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RESEARCH

On some indices of foliations and applications

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Abstract

In this paper, we establish a relationship between the Milnor number, the χ-number and the Tjurina number of a foliation with respect to an effective balanced divisor of separatrices. Moreover, using the Gómez-Mont–Seade–Verjovsky index, we prove that the difference between the multiplicity and the Tjurina number of a foliation with respect to a reduced curve is independent of the foliation. We also derive a local formula for the Tjurina number of a foliation with respect to a reduced curve. From a global point of view, these results lead to the following consequences: We provide a new proof of a global result regarding the multiplicity of a foliation due to Cerveau–Lins Neto and a new proof of a Soares's inequality for the sum of the Milnor number of an invariant curve of a foliation. Additionally, we obtain bounds for the global Tjurina number of a foliation on the complex projective plane. Finally, we provide an answer to the conjecture posed by Alcántara and Mozo-Fernández about foliations on the complex projective plane having a unique singularity.

Keywords: Holomorphic foliations, Milnor number, Tjurina number, Multiplicity of a foliation along a divisor of separatrices, GSV-index.

Mathematics Subject Classification: Primary 32S65, 32M25

1 Introduction

The aim of this paper is to study indices of local and global foliations in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$. We focus particularly on four indices of the foliations: the *Milnor number*, the *Gómez-Mont–Seade–Verjovsky index* (abbreviated as the GSV-index), the *multiplicity* and the *Tjurina number* with respect to an invariant curve.

First, using known local formulas for the GSV-index and the Tjurina number of a singular curve, we present a local formula for the Tjurina number of a foliation (see Proposition 3.1). Next, we prove in Proposition 4.1 that the difference between the multiplicity and the Tjurina number of a foliation with respect to a reduced curve does not depend on the foliation. More specifically, let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$ and let C be any $\mathcal{F}$-invariant reduced curve. Denoting by $\mu_p(\mathcal{F}, C)$ and $\tau_p(\mathcal{F}, C)$ the multiplicity and Tjurina number of $\mathcal{F}$ along C, respectively, we obtain

$$\mu_p(\mathcal{F}, C) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C),$$

where $\mu_p(C)$, $\tau_p(C)$ are the Milnor and Tjurina numbers of C. This result leads to several applications; for instance, we show that

$$GSV_p(\mathcal{F}, C) < 4\tau_p(\mathcal{F}, C) - 3\mu_p(\mathcal{F}, C),$$

where $GSV_p(\mathcal{F}, C)$ denotes the GSV-index of $\mathcal{F}$ along C at p. Some applications also are presented in Sects. 4 and 7; for example, we provide a positive answer to a question posed in [20].

In Sect. 5, following the terminology of balanced divisor of separatrices of a foliation introduced by Genzmer [24], we present Proposition 5.1 that generalizes Corollary B of [19]. We recall that a foliation $\mathcal{F}$ is said to be a *generalized curve* if there are no saddle nodes in its reduction process of singularities. This concept was introduced by Camacho, Lins Neto and Sad in [9] and the non-dicritical case, delimits a family of foliations whose topology is closely related to that of their separatrices (i.e., local invariant curves). Generalized curve foliations are part of the broader family of *second type foliations*, introduced by Mattei and Salem in [33]. Foliations in this family may admit saddle nodes in the reduction process of singularities, provided that they are not *tangent saddle nodes* (see [21, Definition 2.1]). In order to study second type foliations, the χ-number of $\mathcal{F}$ was introduced in [19, Section 3]. This number, denoted by $\chi_p(\mathcal{F})$, can be interpreted as the difference $\mu_p(\mathcal{F}) - \mu_p(\mathcal{F}, \mathcal{B})$, where $\mu_p(\mathcal{F})$ is the Milnor number of $\mathcal{F}$ at p and $\mu_p(\mathcal{F}, \mathcal{B})$ is the multiplicity of $\mathcal{F}$ along a balanced divisor of separatrices $\mathcal{B}$. For more details on balanced divisor of separatrices of a foliation, we refer the reader to [24,26].

Now, we can establish our main result as follows:

Theorem A *Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$ and $\mathcal{B}$ be an effective primitive balanced divisor of separatrices for $\mathcal{F}$ at p. Then*

$$\frac{\mu_p(\mathcal{F})}{\tau_p(\mathcal{F}, \mathcal{B}) + \chi_p(\mathcal{F})} < \frac{4}{3}.$$

Moreover, if $\mathcal{F}$ is of second type then $\frac{\mu_p(\mathcal{F})}{\tau_p(\mathcal{F},\mathcal{B})} < \frac{4}{3}$. In particular, if $\mathcal{F}$ is a non-dicritical foliation of second type, and C is the total union of separatrices, then $\frac{\mu_p(\mathcal{F})}{\tau_p(\mathcal{F},C)} < \frac{4}{3}$.

The proof of Theorem A will be given in Sect. 6. Finally, we apply our local results in three ways. First, we provide a new proof of a result by Cerveau–Lins Neto [12, Proposition 7.1] and a new proof of a Soares's inequality [39, Theorem 7.3] for the sum of the Milnor number of an invariant curve of a foliation. In Theorem 7.5, we establish bounds for the global Tjurina number of a foliation on the complex projective plane. Lastly, we answer a question posed by Alcántara and Mozo-Fernández [1, p. 16] regarding the existence of non-dicritical logarithmic foliations with a unique singular point and an invariant algebraic curve passing through the singularity.

2 Basic tools

To establish the terminology and notation, we recall some basic concepts of the theory of holomorphic foliations. Unless stated otherwise, throughout this text, $\mathcal{F}$ denotes a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$. In local coordinates (x, y) centered at p, the foliation $\mathcal{F}$ is given by a holomorphic 1-form

$$\omega = P(x, y)dx + Q(x, y)dy,$$

or by its dual vector field

$$v = -Q(x, y)\frac{\partial}{\partial x} + P(x, y)\frac{\partial}{\partial y},$$

where $P(x, y), Q(x, y) \in \mathbb{C}\{x, y\}$ are relatively prime and $\mathbb{C}\{x, y\}$ is the ring of complex convergent power series in two variables. The *algebraic multiplicity* $\nu_p(\mathcal{F})$ is the minimum of the orders $\nu_p(P)$, $\nu_p(Q)$ at p of the coefficients of a local generator of $\mathcal{F}$.

Let $f(x, y) \in \mathbb{C}[[x, y]]$, where $\mathbb{C}[[x, y]]$ is the ring of complex formal power series in two variables. We say that $C : f(x, y) = 0$ is *invariant* by $\mathcal{F}$ or $\mathcal{F}$*-invariant* if

$$\omega \wedge df = (f.h)dx \wedge dy,$$

for some $h \in \mathbb{C}[[x, y]]$. If C is an irreducible $\mathcal{F}$-invariant curve, then we will say that C is a *separatrix* of $\mathcal{F}$ at p. The separatrix C is analytical if f is convergent. We denote by $\mathrm{Sep}_p(\mathcal{F})$ the set of all separatrices of $\mathcal{F}$ at p. When $\mathrm{Sep}_p(\mathcal{F})$ is a finite set, we will say that the foliation $\mathcal{F}$ is *non-dicritical* and the union of all elements of $\mathrm{Sep}_p(\mathcal{F})$ is called *total union of separatrices* of $\mathcal{F}$ at p. Otherwise, we will say that $\mathcal{F}$ is a *dicritical* foliation.

2.1 GSV-index

The GSV-index was introduced by Gómez-Mont–Seade–Verjovsky in [28]. This index is a topological invariant for vector fields on singular complex varieties and is closely related to the Euler–Poincaré characteristic of the Milnor fiber (see, for instance, [8, Theorem 3.1]). It also relates to the Schwartz index and local Euler obstruction through a proportionality theorem [5, Theorem 3.1]. Below we will present Brunella's definition of the GSV-index for one-dimensional holomorphic foliations (see [6]).

Let $\mathcal{F} : \omega = 0$ be a singular foliation at $(\mathbb{C}^2, p)$. Let $C : f(x, y) = 0$ be an $\mathcal{F}$-invariant curve, where $f(x, y) \in \mathbb{C}[[x, y]]$ is reduced. Then, as in the convergent case, there are $g, h \in \mathbb{C}[[x, y]]$ (depending on f and ω), with f and g and f and h relatively prime and a 1-form η (see [40, Lemma 1.1 and its proof]) such that

$$g\omega = hdf + f\eta. \tag{1}$$

Expression (1) was first proved by K. Saito [37]. Subsequently, A. Lins Neto [32] obtained the same result in the case where f is irreducible, and T. Suwa [40], in the case where f is reduced, showed that the functions f, g, h can be chosen in such a way that each pair (f, g) and (f, h) is relatively prime.

The *GSV-index* of the foliation $\mathcal{F}$ at $(\mathbb{C}^2, p)$ with respect to an analytic $\mathcal{F}$-invariant curve C is

$$GSV_p(\mathcal{F}, C) = \frac{1}{2\pi i}\int_{\partial C}\frac{g}{h}d\left(\frac{h}{g}\right), \tag{2}$$

where $g, h \in \mathbb{C}\{x, y\}$ are from (1).

2.2 The multiplicity of a foliation along a separatrix

Let $\mathcal{F}$ be a germ of a singular foliation at $(\mathbb{C}^2, p)$ induced by the vector field v and B be a separatrix of $\mathcal{F}$ at p. Let $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, p)$ be a primitive parametrization of

B. Camacho–Lins Neto–Sad [9, Section 4] defined the *multiplicity of $\mathcal{F}$ along B at p* as $\mu_p(\mathcal{F}, B) := \text{ord}_t\theta(t)$, where $\theta(t)$ is the unique vector field at $(\mathbb{C}, 0)$ such that $\gamma_*\theta(t) = v \circ \gamma(t)$. If $\omega = P(x, y)dx + Q(x, y)dy$ is a 1-form inducing $\mathcal{F}$ and $\gamma(t) = (x(t), y(t))$, we have

$$\theta(t) = \begin{cases} -\frac{Q(\gamma(t))}{x'(t)} & \text{if } x(t) \neq 0 \\ \frac{P(\gamma(t))}{y'(t)} & \text{if } y(t) \neq 0. \end{cases}$$

2.3 The Tjurina number of a foliation along an invariant reduced curve

Let $\mathcal{F}$ be the germ of holomorphic foliation defined by $\omega = P(x, y)dx + Q(x, y)dy$ at $(\mathbb{C}^2, p)$, and let $C : f(x, y) = 0$ be an *invariant reduced curve*. The *Tjurina number* of $\mathcal{F}$ with respect to C is by definition

$$\tau_p(\mathcal{F}, C) := \dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(P, Q, f).$$

This number appears for the first time in [27], and the terminology of Tjurina number of $\mathcal{F}$ with respect to C was given in [10, p. 159].

3 Local formulas for the Tjurina number of a foliation

Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$, and let $C = \cup_{j=1}^r C_j$ be an $\mathcal{F}$-invariant reduced curve such that each C_j is irreducible. Then, from [42, Theorem 6.5.1], [30, Proposition 3.2], [31, Equation (2.2), p. 329] and [7, p. 29], we obtain the following formulas, respectively:

$$\mu_p(C) = \sum_{j=1}^{r} \mu_p(C_j) + 2\left(\sum_{1 \leq i < j \leq r} I_p(C_i, C_j)\right) - r + 1, \tag{3}$$

$$\tau_p(C) = \sum_{j=1}^{r} \tau_p(C_j) + \left(\sum_{1 \leq i < j \leq r} I_p(C_i, C_j)\right) + \Theta, \tag{4}$$

$$\mu_p(\mathcal{F}, C) = \sum_{j=1}^{r} \mu_p(\mathcal{F}, C_j) - r + 1, \tag{5}$$

$$GSV_p(\mathcal{F}, C) = \left(\sum_{j=1}^{r} GSV_p(\mathcal{F}, C_j)\right) - 2\sum_{1 \leq i < j \leq r} I_p(C_i, C_j), \tag{6}$$

where $I_p(C_i, C_j)$ denotes the intersection multiplicity of the branches C_i and C_j at p, and Θ is a number that depends on the module of Kähler differentials of C, whose explicit formula can be found in [30, Theorem 5.1]. Moreover, the *Milnor number* $\mu_p(\mathcal{F})$ of the foliation $\mathcal{F}$ at a point p, defined by the 1-form $\omega = P(x, y)dx + Q(x, y)dy$, is given by $\mu_p(\mathcal{F}) = I_p(P, Q)$. Recall that we assume P and Q to be coprime, so $\mu_p(\mathcal{F})$ is a nonnegative integer. In [9, Theorem A], it was proved that the Milnor number of a foliation is a topological invariant.

Analogous to the local formula for the Tjurina number of curves, we provide an local formula for the Tjurina number of a foliation $\mathcal{F}$ with respect to a reduced $\mathcal{F}$-invariant curve C, in terms of the Tjurina numbers of $\mathcal{F}$ with respect to the irreducible components of C.

Proposition 3.1 *Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$ and let $C = \cup_{j=1}^{r} C_j$ be a $\mathcal{F}$-invariant reduced curve. Then*

$$\tau_p(\mathcal{F}, C) = \sum_{j=1}^{r} \tau_p(\mathcal{F}, C_j) - \left(\sum_{1 \leq i < j \leq r} I_p(C_i, C_j) \right) + \Theta.$$

Proof After [19, Proposition 6.2], we get $\tau_p(\mathcal{F}, C) = GSV_p(\mathcal{F}, C) + \tau_p(C)$. Hence, by equalities (6) and (4), we obtain

$$\begin{aligned}\tau_p(\mathcal{F}, C) &= GSV_p(\mathcal{F}, C) + \tau_p(C) \\ &= \sum_{j=1}^{r} GSV_p(\mathcal{F}, C_j) - 2\sum_{i<j} I_p(C_i, C_j) + \sum_{j=1}^{r} \tau_p(C_j) + \left(\sum_{i<j} I_p(C_i, C_j) \right) + \Theta \\ &= \sum_{j=1}^{r} \tau_p(\mathcal{F}, C_j) - \left(\sum_{i<j} I_p(C_i, C_j) \right) + \Theta.\end{aligned}$$

□

We illustrate Proposition 3.1 with the following example.

Example 3.2 We borrow from [19, Example 6.5] the family of foliations $\mathcal{F}_k$, $k \geq 3$, given by the 1-form

$$\omega_k = y\left(2x^{2k-2} + 2(\lambda+1)x^2y^{k-2} - y^{k-1}\right)dx + x\left(y^{k-1} - (\lambda+1)x^2y^{k-2} - x^{2k-2}\right)dy,$$

which is a family of dicritical foliations which are not of second type. An effective balanced divisor of separatrices of $\mathcal{F}_k$ is $\mathcal{B} = \mathcal{B}_0 : xy = 0$. Let $B_1 : x = 0$ and $B_2 : y = 0$. Hence, $\tau_0(\mathcal{F}_k, \mathcal{B}) = 3k - 2$, $\tau_0(\mathcal{F}_k, B_1) = k$, and $\tau_0(\mathcal{F}_k, B_2) = 2k - 1$; since $\Theta = 0$ by (4), we get

$$\begin{aligned}\tau_0(\mathcal{F}_k, \mathcal{B}) &= \tau_0(\mathcal{F}_k, B_1) + \tau_0(\mathcal{F}_K, B_2) - I_0(B_1, B_2) + \Theta \\ &= k + 2k - 1 - 1 + 0 = 3k - 2.\end{aligned}$$

4 The GSV-index, the multiplicity and the Tjurina number of a foliation

In this section, we present some local results involving the GSV-index, the multiplicity and the Tjurina number of a singular foliation $\mathcal{F}$ along an $\mathcal{F}$-invariant curve.

Proposition 4.1 *Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$ and let C be any $\mathcal{F}$-invariant reduced curve. Then*

$$GSV_p(\mathcal{F}, C) = \mu_p(\mathcal{F}, C) - \mu_p(C) \tag{7}$$

and

$$\mu_p(\mathcal{F}, C) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C). \tag{8}$$

Proof Let $C = \cup_{j=1}^{r} C_j$. It follows from [20, Eq. 5] that

$$GSV_p(\mathcal{F}, C_j) = \mu_p(\mathcal{F}, C_j) - \mu_p(C_j) \quad \text{for all} \quad j = 1, \ldots, r. \tag{9}$$

By combining (9), (3), (5) and (6), we obtain $GSV_p(\mathcal{F}, C) = \mu_p(\mathcal{F}, C) - \mu_p(C)$. Finally, using [19, Proposition 6.2], we get

$$\mu_p(\mathcal{F}, C) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C).$$

□

After Proposition 4.1 and [2, Theorem 3.2], we prove Corollary 4.2 which provides a positive answer to Question 1 posed in [20].

Corollary 4.2 *Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$ and let C be any $\mathcal{F}$-invariant reduced curve. Then*

$$GSV_p(\mathcal{F}, C) < 4\tau_p(\mathcal{F}, C) - 3\mu_p(\mathcal{F}, C).$$

Proof From Proposition 4.1, Eq. (8) and [2, Theorem 3.2], we obtain

$$\mu_p(\mathcal{F}, C) - \tau_p(\mathcal{F}, C) = \mu_p(C) - \tau_p(C) < \mu_p(C)/4.$$

and proof follows from formula (7). □

Corollary 4.3 *Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$ and let C be a $\mathcal{F}$-invariant reduced curve. Then $\mu_p(\mathcal{F}, C) = \tau_p(\mathcal{F}, C)$ if and only if after a suitable holomorphic coordinate transformation C is quasi-homogeneous.*

Proof It follows from Proposition 4.1 and [36, Satz p.123, (a) and (d)]. □

Corollary 4.4 *Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$ and let $C = \cup_{j=1}^{r} C_j$ be a $\mathcal{F}$-invariant reduced curve. Then*

$$\sum_{i=1}^{r} \big(\mu_p(\mathcal{F}, C_i) - \tau_p(\mathcal{F}, C_i)\big) \le \mu_p(\mathcal{F}, C) - \tau_p(\mathcal{F}, C).$$

Proof By [30, Theorem 4.3], we have

$$\sum_{i=1}^{\ell} \big(\mu_p(C_i) - \tau_p(C_i)\big) \le \mu_p(C) - \tau_p(C). \tag{10}$$

Applying Proposition 4.1 to each C_i and to C, and substituting into (10), we obtain the desired inequality. □

4.1 Values set of fractional ideals and the Tjurina number of a foliation

In this subsection, we follow the notations from [3]. Let $C : f(x, y) = 0$ be a (reduced) germ of complex plane curve singularity at $p \in \mathbb{C}^2$ with equation $f = \Pi_{i=1}^{r} f_i = 0$, where $f_i \in \mathbb{C}\{x, y\}$ is irreducible. Each f_i defines a branch C_i and its analytic type is characterized by the local ring $\mathcal{O}_i := \mathbb{C}\{x, y\}/(f_i)$ up to $\mathbb{C}$-algebra isomorphism. The field of fractions $\mathcal{H}_i$ of $\mathcal{O}_i$ is isomorphic to $\mathbb{C}(t_i)$ and associated with it we have a canonical discrete valuation $\nu_i : \mathcal{H}_i \to \overline{\mathbb{Z}} := \mathbb{Z} \cup \{\infty\}$. If $\overline{h} \in \mathcal{O}_i$ with $h \in \mathbb{C}\{x, y\}$, then we have $\nu_i(\overline{h}) = I_p(f_i, h)$. The image of ν_i of $\mathcal{O}_i \setminus \{0\}$, that is,

$$S(C_i) := \{\nu_i(\overline{h}) \in \mathbb{N} : \overline{h} \in \mathcal{O}_i, \ \ \overline{h} \neq 0\}$$

is the semigroup of values of C_i. Let $\mathcal{O} := \mathbb{C}\{x, y\}/(f)$. The semigroup of values of C is

$$S(C) := \{(\nu_1(h), \dots, \nu_r(h)) \ : \ h \in \mathcal{O}, h \text{ is not a zero divisor}\} \subseteq \mathbb{N}^r.$$

Let $\Omega^1 = \mathbb{C}\{x, y\}dx + \mathbb{C}\{x, y\}dy$ be the $\mathbb{C}\{x, y\}$-module of holomorphic forms on $\mathbb{C}^2$ and consider the submodule $\mathcal{F}(f) := \mathbb{C}\{x, y\}df + f\Omega^1$. The module of Kähler differentials of C is $\Omega_f := \frac{\Omega^1}{\mathcal{F}(f)}$, that is, the $\mathcal{O}$-module, $\mathcal{O}dx + \mathcal{O}dy$ module and the relation $df = 0$.

If $\varphi_i = (x_i, y_i) \in \mathbb{C}\{t_i\} \times \mathbb{C}\{t_i\}$ is a parametrization (non-necessarily a Newton–Puiseux parametrization) of the branch C_i and $h(x, y) \in \mathcal{O}$, then we denote $\varphi_i^*(h) := h(x_i, y_i) \in \mathbb{C}\{t_i\}$. In addition, given $\omega = A(x, y)dx + B(x, y)dy \in \Omega_f$, we set

$$\varphi_i^*(\omega) := t_i \cdot (A(\varphi_i) \cdot x_i' + B(\varphi_i) \cdot y_i') \in \mathbb{C}\{t_i\},$$

where x_i', y_i' denote, respectively, the derivate of $x_i, y_i \in \mathbb{C}\{t_i\}$ with respect to t_i. Let $\mathcal{H} = \prod_{i=1}^r \mathcal{H}_i$, that is, the total ring of fractions of $\mathcal{O}$ and

$$\varphi^*(\Omega_f) = \{(\varphi_1^*(\omega), \ldots, \varphi_r^*(\omega)) : \omega \in \Omega_f\} \subset \mathcal{H}.$$

By [4, Theorem 3.1],

$$\varphi^*(\Omega_f) \simeq \frac{\Omega_f}{Tor(\Omega_f)}$$

where $Tor(\Omega_f)$ is the torsion submodule of Ω_f. In this way, $\varphi^*(\Omega_f)$ is a fractional ideal of $\mathcal{O}$ and, considering $\nu_i(\omega) := \nu_i(\varphi_i^*(\omega))$, its values set is given by

$$\Lambda_f = \underline{\nu}(\varphi^*(\Omega_f)) = \{\underline{\nu}(\omega) := (\nu_1(\omega), \ldots, \nu_r(\omega)) : \omega \in \Omega_f\}.$$

We state the following theorem of [3, Theorem 2.9].

Theorem 4.5 *Let C be a germ of reduced plane curve at $p \in \mathbb{C}^2$. With the previous notation, let $\overline{\Lambda} = \Lambda_f \cup \{0\}$. Then, we have*

$$\mu_p(C) - \tau_p(C) = d(\overline{\Lambda} \setminus S(C)),$$

where d denotes the distance function defined in [3, Section 2.1].

In the context of foliations, Proposition 4.1 implies the following corollary.

Corollary 4.6 *Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$, and let $C := f(x, y) = 0$ be a $\mathcal{F}$-invariant reduced plane curve. Then*

$$\mu_p(\mathcal{F}, C) - \tau_p(\mathcal{F}, C) = d(\overline{\Lambda} \setminus S(C)),$$

where $\overline{\Lambda} = \Lambda_f \cup \{0\}$.

5 Multiplicity of a foliation along a divisor of separatrices

Following [20, Section 2], we define the multiplicity of $\mathcal{F}$ along any non-empty divisor $\mathcal{B} = \sum_B a_B \cdot B$ of separatrices of $\mathcal{F}$ at p as follows:

$$\mu_p(\mathcal{F}, \mathcal{B}) = \left(\sum_B a_B \cdot \mu_p(\mathcal{F}, B)\right) - \deg(\mathcal{B}) + 1. \quad (11)$$

The notion of balanced divisor of separatrices of a foliation was introduced by Genzmer in [24], see also [26]. By convention, we set $\mu_p(\mathcal{F}, \mathcal{B}) = 1$ for any empty divisor $\mathcal{B}$. In particular, when $\mathcal{B} = B_1 + \cdots + B_r$ is an *effective primitive divisor of separatrices* of the singular foliation $\mathcal{F}$ at p, we recover [31, Equation (2.2), p. 329] (for the reduced plane curve $C := \cup_{i=1}^r B_i$). Hence, if we write $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ where $\mathcal{B}_0$ and $\mathcal{B}_\infty$ are effective divisors we obtain

$$\mu_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}, \mathcal{B}_0) - \mu_p(\mathcal{F}, \mathcal{B}_\infty) + 1. \tag{12}$$

Since $\mu(\mathcal{F}, \mathcal{B}_\infty) \geq 1$, it follows that $\mu_p(\mathcal{F}, \mathcal{B}_0) \geq \mu_p(\mathcal{F}, \mathcal{B})$.

In [19, Section 3], the χ-number of the foliation $\mathcal{F}$ at p was introduced as $\chi_p(\mathcal{F}) = \left(\sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(\mathcal{F}) \xi_q(\mathcal{F})\right) - \xi_p(\mathcal{F})$, where $\mathcal{I}_p(\mathcal{F})$ denotes the set of infinitely near points of $\mathcal{F}$ at p (see [19, p. 7]), $\nu_q(\mathcal{F})$ is the algebraic multiplicity of the strict transform of $\mathcal{F}$ passing by q and $\xi_p(\mathcal{F})$ is the tangency excess of the foliation $\mathcal{F}$ (see [19, Definition 2.3]). By [19, Proposition 4.7], we get

$$\mu_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}) - \chi_p(\mathcal{F}). \tag{13}$$

Since $\chi_p(\mathcal{F})$ is a nonnegative number (see [19, Proposition 3.1]), we have

$$\mu_p(\mathcal{F}) \geq \mu_p(\mathcal{F}, \mathcal{B}). \tag{14}$$

Observe that, if $\mathcal{B}$ is a primitive balanced divisor of separatrices for $\mathcal{F}$ at p and $\mu_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F})$ then, by [19, Proposition 3.1]), $\mathcal{F}$ is of second type or has algebraic multiplicity equal to one. Moreover, if $\mathcal{F}$ is of second type and $\mathcal{B}$ is a primitive balanced divisor of separatrices of $\mathcal{F}$ then $\mu_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F})$.

The following proposition generalizes [19, Corollary B].

Proposition 5.1 *Let $\mathcal{F}$ be a germ of a singular foliation at $(\mathbb{C}^2, p)$. Let $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a primitive balanced divisor of separatrices for $\mathcal{F}$ at p. Then,*

$$\mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, \mathcal{B}_0) = \mu_p(\mathcal{B}_0) - \tau_p(\mathcal{B}_0) - \mu_p(\mathcal{F}, \mathcal{B}_\infty) + \chi_p(\mathcal{F}) + 1.$$

Moreover, $\mu_p(\mathcal{F}) = \tau_p(\mathcal{F}, \mathcal{B}_0)$ if and only if

$$\mu_p(\mathcal{B}_0) - \tau_p(\mathcal{B}_0) = \mu_p(\mathcal{F}, \mathcal{B}_\infty) - \chi_p(\mathcal{F}) - 1.$$

In particular, if $\mathcal{B}$ is an effective divisor then $\mu_p(\mathcal{F}) = \tau_p(\mathcal{F}, \mathcal{B}_0)$ if and only if $\mu_p(\mathcal{B}_0) = \tau_p(\mathcal{B}_0)$ and $\chi_p(\mathcal{F}) = 0$.

Proof It follows from (13) and (12) that $\mu_p(\mathcal{F}) = \mu_p(\mathcal{F}, \mathcal{B}_0) - \mu_p(\mathcal{F}, \mathcal{B}_\infty) + \chi_p(\mathcal{F}) + 1$. The proof follows by applying Proposition 4.1 to $\mathcal{B}_0$ and substituting into the last equation. Moreover, $\mu_p(\mathcal{F}) = \tau_p(\mathcal{F}, \mathcal{B}_0)$ if and only if

$$\mu_p(\mathcal{B}_0) - \tau_p(\mathcal{B}_0) = \mu_p(\mathcal{F}, \mathcal{B}_\infty) - \chi_p(\mathcal{F}) - 1.$$

If $\mathcal{B}$ is effective, then the last part of the proposition follows since $\mu_p(\mathcal{B}_0) - \tau_p(\mathcal{B}_0)$ and $\chi_p(\mathcal{F})$ are nonnegative integers. □

The following corollary generalizes [22, Corollaries 3.4].

Corollary 5.2 *Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, p)$. Let $\mathcal{B}$ be an effective primitive balanced divisor of separatrices. If $\mu_p(\mathcal{F}) = \tau_p(\mathcal{F}, \mathcal{B})$, then, after a suitable holomorphic coordinate transformation, $\mathcal{B}$ is quasi-homogeneous and either $\mathcal{F}$ has algebraic multiplicity* 1 *at p or $\mathcal{F}$ is of second type.*

Proof Since $\mu_p(\mathcal{F}) = \tau_p(\mathcal{F}, \mathcal{B})$ and $\mathcal{B}$ is effective, by Proposition 5.1 we obtain $\mu_p(\mathcal{B}) = \tau_p(\mathcal{B})$. Hence, $\mathcal{B}$ belongs to its Jacobian ideal. The corollary follows from [19, Proposition 3.1 (3)] and [36, Satz p.123, (a) and (d)]. □

6 Proof of Theorem A

In this section, we prove Theorem A. First, we present the following proposition.

Proposition 6.1 *Let $\mathcal{F}$ be a germ of a singular foliation at $(\mathbb{C}^2, p)$ and $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ be a balanced divisor of separatrices for $\mathcal{F}$ at p. If $\mu_p(\mathcal{B}_0) \leq \mu_p(\mathcal{F})$, then*

$$\frac{\mu_p(\mathcal{F})}{\tau_p(\mathcal{F}, \mathcal{B}_0) + \chi_p(\mathcal{F}) - \mu_p(\mathcal{F}, \mathcal{B}_\infty) + 1} < \frac{4}{3}. \tag{15}$$

Proof By [2, Theorem 3.2], we have $\mu_p(\mathcal{B}_0) - \tau_p(\mathcal{B}_0) < \frac{\mu_p(\mathcal{B}_0)}{4}$ so after Proposition 5.1 we get

$$\begin{aligned} \mu_p(\mathcal{F}) - \tau_p(\mathcal{F}, \mathcal{B}_0) &< \frac{\mu_p(\mathcal{B}_0)}{4} - \mu_p(\mathcal{F}, \mathcal{B}_\infty) + \chi_p(\mathcal{F}) + 1 \\ &\leq \frac{\mu_p(\mathcal{F})}{4} - \mu_p(\mathcal{F}, \mathcal{B}_\infty) + \chi_p(\mathcal{F}) + 1, \end{aligned}$$

where we use the hypothesis for the last inequality. Hence,

$$3\mu_p(\mathcal{F}) - 4\tau_p(\mathcal{F}, \mathcal{B}_0) < 4\left(\chi_p(\mathcal{F}) - \mu_p(\mathcal{F}, \mathcal{B}_\infty)\right) + 4$$

and the proposition follows. □

Remark 6.2 The hypothesis $\mu_p(\mathcal{B}_0) \leq \mu_p(\mathcal{F})$ of Proposition 6.1 is essential as the following example illustrates: We consider the Suzuki's foliation (see, for instance, [13, p. 75]) given by

$$\mathcal{F} : \omega = (2y^2 + x^3)dx - 2xydy$$

whose separatrices are $C : x = 0$ and $C_c : y^2 - cx^2 - x^3 = 0$, where $c \in \mathbb{C}$. Consider the balanced divisor of separatrices $\mathcal{B} := C + C_0 + C_1 - C_2$. After some computations, we have $17 = \mu_0(\mathcal{B}_0) \nleq \mu_0(\mathcal{F}) = 5$, $\mu_0(\mathcal{F}, \mathcal{B}_\infty) = 3$, $\tau_0(\mathcal{F}, \mathcal{B}_0) = 5$, and $\chi_0(\mathcal{F}) = 0$ since $\mathcal{F}$ is a curve generalized foliation. So the left term of (15) is $\frac{5}{3}$ and Proposition 6.1 does not hold.

6.1 Proof of Theorem A

Since $\mathcal{B}$ is an effective, primitive, balanced divisor of separatrices for $\mathcal{F}$ at p, we have $\mathcal{B}_\infty = \emptyset$ and $GSV_p(\mathcal{F}, \mathcal{B}) = \Delta_p(\mathcal{F}, \mathcal{B})$, where $\Delta_p(\mathcal{F}, \mathcal{B})$ denotes the polar excess number of $\mathcal{F}$ with respect to $\mathcal{B}$, and this number is nonnegative because it is a sum of nonnegative numbers (see [21, Equality(3.12)]). Hence, $GSV_p(\mathcal{F}, \mathcal{B}) = \mu_p(\mathcal{F}, \mathcal{B}) - \mu_p(\mathcal{B}) \geq 0$ by (7), which implies that $\mu_p(\mathcal{F}, \mathcal{B}) \geq \mu_p(\mathcal{B})$. From (14), we get $\mu_p(\mathcal{F}) \geq \mu_p(\mathcal{F}, \mathcal{B}) \geq \mu_p(\mathcal{B})$. Thus, Theorem A is a consequence of Proposition 6.1. □

Example 6.3 Returning to Example 3.2, the family $\mathcal{F}_k$ of dicritical foliations which are not of second type with $k \geq 3$. For the effective balanced divisor of separatrices $\mathcal{B} = \mathcal{B}_0 :$ $xy = 0$ of $\mathcal{F}_k$, we have $\tau_0(\mathcal{F}_k, \mathcal{B}_0) = 3k - 2$, $\mu_0(\mathcal{F}_k) = k(2k - 1)$ and $\chi_0(\mathcal{F}_k) = 2(k - 1)^2$. Theorem A is verified.

7 Applications to global foliations

In this section, we work with global foliations on the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$. Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^2_{\mathbb{C}}$. The number of points of tangency counted with multiplicities, between $\mathcal{F}$ and a non-invariant $L \subset \mathbb{P}^2_{\mathbb{C}}$ is the *degree* of the foliation, denoted by d.

A holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ of degree d is a homogeneous $1-$ form $\omega = A(x, y, z)dx + B(x, y, z)dy + C(x, y, z)dz$ with A, B and C homogeneous polynomials of degree $d+1$ such that $xA + yB + zC = 0$.

7.1 New proof of a global result of Cerveau–Lins Neto

In [12, p. 885], Cerveau and Lins Neto established the following proposition. The main ingredients in their proof are the reduction of singularities of foliations and the Poincaré–Hopf formula. Here, we provide an alternative proof using our previous results and some known local formulas.

Proposition 7.1 *(Cerveau–Lins Neto [12]) Let C be an irreducible curve on $\mathbb{P}^2_{\mathbb{C}}$ of degree d_0 and $\mathcal{F}$ be a holomorphic foliation of degree d having C as a separatrix. Then*

$$2 - 2g(C) = \sum_p \sum_i \mu_p(\mathcal{F}, C_i) - d_0(d-1),$$

where $g(C)$ is the geometric genus of C and the sum is taken over all local branches C_i of C passing through the singularities p of $\mathcal{F}$ in C.

Proof For each singularity p of $\mathcal{F}$ such that $p \in C$, we have by (7),

$$\mu_p(\mathcal{F}, C) = GSV_p(\mathcal{F}, C) + \mu_p(C).$$

Therefore, by (5)

$$\sum_p \sum_i \mu_p(\mathcal{F}, C_i) - \sum_p (r_p - 1) = \sum_p GSV_p(\mathcal{F}, C) + \sum_p \mu_p(C), \tag{16}$$

where r_p is the number of branches at p. Since C is irreducible, the geometric genus formula gives

$$g(C) = (d_0 - 1)(d_0 - 2)/2 - \sum_p \delta_p, \tag{17}$$

where $\delta_p = \dim_{\mathbb{C}} \widetilde{\mathcal{O}}_p / \mathcal{O}_p$ is the co-length of the local ring $\mathcal{O}_p$ of (C, p) in its normalization. It follows from the *Milnor formula* [34, Theorem 10.5] that

$$\mu_p(C) = 2\delta_p - r_p + 1. \tag{18}$$

The proof follows by combining (17), (18) and the GSV-index formula [7, Proposition 2.3]

$$\sum_p GSV_p(\mathcal{F}, C) = (d+2)d_0 - d_0^2$$

and then substituting these expressions into (16). □

7.2 New proof of a global result of Soares

In [39], Soares studied the problem of bounding the degree of an algebraic curve C invariant for a foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ in terms of the degree of $\mathcal{F}$, in order to give an answer to the *Poincaré Problem*. For a recent account of this problem, we refer to [14] and the references therein. For a local version of the Poincaré problem, see [23].

Before stating the main result of this subsection, we need the following remark.

Remark 7.2 We note that $\tau_p(\mathcal{F}, C) \geq 1$ for all singularity p of $\mathcal{F}$. Indeed, there would exist a point $q \in C$ which is a singularity of $\mathcal{F}$ with $\tau_q(\mathcal{F}, C) = 0$; then, we can take local coordinates centered at q such that $q = (0, 0)$, $\mathcal{F}$ is given by a 1-form $\omega_q = P(x, y)dx + Q(x, y)dy$ and $C = \{f(x, y) = 0\}$. Since $\tau_q(\mathcal{F}, C) = 0$, it follows that $1 \in (P, Q, f)$ so there exist $g, h, k \in \mathbb{C}\{x, y\}$ such that

$$1 = g \cdot P + h \cdot Q + k \cdot f. \tag{19}$$

However, since $q \in C$ is a singularity of $\mathcal{F}$, we have $P(0, 0) = Q(0, 0) = f(0, 0) = 0$. Substituting these values into Eq. (19) leads to a contradiction.

M. Soares proved the following theorem using polar varieties and characteristic classes. In [17], the author obtained a characterization of Soares's formula (in higher dimensions), in terms of the Schwartz and GSV indices of the foliation. Also, in [16], the author obtained another generalization of Soares's theorem for foliations on hypersurfaces with isolated singularities. In [15], the author presents a more general version of Soares's theorem in higher dimensions. Here, we present a simpler proof based on the indices studied above.

Theorem 7.3 *(Soares [39, Theorem II]) Let C be an irreducible curve of degree $d_0 > 1$ that is invariant by a foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geq 2$ with isolated singularities. Then*

$$d_0(d_0 - 1) - \sum_p (\mu_p(C) - 1) \leq (d + 1)d_0,$$

where the sum runs over all singularities p of C.

Proof For each singularity p of $\mathcal{F}$ such that $p \in C$, we have, by (7),

$$\mu_p(C) = \mu_p(\mathcal{F}, C) - GSV_p(\mathcal{F}, C). \tag{20}$$

It follows from (8) that $\mu_p(\mathcal{F}, C) \geq \tau_p(\mathcal{F}, C)$, since $\mu_p(C) \geq \tau_p(C)$, and therefore, $\mu_p(\mathcal{F}, C) \geq 1$ by Remark 7.2. In particular, from (20), we obtain

$$\mu_p(C) - 1 \geq -GSV_p(\mathcal{F}, C). \tag{21}$$

The proof follows by summing over all singularities of $\mathcal{F}$ in C and using the GSV-index formula [7, Proposition 2.3] $\sum_p GSV_p(\mathcal{F}, C) = (d + 2)d_0 - d_0^2$, and then substituting this equality into (21). □

7.3 Bounds of the global Tjurina number of a foliation

Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^2_{\mathbb{C}}$ of degree d, having an invariant reduced algebraic curve C of degree d_0. We define the global Tjurina number of $\mathcal{F}$ with respect to C as $\tau(\mathcal{F}, C) := \sum_p \tau_p(\mathcal{F}, C)$, which is the sum of the Tjurina number of $\mathcal{F}$ at all singularities p of $\mathcal{F}$ on C. Similarly, we denote $\mu(C) := \sum_p \mu_p(C)$, the *global Milnor number* of C and $\tau(C) := \sum_p \tau_p(C)$, the *global Tjurina number* of C.

Remark 7.4 Teissier's Proposition [41, Chapter II, Proposition 1.2] implies that

$$I_p(\mathcal{P}^{df}, C) = \mu_p(C) + \nu_p(C) - 1, \tag{22}$$

and by [29, Theorem 4.18], we get

$$I_p(\mathcal{P}^{df}, C) \geq \nu_p(df) \cdot \nu_p(C) = (\nu_p(C) - 1) \cdot \nu_p(C). \tag{23}$$

Using (22) and (23), we obtain $\mu_p(C) \geq (\nu_p(C) - 1)^2$.

Now, we present bounds for the global Tjurina number of a foliation on $\mathbb{P}^2_{\mathbb{C}}$.

Theorem 7.5 *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geq 2$ having an invariant irreducible algebraic curve C of degree $d_0 \geq 2$. Then*

$$\tau(\mathcal{F}, C) = d_0(d-1) - 2g(C) - \sum_p (r_p - 1) - \mu(C) + \tau(C) + 2,$$

where $g(C)$ is the geometric genus of C and r_p is the number of branches of C at p. Moreover,

$$\tau(\mathcal{F}, C) \leq d_0(d_0 - 3) + d(d+1) - 2g(C) - \sum_p (r_p - 1) - \sum_p (\nu_p(C) - 1)^2 + 3,$$

and if C is not concurrent lines, then

$$2 - 2g(C) + \left[\frac{d_0}{2}\right] + d - d_0 - \sum_p (r_p - 1) \leq \tau(\mathcal{F}, C).$$

Proof For each singularity p of $\mathcal{F}$ such that $p \in C$, we have

$$\tau_p(\mathcal{F}, C) = \mu_p(\mathcal{F}, C) - \mu_p(C) + \tau_p(C)$$

by Proposition 4.1. Then, using the definition of $\tau(\mathcal{F}, C)$, Eq. (5) and Proposition 7.1, we get

$$\begin{aligned}\tau(\mathcal{F}, C) = \sum_p \tau_p(\mathcal{F}, C) &= \sum_p \mu_p(\mathcal{F}, C) - \mu(C) + \tau(C) \\ &= \sum_p \left(\sum_i \mu_p(\mathcal{F}, C_i) - r_p + 1 \right) - \mu(C) + \tau(C) \\ &= \sum_p \sum_i \mu_p(\mathcal{F}, C_i) - \sum_p (r_p - 1) - \mu(C) + \tau(C) \\ &= d_0(d-1) + 2 - 2g(C) - \sum_p (r_p - 1) - \mu(C) + \tau(C). \end{aligned} \tag{24}$$

Moreover, by [38, Corollary 3.4], if C is not concurrent lines, we have

$$\mu(C) \leq (d_0 - 1)^2 - \left[\frac{d_0}{2}\right]. \tag{25}$$

Furthermore, it follows from [18, Theorem 3.2] that

$$(d_0 - d - 1)(d_0 - 1) \leq \tau(C) \leq (d_0 - 1)^2 - d(d_0 - d - 1). \tag{26}$$

Finally, the theorem follows by substituting (25), (26) and Remark 7.4 into (24). □

7.4 Alcántara–Mozo-Fernández's conjecture

Now, we address the following conjecture posed by Alcántara–Mozo-Fernández [1, p. 16]: *Does there exist a logarithmic, non-dicritical and quasi-homogeneous foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$, with a unique singular point p, having an invariant reduced algebraic curve C passing through p?*

These authors indicate that they have some evidence suggesting that this type of foliation does not exist; however, they state that they do not yet have a solution to the problem.

Recall that a foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ is *logarithmic* if there exist homogeneous, irreducible polynomials $f_1, \ldots, f_m \in \mathbb{C}[x, y, z]$, of degrees $d_1, \ldots, d_m$, respectively, such that a 1-form describing the foliation is:

$$\Omega = f_1 \cdot \ldots \cdot f_m \sum_{i=1}^{m} \lambda_i \frac{df_i}{f_i},$$

for some nonzero complex numbers $\lambda_1, \ldots, \lambda_m$ satisfying $\sum_{i=1}^{m} \lambda_i d_i = 0$. Note that the foliation has degree $\deg(\mathcal{F}) = \sum_{i=1}^{m} d_i - 2$. In particular, taking $C = \{f_1 \cdots f_m = 0\}$, we obtain

$$\deg(C) = \deg(\mathcal{F}) + 2. \tag{27}$$

A logarithmic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ is always a generalized curve foliation, by [21, Section 6].

We also recall that a non-dicritical foliation $\mathcal{F}$ is *quasi-homogeneous* at p, if the set of all separatrices of $\mathcal{F}$ passing by p is a germ of curve given in some coordinates by a quasi-homogeneous polynomial, see [25, p. 1657].

We propose the following answer in the case where C is irreducible. Note that in this situation it is not necessarily assume that the foliation is quasi-homogeneous.

Theorem 7.6 *There does not exist a logarithmic, non-dicritical foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ of degree d, with a unique singular point p, having an invariant irreducible algebraic curve C of degree $d_0 \geq 2$ passing through p.*

Proof Suppose, for contradiction, that there exists a logarithmic, non-dicritical foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ of degree d, with a unique singular point p, having an invariant irreducible algebraic curve C of degree $d_0 \geq 2$ passing through p. Since $\mathcal{F}$ is logarithmic, it follows that $\mathcal{F}$ is a generalized curve foliation at p and $d_0 = d + 2$ (see [21, Section 6]). Moreover, since $\mathcal{F}$ is non-dicritical at p, we have

$$\mu_p(C) = \mu_p(\mathcal{F}) = d^2 + d + 1 \tag{28}$$

by [9, Theorem 4] and [7, p. 19]. Now, since C is irreducible, we can apply Płoski's inequality [35, Lemma 2.2] to C at p, obtaining

$$\mu_p(C) \leq (d_0 - 1)(d_0 - 2). \tag{29}$$

Combining (28) and (29), and using $d = d_0 - 2$, we get

$$d^2 + d + 1 = d_0^2 - 3d_0 + 3 \leq d_0^2 - 3d_0 + 2,$$

implying $1 \leq 0$, which is a contradiction. This completes the proof. □

Now, we present an answer to the conjecture posed by Alcántara and Mozo-Fernández in the case where C is reduced.

Theorem 7.7 *There does not exist a logarithmic, non-dicritical, quasi-homogeneous foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ of degree d, with a unique singular point p, that admits an invariant reduced algebraic curve C of degree $d_0 \geq 2$, containing all separatrices passing through p.*

Proof Suppose, by contradiction, that there exists a logarithmic, non-dicritical, quasi-homogeneous foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ of degree d, with a unique singular point p, and an invariant reduced algebraic curve C of degree $d_0 \geq 2$ which contains all the separatrices passing through p.

We fix the singular point at $p = [0:0:1]$ and consider local coordinates $(x, y) \in \mathbb{C}^2$ centered at p. Since C contains all the separatrices passing through p, and $\mathcal{F}$ is non-dicritical at p, it follows that C is the total union of the separatrices of $\mathcal{F}$ at p. Moreover, if the germ of C at p is given by $C_p = \{f(x, y) = 0\}$, then, by hypothesis, f is quasi-homogeneous. Hence, after a suitable change of variables, we can write

$$f(x, y) = u \cdot x^m \cdot y^n \Pi_{j=1}^s (y^k - \zeta_j x^q),$$

where u is a unit, $m, n \in \{0, 1\}$, $\gcd(k, q) = 1$, and $\zeta_j \in \mathbb{C}^*$ for all $j = 1, \ldots, s$. Without loss of generality, we may assume that $k < q$. Now, we analyze the possible branches of f. First, note that $m = 0$, because the intersection $\{x = 0\} \cap \{y^k z^{q-k} - \zeta_j x^q = 0\}$ consists of two points, namely $[0:0:1]$ and $[0:1:0]$, which would yield two singularities of $\mathcal{F}$, contradicting our hypothesis. Additionally, C cannot have two different branches of the form $C_j = y^k - \zeta_j x^q$; otherwise, taking two such branches, say $C_i = y^k - \zeta_i x^q$ and $C_j = y^k - \zeta_j x^q$, we obtain, by Bézout's theorem

$$I_p(C_i, C_j) = kq = q^2,$$

which implies that $k = q$, yielding a contradiction. Consequently, we conclude that $f(x, y) = uy(y^k - \zeta x^q)$, where $\zeta \in \mathbb{C}^*$. In particular, $d_0 = q + 1$, and since $\mathcal{F}$ is logarithmic, $d = d_0 - 2 = q - 1$ by (27). Moreover, since $\mathcal{F}$ is a non-dicritical generalized curve foliation, we obtain by [7, p. 19] and [9, Theorem 4]

$$d^2 + d + 1 = \mu_p(\mathcal{F}) = \mu_p(C) = \mu_p(y) + \mu_p(y^k - \zeta x^q) + 2q - 1,$$

which implies that $q = k+1$. Thus, we conclude that $f(x, y) = uy(y^k - \zeta x^{k+1})$. In particular, f has two branches and contains all separatrices of $\mathcal{F}$ passing through p. Now, observing that in homogeneous coordinates $[x:y:z]$ of $\mathbb{P}^2_{\mathbb{C}}$, the curve $C = \{y(y^k z - \zeta x^{k+1}) = 0\}$ is an algebraic curve invariant by $\mathcal{F}$, and since $d_0 = q + 1 = k + 2$ and $d = q - 1 = k$, we conclude that $\mathcal{F}$ can be defined by a closed logarithmic 1-form with poles along C, by Cerveau's theorem [11, Theorem 1.4]. That is, there exist $\lambda_1, \lambda_2 \in \mathbb{C}^*$ satisfying $\lambda_1 + (k+1)\lambda_2 = 0$, and

$$\mathcal{F}: \quad \Omega = \lambda_1 \frac{dy}{y} + \lambda_2 \frac{d(y^k z - \zeta x^{k+1})}{y^k z - \zeta x^{k+1}}.$$

However, since $\lambda_1 + (k+1)\lambda_2 = 0$, we can write Ω as:

$$\Omega = \lambda_2 \frac{y^{k+1}}{(y^k z - \zeta x^{k+1})} d\left(\frac{y^k z - \zeta x^{k+1}}{y^{k+1}}\right),$$

which implies that $\mathcal{F}$ admits a rational first integral and, consequently, has a unique dicritical singularity at $p = [0 : 0 : 1]$. This contradicts the assumption that $\mathcal{F}$ is a non-dicritical foliation at p. □

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Data Availability
We do not analyze or generate any datasets, because our work proceeds within a theoretical approach.

Declarations

Conflict of interest
The authors declare that they have no conflict of interest.

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