$ when we consider f_y and f_x respectively. If $(i,j) \in \mathcal{I}_\lambda$ then $i > p + \left(\frac{(q-j)m}{n}\right)$.

Therefore, $i = p + \left[\frac{(q-j)m}{n}\right]$ is the lowest possible value of i such that $(i,j) \in \text{supp}\,(f)$.

Hence, for $(i,j) \in T_\lambda$ we get:

- the minimal value of i such that $(i,j-1) \in \text{supp}\,(f_y)$ is $p + \left[\frac{(q-j)m}{n}\right]$;
- the minimal value of i for $(i-1,j) \in supp(f_x)$ is $p + \left[\frac{(q-j)m}{n}\right] - 1$.

Since for every $(i,j) \in \mathcal{I}_\lambda$ we get

$$p + \left[\frac{(q-j-1)m}{n}\right] \le p + \left[\frac{(q-j)m}{n}\right] - 1,$$

and it follows that $i(\rho) = p + \left[\frac{(q-\rho-1)m}{n}\right]$, is the minimal value of i such that $(i(\rho),\rho) \in \text{supp}\,(P(f))$ for $0 \le \rho \le q-2$, that is, the Newton polygon of $P(f)$ equals to the Newton polygon of the set $E_\lambda \ \cup \ \{(i(\rho),\rho) : \ 0 \le$

$\rho \leq q-2\}$. However, since the point $\left(p+\left[\frac{(q-\rho-1)m}{n}\right],\rho\right)$ arise the point $\left(p+\left[\frac{(q-j)m}{n}\right],j\right) \in \mathcal{I}_\lambda$ for $1 \leq j \leq q-1$, it is sufficient to consider the case when the point $(i(\rho),\rho)$ belongs to T_λ.

Since $q \neq 1$, by (6), we have $(i,j) \in T_\lambda$ if and only if

$$p+\frac{(q-j)m}{n} < i \leq p+\frac{(q-j)(m-p-1)}{q-1}, \ 1 \leq j \leq q-1 \ \text{ and } i \neq m-1.$$

Thus, for each $1 \leq j \leq q-1$ we get that $\{i \in \mathbb{N} : (i,j) \in T_\lambda\}$ is the emptyset when $\left\lceil\frac{(q-j)m}{n}\right\rceil > \frac{(q-j)(m-p-1)}{q-1}$; otherwise $\min\{i \in \mathbb{N} : (i,j) \in T_\lambda\} = p+\left\lceil\frac{(q-j)m}{n}\right\rceil$.

Hence, $\mathcal{NP}(P(f))$ coincides with the Newton polygon of the set $E_\lambda \cup \{\left(p+\left[\frac{(q-j-1)m}{n}\right],j\right) : 0 \leq j \leq q-2\}$ such that $\left\lceil\frac{(q-j)m}{n}\right\rceil \leq \frac{(q-j)(m-p-1)}{q-1}$ and we get the proposition. □

By Proposition 4.3, for any $f \in \mathcal{L}(n,m,\lambda)$ satisfying supp $(f) \cap T_\lambda = \emptyset$ we get $\mathcal{NP}(P(f)) = \mathcal{NP}(E_\lambda)$. In what follows we will determine the topological type of $P(f)$ when its Newton polygon equals the Newton polygon of E_λ. Notice that this corresponds to considering, for instance, the curve of equation

$$f = y^n - x^m + x^p y^q \in \mathcal{L}(n,m,\lambda), \tag{9}$$

where $\Phi(\lambda) = (p,q)$.

Proposition 4.5. *Given $f = y^n - x^m + x^p y^q \in \mathcal{L}(n,m,\lambda)$ with $\Phi(\lambda) = (p,q)$ and $q > 1$ the topological type of the generic polar $P(f)$ is determined by n, m and λ as follows:*

1) *If $\lambda < p+q+m-n$ then $P(f)$ has $d_1 := \gcd(n-q,p)$ branches $\{P_i\}_{i=1}^{d_1}$ with semigroup of values $\langle \alpha_1, \beta_1 \rangle$ where $\alpha_1 = \frac{n-q}{d_1}$ and $\beta_1 = \frac{p}{d_1}$; and $d_2 := \gcd(q-1, m-p-1)$ branches $\{Q_i\}_{i=1}^{d_2}$ with semigroup of values $\langle \alpha_2, \beta_2 \rangle$ where $\alpha_2 := \frac{q-1}{d_2}$ and $\beta_2 := \frac{m-p-1}{d_2}$. Moreover, $\mathrm{I}(P_i,P_j) = \alpha_1\beta_1$, $\mathrm{I}(Q_i,Q_j) = \alpha_2\beta_2$ and $\mathrm{I}(P_i,Q_j) = \min\{\alpha_1\beta_2, \alpha_2\beta_1\}$ for $i \neq j$.*

2) *If $\lambda \geq p+q+m-n$ then $P(f)$ is the union of $d := \gcd(n-1,m-1)$ equisingular branches P_i with semigroup of values $\langle \alpha, \beta \rangle$ where $\alpha = \frac{n-1}{d}$ and $\beta = \frac{m-1}{d}$ and such that $\mathrm{I}(P_i,P_j) = \alpha\beta$, for $i \neq j$.*

Proof. By Proposition 4.3, the Newton polygon of the generic polar curve $P(f)$ equals the Newton polygon of the set E_λ. We distinguish two cases:

1) $\lambda < p+q+m-n$ that is, the point $(p,q-1)$ is below the line $l_{N,M}$.

In this case the Newton polygon of $P(f)$ has two compact edges contained in $l_{N,B}$ and $l_{B,M}$ respectively.

According to Proposition 2.1, associated with the compact edge contained in $l_{N,B}$ we get $d_1 := \gcd(n-q,p)$ branches $\{P_i\}_{i=1}^{d_1}$ with semigroup of values $\langle\alpha_1,\beta_1\rangle$ where $\alpha_1 = \frac{n-q}{d_1}$ and $\beta_1 = \frac{p}{d_1}$. Associated with the compact edge contained in $l_{B,M}$ we get $d_2 = \gcd(q-1, m-p-1)$ branches $\{Q_i\}_{i=1}^{d_2}$ with semigroup of values $\langle\alpha_2,\beta_2\rangle$ where $\alpha_2 = \frac{q-1}{d_2}$ and $\beta_2 = \frac{m-p-1}{d_2}$. Moreover, for $i \neq j$, $\mathrm{I}(P_i,P_j) = \alpha_1\beta_1$, $\mathrm{I}(Q_i,Q_j) = \alpha_2\beta_2$ and $\mathrm{I}(P_i,Q_j) = \min\{\alpha_1\beta_2, \alpha_2\beta_1\}$.

2) $\lambda \geq p+q+m-n$, that is the point $(p,q-1)$ is on or above the line $l_{N,M}$.

In this case the Newton polygon of the generic polar curve $P(f)$ has only one compact edge L contained in the line $l_{N,M}$. The univariate polynomial associated with L is $(P(f))_L(1,t) = bnt^{n-1} - am$ (when $(p,q-1)$ is above $l_{N,M}$) or $(P(f))_L(1,t) = bnt^{n-1} + bqt^{q-1} - am$ (when $(p,q-1)$ is on $l_{N,M}$). Since we are considering the generic polar of f in both cases there exists an open set $U \subset \mathbb{P}^1$ such that for any $(a:b) \in U$ the polar $P(f)$ is Newton non-degenerate. In particular, by Proposition 2.1, the generic polar $P(f)$ is the union of $d = \gcd(n-1,m-1)$ equisingular branches $\{P_i\}_{i=1}^{d}$ with $\Gamma_{P_i} = \langle\alpha,\beta\rangle$, for any i, where $\alpha = \frac{n-1}{d}, \beta = \frac{m-1}{d}$ and $\mathrm{I}(P_i,P_j) = \alpha\beta$ for $i \neq j$. $\square$

As a consequence of Lemma 4.2 and Proposition 4.5 we have

Corollary 4.6. *Let $f \in \mathcal{L}(n,m,\lambda)$ where $\lambda \in \mathcal{Z}(n,m)\cup\{\infty\}$ and with $\Phi(\lambda) = (p,q)$ if $\lambda \neq \infty$.*

1. *If $\lambda > 2m-n$ (admitting $\lambda = \infty$) then the generic polar curve $P(f)$ is equisingular to the monomial curve $y^{n-1} - x^{m-1} = 0$. Moreover, if $d := \gcd(n-1,m-1)$ then $P(f)$ have d irreducible equisingular components $\{P_i\}_i^d$ such that $\Gamma_{P_i} = \left\langle\frac{n-1}{d}, \frac{m-1}{d}\right\rangle$ for any $i \in \{1,\ldots,d\}$ and $\mathrm{I}(P_i,P_j) = \frac{(n-1)(m-1)}{d^2}$ for any $i \neq j$.*
2. *If $q = 1$ then the generic polar curve $P(f)$ is equisingular to the monomial curve $y^{n-1} - x^p = 0$. Moreover, if $d := \gcd(n-1,p)$ then $P(f)$ have d irreducible equisingular components $\{P_i\}_i^d$ such that $\Gamma_{P_i} = \left\langle\frac{n-1}{d}, \frac{p}{d}\right\rangle$ for any $i \in \{1,\ldots,d\}$ and $\mathrm{I}(P_i,P_j) = \frac{(n-1)p}{d^2}$ for any $i \neq j$.*
3. *For any $f \in K(3,m)$ the generic polar curve $P(f)$ has the topological type described in item 1) if $\lambda = \infty$ or as described in item 2) if $\lambda \neq \infty$.*
4. *For any $f \in K(n,n+1)$ the generic polar curve $P(f)$ has the topological type described in item 1).*

Proof. First of all, notice that in all cases we get, by Lemma 4.2, $T_\lambda = \emptyset$. In this way, by Theorem 4.3, it follows that $\mathcal{NP}(P(f)) = \mathcal{NP}(E_\lambda)$ where $E_\lambda =$

$\{(0, n-1), (m-1, 0), (p, q-1)\}$. So, the topological type of $P(f)$ is the same of the topological type of the generic polar curve of $y^n - x^m + x^p y^q$ as given in (9).

If $\lambda > 2m - n$ then $(p, q-1)$ is above the line $l_{N,M}$ and the result follows by item 2 of Proposition 4.5.

If $q = 1$ then $\mathcal{NP}(P(f)) = \mathcal{NP}(E_\lambda)$ is equal to the Newton polygon of the set $\{(0, n-1), (p, 0)\}$ and the result follows from Proposition 2.1.

If $f \in K(3, m)$ and $\lambda_f = \infty$ the item 1) gives the result. On the other hand, if $\lambda_f \neq \infty$ then we get $q = 1$ for any $\lambda \in \mathcal{Z}(3, m)$ and we are in the hypothesis of item 2).

If $f \in K(n, n+1)$ then, as we have verified in the proof of Lemma 4.2, any $f \in K(n, n+1)$ satisfies $\lambda_f > 2m - n$ and item 1) give us the result. □

In the following theorem, we summarize all the previous results concerning the topological type of the generic polar $P(f)$ for $f \in \mathcal{L}(n, m, \lambda)$.

Theorem 4.7. *Let $f = y^n - x^m + x^p y^q + \sum_{(i,j)\in\mathcal{I}_\lambda} a_{i,j} x^i y^j \in \mathcal{L}(n, m, \lambda)$ for $\lambda \in \mathcal{Z}(n, m)$ such that $\Phi(\lambda) = (p, q)$. Then*

1. *If $q = 1$, that is, $\lambda = 2m - (m-p)n$ then $P(f)$ is topologically equivalent to $y^{n-1} - x^p$.*
2. *If $\lambda > 2m - n$ then $P(f)$ is topologically equivalent to $y^{n-1} - x^{m-1}$.*
3. *If $\lambda < 2m - n$ and $q > 1$ then the Newton polygon of $P(f)$ is equal to the Newton polygon of the generic polar of*

$$y^n - x^m + x^p y^q + \sum_{(i,j)\in T_\lambda \cap \operatorname{supp}(f)} a_{i,j} x^i y^j.$$

Moreover, if $T_\lambda = \emptyset$ then the topological type of $P(f)$ is given by Proposition 4.5.

Example 4.8. *In the sequel we apply our results to describe the topological type for plane branches in $K(5, 12)$, that is, $n = 5$ and $m = 12$.*
The set of possible finite Zariski invariants in this equisingularity class is (see the lower part of Figure 1)

$$\mathcal{Z}(5, 12) = \{13, 14, 16, 18, 21, 23, 26, 28, 33, 38\}.$$

Since $\Phi(14) = (10, 1)$, by item 1 of Theorem 4.7, for any $f \in \mathcal{L}(5, 12, 14)$ the generic polar $P(f)$ is topologically equivalent to $y^4 - x^{10} = (y^2 - x^5)(y^2 + x^5)$, that is, $P(f)$ has two equisingular branches with semigroup $\langle 2, 5\rangle$ and the intersection multiplicity of them is 10.

By item 2 of Theorem 4.7, for any $f \in \mathcal{L}(5,12,\lambda)$ *with* $\lambda > 19 = 2m - n$, *that is* $\lambda \in \{21,23,26,28,33,38\}$, *the generic polar* $P(f)$ *is topologically equivalent to* $y^4 - x^{11}$. *So,* $P(f)$ *is irreducible with semigroup* $\langle 4,11\rangle$.

After the description of T_λ *in (6) we get* $T_{16} = \emptyset$. *Since* $\Phi(16) = (8,2)$ *and* $\lambda = 16 < 17 = 8+2-5+12 = p+q-n+m$, *according to item 1 of Proposition 4.5, the generic polar for any* $f \in \mathcal{L}(5,12,16)$ *has a branch* P_1 *with semigroup* $\langle 3,8\rangle$ *and a smooth branch* Q_1 *with* $\mathrm{I}(P_1,Q_1) = 8$.

For $\lambda = 18$ *we also get* $T_{18} = \emptyset$. *Since* $\Phi(18) = (6,3)$ *and* $\lambda = 18 > 16 = 6+3-5+12 = p+q-n+m$ *then, by item 2 of Proposition 4.5 the generic polar for any* $f \in \mathcal{L}(5,12,18)$ *is irreducible with semigroup* $\langle 4,11\rangle$.

Consider now $\lambda = 13$. *We get* $\Phi(13) = (5,3)$ *and* $T_{13} = \{(10,1),(8,2)\}$ *(see Figure 6). So, by item 3 of Theorem 4.7, for any* $f \in \mathcal{L}(5,12,13)$ *the Newton polygon of* $P(f)$ *is equal to the Newton polygon of the generic polar of* $y^5 - x^{12} + x^5y^3 + a_{10,1}x^{10}y + a_{8,2}x^8y^2$. *We have the following possibilities:*

- *If* $a_{10,1} = a_{8,2} = 0$, *then* $T_{13} \cap \mathrm{supp}\,(f) = \emptyset$. *Since* $\lambda = 13 < 15 = p+q-n+m$, *the item 1 of Proposition 4.5 allows us to conclude that the generic polar* $P(f)$ *has one branch* P_1 *with semigroup* $\langle 2,5\rangle$, *two branches* Q_1 *and* Q_2 *with semigroup* $\langle 1,3\rangle = \mathbb{N}$ *with* $\mathrm{I}(Q_1,Q_2) = 3$ *and* $\mathrm{I}(P_1,Q_i) = 5$ *for* $i = 1,2$.

- *If* $a_{10,1} \neq 0$ *then* $\mathcal{ND}(P(f)) = \mathcal{ND}(\{(0,4),(0,10),(5,2)\})$. *For* $a_{10,1} \neq 9/20$ *the generic polar has two equisingular branches* Q_1 *and* Q_2 *with semigroup* $\langle 2,5\rangle$ *and* $\mathrm{I}(Q_1,Q_2) = 10$. *For* $a_{10,1} = 9/20$ *the generic polar is degenerate and it corresponds to a branch with semigroup* $\langle 4,10,21\rangle$. *In both cases, we get the topological type by the computation of the first terms of the Puiseux parametrizations of* $P(f)$.

- *If* $a_{10,1} = 0$ *and* $a_{8,2} \neq 0$ *then the Newton diagram of* $P(f)$ *is*

$$\mathcal{ND}(\{(0,4),(5,2),(8,1),(11,0\})$$

  and computing the Puiseux parametrizations of $P(f)$ *we conclude that the topological type of* $P(f)$ *is equal to the case* $a_{10,1} = a_{8,2} = 0$.

In [3] is presented the description of the all possible topological type of the polar generic curve for any plane curve in $K(5,12)$. The method employed consists in, fixing an analytical invariant denoted by Λ and for each associated normal form of Λ given by a Puiseux parametrization, considering the equation of each normal formal to analyze the possible topological type of the generic polar curve. That result is in line with our study when considering $f \in \mathcal{L}(n,m,\lambda)$, but differ for the description of particular value of parameter since the parameters in the parametrization are related by are not the same of parameters in the expression $f \in \mathbb{C}\{x, y\}$. Notice that for $f \in \mathcal{L}(5, 12, \lambda)$, except for the case $\lambda = 13$, we have the description of the topological type of $P(f)$ directly by n, m and λ, that is can be obtained independently how the curve is presented, by Puiseux parametrization or by an equation.

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